Symbolic Dynamics:
Entropy = Dimension = Complexity

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Symbolic dynamics.

Let G be $(\mathbb{N}^d, +)$ or $(\mathbb{Z}^d, +)$ where $d \geq 1$.

Let A be a finite set of symbols.

We endow A with the discrete topology and A^G with the product topology.

The *shift action* of G on A^G is given by $(S^g x)(h) = x(g+h)$ for $g, h \in G$ and $x \in A^G$.

A *subshift* is a nonempty set $X \subseteq A^G$ which is topologically closed and *shift-invariant*, i.e., $x \in X$ implies $S^g x \in X$ for all $g \in G$.

Symbolic dynamics is the study of subshifts.

If $X \subseteq A^G$ and $Y \subseteq B^G$ are subshifts, a *shift morphism* from X to Y is a continuous mapping $\Phi : X \to Y$ such that $\Phi(S^g x) = S^g \Phi(x)$ for all $x \in X$ and $g \in G$.

By compactness, any shift morphism Φ is given by a *block code*, i.e., a finite mapping $\phi : A^F \to B$ where F is a finite subset of G and $\Phi(x)(g) = \phi(S^g x \restriction F)$ for all $x \in X$ and $g \in G$.

I have been applying recursion-theoretic concepts such as Muchnik degrees and Kolmogorov complexity to obtain new results in symbolic dynamics.

Muchnik degrees of subshifts.

A subshift X is *of finite type* if it is given by a finite set of excluded finite configurations:

$$X = \{x \in A^G \mid (\forall g \in G)\,(S^g x{\upharpoonright}F \notin E)\}$$

where E and F are finite.

Recall that $\mathcal{E}_{\mathsf{w}}$ is the lattice of Muchnik degrees of nonempty Π^0_1 classes, in Cantor space (or in Euclidean space).

Recall also that $\mathcal{E}_{\mathsf{w}}$ includes many specific, natural degrees which are associated with foundationally interesting topics.

A picture of $\mathcal{E}_{\mathsf{w}}$ is on slides 5 and 6.

