Strong separations and Kolmogorov complexity

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Recursion theory

A string is a finite or infinite sequence of 0's and 1's. The set of all finite strings is $\{0,1\}^{<\mathbb{N}}$; elements are usually denoted as σ or τ. The set of all infinite strings is $\{0,1\}^{\mathbb{N}}$, also known as Cantor space; elements are usually denoted as X or Y.

A function $\varphi :\subseteq \{0,1\}^{<\mathbb{N}} \to \{0,1\}^{<\mathbb{N}}$ is **partial recursive** if it can be emulated by a Turing machine, and **recursive** if it is partial recursive and total.

A set $A \subseteq \{0,1\}^{<\mathbb{N}}$ is **recursively enumerable (r.e.)** if there is a partial recursive φ such that $A = \mathrm{rng}(\varphi)$.

Prefix-free complexity

A **prefix-free machine** is a partial recursive function $M :\subseteq \{0,1\}^{<\mathbb{N}} \to \{0,1\}^{<\mathbb{N}}$ such that if $M(\sigma)\downarrow$ and $\sigma \subset \tau$ then $M(\tau)\uparrow$.

We can view the string τ as a description of the string σ if $M(\tau) = \sigma$.

There is a universal prefix-free machine U: for any prefix-free machine M there is a τ_M such that $U(\tau_M{}^\frown\sigma) = M(\sigma)$ for all σ.

Definition

The **prefix-free complexity** of σ is $\mathrm{K}(\sigma) = \min\{|\tau| \mid U(\tau) = \sigma\}$.

A notion of partial randomness

$X \upharpoonright n$ means the length n initial segment of X.

Theorem

X is **Martin-Löf random** if $\mathrm{K}(X \upharpoonright n) \geq n - O(1)$.

Let $f : \mathbb{N} \to \mathbb{N}$ be recursive.

Definition

X is ***f*-complex** or ***f*-random** if $\mathrm{K}(X \upharpoonright n) \geq f(n) - O(1)$.

Motivating strong separations

Let $f_s : \mathbb{N} \to [0, \infty)$ be defined by $f(n) = sn$ for all n, and fixed recursive s.

If X is Martin-Löf random, then $Y = X \oplus 0^{\mathbb{N}}$ is $f_{1/2}$-random but not f_s-random for any $s > 1/2$.

It is clear in this example that the constructed Y still computes X, and that X is f_s-random for all $1/2 < s \leq 1$.

Theorem (Miller [5])

There is an X which is $f_{1/2}$-random and does not compute any Y which is f_s-random for any $s > 1/2$.

Question

For which recursive f and g does there exist an X which is f-random but does not compute any Y which is g-random?

Convex order functions

An **order function** (or simply an **order**) is a recursive function $f : \mathbb{N} \to \mathbb{N}$ which is unbounded and non-decreasing.

An order f is **convex** if additionally it does not grow too fast:

1. $f(n+1) \leq f(n) + 1$ for all n, and
2. $f(n+1) = f(n)$ infinitely often.

Condition (2) above is equivalent to saying that $\lim_{n \to \infty} n - f(n) = \infty$.

The function $\mathrm{id}_{\mathbb{N}} : \mathbb{N} \to \mathbb{N}$ is an upper bound (modulo a constant) of every convex order function.

Strong domination

Let f be an order. The **increasing set** of f is the set $\mathcal{I}(f) = \{n \mid (\forall m < n)(f(m) < f(n))\}$.

Let $f, g : \mathbb{N} \to \mathbb{N}$ be orders. We say that f is **strongly dominated** by g, written $f \ll g$, if the sum

$$\sum_{n \in \mathcal{I}(f)} 2^{f(n)-g(n)} < \infty$$

is a recursive number.

Equivalently $f \ll g$ if there is a function $h : \mathbb{N} \to \mathbb{N}$ such that $g(n) \geq f(n) + h(f(n))$ for all n and

$$\sum_{n \in \mathbb{N}} 2^{-h(n)} < \infty$$

is a recursive number.

Extracting additional randomness

Theorem

Let f be a convex order. Then there is an X which is f-random, but which does not compute any Y which is g-random for any order $g \gg f$.

Note that we defined ***f*-random** in terms of prefix-free complexity. If we define ***f*-random** instead in terms of either **monotone complexity** or **a priori complexity**, the theorem is still true.

Strong separation examples

Choices of f might include*:

- $f(n) = sn$ for recursive $0 < s < 1$,
- $f(n) = \sqrt{n}$,
- $f(n) = \log_2 n$, or
- $f(n) = n - 2\log_2 n$.

Choices of g might include*:

- $g(n) = (1+\epsilon)f(n)$ for recursive $\epsilon > 0$,
- $g(n) = f(n) + \sqrt{f(n)}$.
- $g(n) = f(n) + 2\log_2 f(n)$, or
- $g(n) = f(n) + \log_2 f(n) + 2\log_2 \log_2 f(n)$.

$\ast$ The theorem holds for real-valued functions in addition to integer-valued.

Effective Hausdorff dimension

The **effective Hausdorff dimension** of X is the quantity (Mayordomo [4])

$$\dim(X) = \liminf_{n\to\infty} \frac{\mathrm{K}(X \upharpoonright n)}{n}.$$

Miller's result essentially shows that there exists a Turing degree of effective Hausdorff dimension 1/2.

The improved strong separation result can also be used to prove the following, originally a theorem of Greenberg and Miller [2]:

Theorem

There exists an X such that $\dim(X) = 1$ but X does not compute a Martin-Löf random.

Proof.

Let $f(n) = |\sigma| - 2\log_2 |\sigma|$ and let $g(n) = |\sigma|$. □

An application to DNR functions

$p : \mathbb{N} \to \mathbb{N}$ is **diagonally non-recursive (DNR)** if $\varphi_n(n) \neq p(n)$ for all n.

Theorem (Kjos-Hanssen/Merkle/Stephan [3])

X is f-random for some order f if and only if X wtt-computes a DNR function p which is bounded by some recursive function g.

Then we can convert back and forth between recursively-bounded DNR and f-random. Then our main theorem on non-extraction of randomness gives a new, bushy-tree free proof of the following theorem:

Theorem (Ambos-Spies/Kjos-Hanssen/Lempp/Slaman [1])

Let $f : \mathbb{N} \setminus \{0, 1\} \to \mathbb{N}$ be recursive. There is a DNR function p which is recursively-bounded, yet which does not compute any f-bounded DNR function q.

A few words on proving the main theorem

Large parts of the proof follow Miller's construction of a Turing degree of dimension 1/2. Some key techniques we use are the following:

- Equivalence between test and complexity definitions of f-randomness
- An expanded notion of **optimal covers**
- Forcing conditions which are Π^0_1 sets of positive measure
- The **increasing set** and **strong domination** allow us to look at non-extraction in a meaningful way even when the base amount of randomness is small (sublinear).

Thank you for listening!

References

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