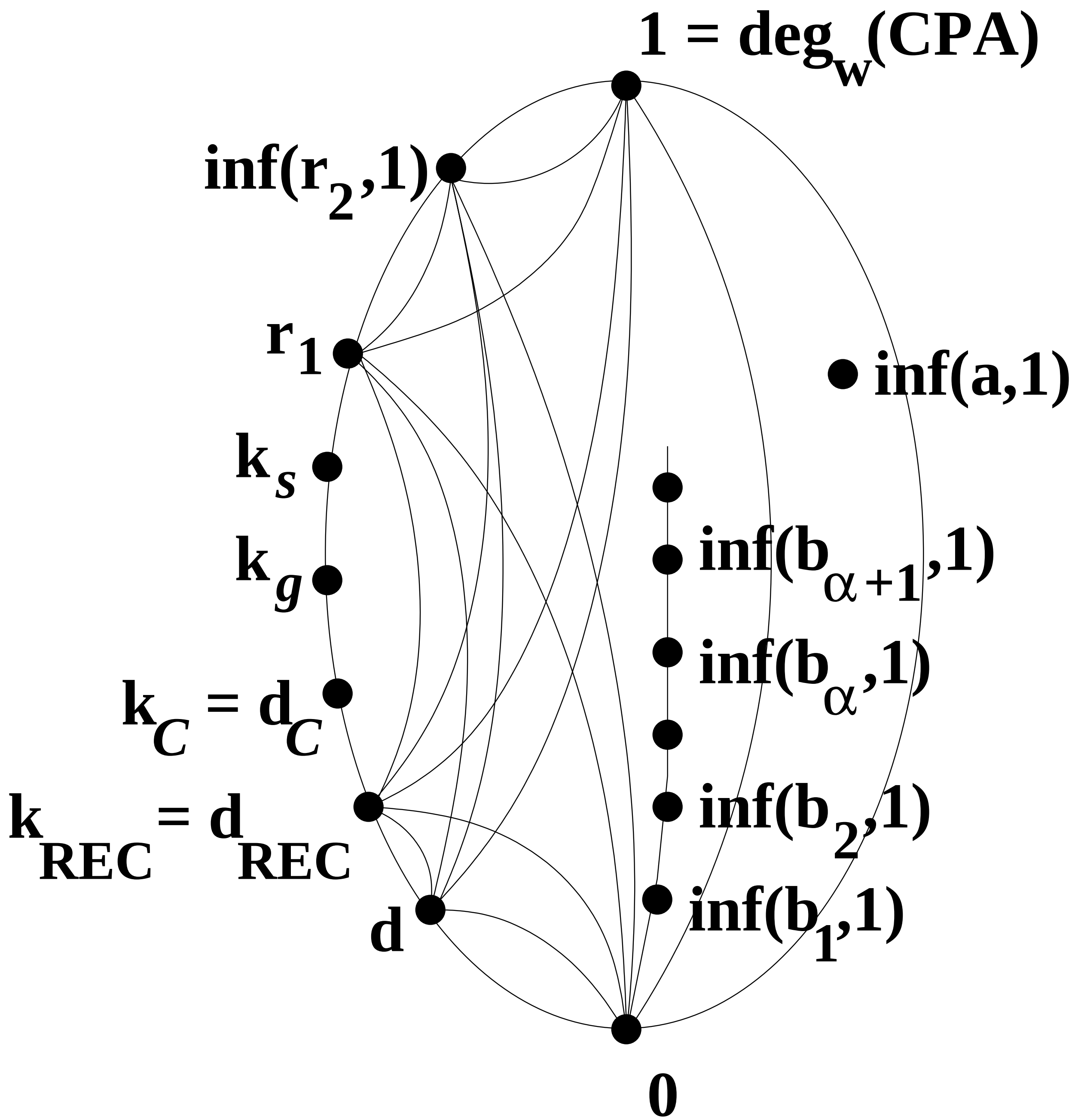
Theorem (Simpson 2007)**.** The Muchnik degrees in $\mathcal{E}_{\mathsf{w}}$ are precisely the Muchnik degrees of $\mathbb{Z}^2$-subshifts of finite type.

Proof. One direction is trivial, because subshifts of finite type may be viewed as Π^0_1 classes. My proof of the converse uses tiling techniques which go back to Berger 1966, Robinson 1971, and Myers 1974. Another proof, due to Durand/Romashchenko/Shen, uses "self-replicating tile sets."

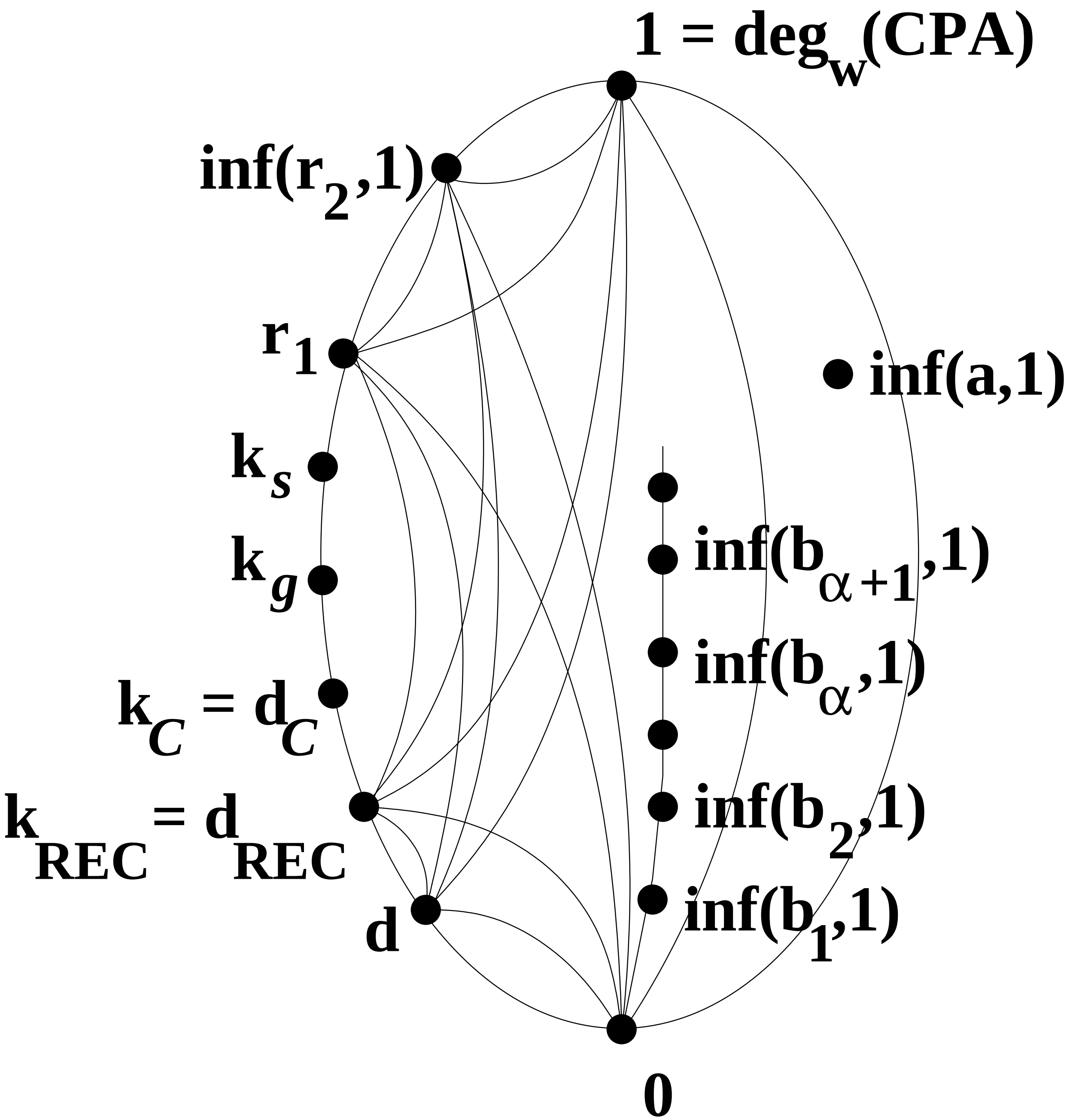
Corollary (Simpson 2007)**.** We can construct an infinite family of $\mathbb{Z}^2$-subshifts of finite type which are strongly independent with respect to shift morphisms, etc.

Proof. This follows from the existence of an infinite independent set of degrees in $\mathcal{E}_{\mathrm{w}}$, which is proved by means of a priority argument.

Thus we have an application of recursion theory (tiling methods plus priority argument) to prove a result in symbolic dynamics which does not mention computability concepts.



A picture of $\mathcal{E}_\mathsf{w}$. Each black dot except inf$(\mathbf{a}, \mathbf{1})$ represents a specific, natural degree in $\mathcal{E}_\mathsf{w}$. We shall explain some of these degrees.



A picture of $\mathcal{E}_\mathrm{w}$. Here $\mathbf{a}=$ any r.e. degree, $\mathbf{r}=$ randomness, $\mathbf{b}=$ LR-reducibility, $\mathbf{k}=$ complexity, $\mathbf{d}=$ diagonal nonrecursiveness.

We now explain some degrees in $\mathcal{E}_\mathsf{w}$.

The top degree in $\mathcal{E}_\mathsf{w}$ is $\mathbf{1} = \mathrm{deg}_\mathsf{w}(\mathrm{CPA})$ where CPA is the problem of finding a complete consistent theory which includes Peano arithmetic (or ZFC, etc.).

We also have $\inf(\mathbf{a}, \mathbf{1}) \in \mathcal{E}_\mathsf{w}$ where $\mathbf{a}$ is any recursively enumerable Turing degree. Moreover, $\mathbf{a} < \mathbf{b}$ implies $\inf(\mathbf{a}, \mathbf{1}) < \inf(\mathbf{b}, \mathbf{1})$

We have $\mathbf{r}_1 \in \mathcal{E}_\mathsf{w}$ where $\mathbf{r}_1 = \mathrm{deg}_\mathsf{w}(\mathrm{MLR})$, $\mathrm{MLR} = \{x \in 2^{\mathbb{N}} \mid x \text{ is Martin-Löf random}\}$).

We also have $\inf(\mathbf{r}_2, \mathbf{1}) \in \mathcal{E}_\mathsf{w}$ where $\mathbf{r}_2 = \mathrm{deg}_\mathsf{w}(\{x \in 2^{\mathbb{N}} \mid x \text{ is } 2\text{-}\textit{random}\})$, i.e., random relative to the halting problem.

Also $\mathbf{d} \in \mathcal{E}_\mathsf{w}$ where $\mathbf{d} = \mathrm{deg}_\mathsf{w}(\{f \mid f \text{ is } \textit{diagonally nonrecursive}\})$, i.e., $\forall n\,(f(n) \neq \varphi_n(n))$.

Let REC $= \{g \in \mathbb{N}^{\mathbb{N}} \mid g$ is recursive$\}$.
Let C be any "nice" subclass of REC.
For instance $C =$ REC, or $C = \{g \in \text{REC} \mid g$ is primitive recursive$\}$. We have $\mathbf{d}_C \in \mathcal{E}_{\mathsf{w}}$ where $\mathbf{d}_C = \deg_{\mathsf{w}}(\{f \in \mathbb{N}^{\mathbb{N}} \mid f$ is diagonally nonrecursive and C-*bounded*$\})$,
i.e., $(\exists g \in C)\, \forall n\, (f(n) < g(n))$.

Also, $\mathbf{d}_C = \deg_{\mathsf{w}}(\{x \in 2^{\mathbb{N}} \mid x$ is C-*complex*$\}$,
i.e., $(\exists g \in C)\, \forall n\, (\mathsf{K}(x{\upharpoonright}\{1,\ldots,g(n)\}) \geq n)\})$.
Moreover, $\mathbf{d}_{C'} < \mathbf{d}_C$ whenever C' contains a function which dominates all functions in C.

For $x \in 2^{\mathbb{N}}$ let effdim$(x) =$ the *effective Hausdorff dimension* of x, i.e.,

$$\text{effdim}(x) = \liminf_{n\to\infty} \frac{\mathsf{K}(x{\upharpoonright}\{1,\ldots,n\})}{n}.$$

Given a right recursively enumerable real number $s < 1$, we have $\mathbf{k}_s \in \mathcal{E}_{\mathsf{w}}$ where

$$\mathbf{k}_s = \deg_{\mathsf{w}}(\{x \in 2^{\mathbb{N}} \mid \text{effdim}(x) > s\}).$$

Moreover, $s < t$ implies $\mathbf{k}_s < \mathbf{k}_t$ (Miller).

More generally, let $g : \mathbb{N} \to [0, \infty)$ be an unbounded computable function such that $g(n) \leq g(n+1) \leq g(n) + 1$ for all n.

For example, $g(n)$ could be $n/2$ or $n/3$ or $\sqrt{n}$ or $\sqrt[3]{n}$ or $\log n$ or $\log n + \log\log n$ or $\log\log n$ or the inverse Ackermann function.

Define $\mathbf{k}_g = \deg_{\mathsf{w}}(\{x \in 2^{\mathbb{N}} \mid x$ is g-*complex*$\})$, i.e., $\exists c\, \forall n\, (\mathsf{K}(x{\upharpoonright}\{1, \ldots, n\} \geq g(n) - c)$.

Theorem (Hudelson 2010). We have $\mathbf{k}_g < \mathbf{k}_h$ provided $g(n) + 2\log g(n) \leq h(n)$ for all n.

In other words, there exists a g-complex real with no h-complex real Turing reducible to it.

This is a generalization of Miller's theorem on the difficulty of information extraction.

References:

Phil Hudelson, Mass problems and initial segment complexity, in preparation.

Joseph S. Miller, Extracting information is hard, to appear in *Advances in Mathematics*.

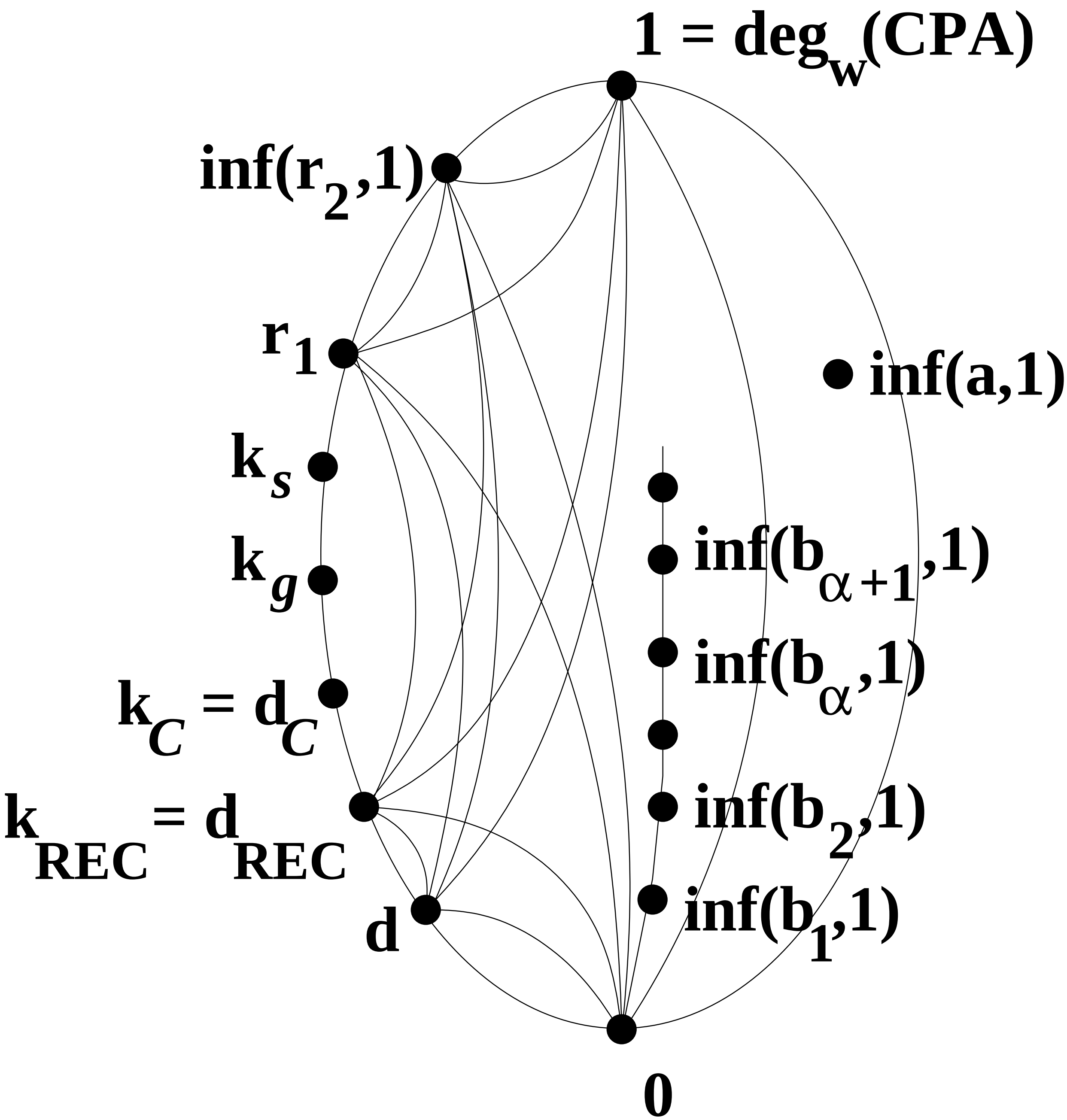
Letting z be a Turing oracle, define
$\mathsf{MLR}^z = \{x \in 2^{\mathbb{N}} \mid x \text{ is random relative to } z\}$
and $\mathsf{K}^z(n) =$ the prefix-free Kolmogorov complexity of n relative to z.

Define $y \leq_{\mathsf{LR}} z \Longleftrightarrow \mathsf{MLR}^z \subseteq \mathsf{MLR}^y$
and $y \leq_{\mathsf{LK}} z \Longleftrightarrow \exists c \, \forall n \, (\mathsf{K}^z(n) \leq \mathsf{K}^y(n) + c)$.

Theorem (Miller/Kjos-Hanssen/Solomon)**.**
We have $y \leq_{\mathsf{LR}} z \Longleftrightarrow y \leq_{\mathsf{LK}} z$.

For each recursive ordinal number α, let $0^{(\alpha)} =$ the αth iterated Turing jump of 0. Thus $0^{(1)} =$ the halting problem, and $0^{(\alpha+1)} =$ the halting problem relative to $0^{(\alpha)}$, etc. This is the hyperarithmetical hierarchy. We embed it naturally into $\mathcal{E}_{\mathsf{w}}$ as follows.

Theorem (Simpson 2009)**.** $0^{(\alpha)} \leq_{\mathsf{LR}} z$
$\Longleftrightarrow$ every $\Sigma^0_{\alpha+2}$ set includes a $\Sigma^{0,z}_2$ set of the same measure. Moreover, letting $\mathbf{b}_\alpha = \deg_{\mathsf{w}}(\{z \mid 0^{(\alpha)} \leq_{\mathsf{LR}} z\})$ we have $\inf(\mathbf{b}_\alpha, \mathbf{1}) \in \mathcal{E}_{\mathsf{w}}$ and $\inf(\mathbf{b}_\alpha, \mathbf{1}) < \inf(\mathbf{b}_{\alpha+1}, \mathbf{1})$.



A picture of $\mathcal{E}_{\mathrm{w}}$. Here $\mathbf{a}=$ any r.e. degree, $\mathbf{r}=$ randomness, $\mathbf{b}=$ LR-reducibility, $\mathbf{k}=$ complexity, $\mathbf{d}=$ diagonal nonrecursiveness.

Two subshifts are said to be *conjugate* if they are topologically isomorphic, i.e., there is a shift isomorphism between them.

The basic problem of symbolic dynamics is: classify subshifts up to conjugacy invariance.

Muchnik degrees can help, because the Muchnik degree of a subshift is a conjugacy invariant. Each of the Muchnik degrees in $\mathcal{E}_{\mathsf{w}}$ including $\mathbf{0}$, $\mathbf{1}$, $\mathbf{r}_1$, $\mathbf{d}$, $\mathbf{d}_{\mathsf{REC}}$, $\mathbf{d}_C$, $\mathbf{k}_s$, $\mathbf{k}_g$, $\inf(\mathbf{r}_2, \mathbf{1})$, $\inf(\mathbf{b}_\alpha, \mathbf{1})$, and even $\inf(\mathbf{a}, \mathbf{1})$ may be viewed as a conjugacy invariant for subshifts of finite type.

It is interesting to compare the Muchnik degree of a subshift X with other conjugacy invariants, e.g., the entropy of X.

Generally speaking, the Muchnik degree of X represents a lower bound on the complexity of the orbits, while the entropy of X is an upper bound on the complexity of these same orbits.

History:

Kolmogorov 1932 developed his "calculus of problems" as a nonrigorous yet compelling explanation of Brouwer's intuitionism. Later Medvedev 1955 and Muchnik 1963 proposed Medvedev degrees and Muchnik degrees as rigorous explications of Kolmogorov's idea.

Some references:

Stephen G. Simpson, Mass problems and randomness, Bulletin of Symbolic Logic, 11, 2005, pages 1–27.

Stephen G. Simpson, An extension of the recursively enumerable Turing degrees, Journal of the London Mathematical Society, 75, 2007, pages 287–297.

Stephen G. Simpson, Mass problems and intuitionism, Notre Dame Journal of Formal Logic, 49, 2008, pages 127–136.

Stephen G. Simpson, Mass problems and measure-theoretic regularity, Bulletin of Symbolic Logic, 15, 2009, pages 385–409.

Stephen G. Simpson, Medvedev degrees of 2-dimensional subshifts of finite type, to appear in Ergodic Theory and Dynamical Systems.

Stephen G. Simpson, Entropy equals dimension equals complexity, 2010, in preparation.

A possibly interesting research program:

Given a subshift X, explore the relationship between the <u>*dynamical properties* of X</u> and the <u>*degree of unsolvability* of X</u>, i.e., its Muchnik degree, $\deg_{\mathrm{w}}(X)$.

For example, the *entropy* of X is a well-known dynamical property which serves as an <u>upper bound</u> on the complexity of orbits. In particular $\mathrm{ent}(X) > 0$ implies $(\exists x \in X)\,(x$ is not computable$)$.

By contrast, the degree of unsolvability of X serves as a <u>lower bound</u> on the complexity of orbits. For instance, $\deg_{\mathrm{w}}(X) > \mathbf{0} \iff$ $(\forall x \in X)\,(x$ is not computable$)$.

Theorem (Hochman)**.** If X is of finite type and *minimal* (i.e., every orbit is dense), then $\deg_{\mathrm{w}}(X) = \mathbf{0}$.

More generally, the theorem holds for all Π^0_1 subshifts, not necessarily of finite type.

Some new (?) results on subshifts:

Let d be a positive integer, let A be a finite set of symbols, and let X be a nonempty subset of A^G where G is $\mathbb{N}^d$ or $\mathbb{Z}^d$.

The *Hausdorff dimension*, $\text{dim}(X)$, and the *effective Hausdorff dimension*, $\text{effdim}(X)$, are defined as usual with respect to the standard metric $\rho(x,y) = 2^{-|F_n|}$ where n is as large as possible such that $x{\upharpoonright}F_n = y{\upharpoonright}F_n$.

Here F_n is $\{1,\ldots,n\}^d$ if $G = \mathbb{N}^d$,
or $\{-n,\ldots,n\}^d$ if $G = \mathbb{Z}^d$.

We first state some old results.

1. $\text{effdim}(X) = \sup\limits_{x \in X} \text{effdim}(x)$.

2. $\text{effdim}(x) = \liminf\limits_{n \to \infty} \dfrac{\mathsf{K}(x{\upharpoonright}F_n)}{|F_n|}$.

3. $\text{effdim}(X) = \text{dim}(X)$
   provided X is <u>effectively closed</u>, i.e., Π^0_1.

Here K denotes Kolmogorov complexity.

Theorem (Simpson 2010)**.** Assume that X is a subshift, i.e., X is closed and shift-invariant. Then

$$\mathsf{effdim}(X) = \mathsf{dim}(X) = \mathsf{ent}(X).$$

Moreover

$$\mathsf{dim}(X) \geq \limsup_{n\to\infty} \frac{\mathsf{K}(x{\restriction}F_n)}{|F_n|} \quad \text{for all } x \in X,$$

and

$$\mathsf{dim}(X) = \lim_{n\to\infty} \frac{\mathsf{K}(x{\restriction}F_n)}{|F_n|} \quad \text{for some } x \in X.$$

Note. In the above theorem, there is no finiteness or computability hypothesis on the subshift X. Moreover, X can be a G-subshift where G is $\mathbb{N}^d$ or $\mathbb{Z}^d$ for any positive integer d.

Remark. Here $\mathsf{ent}(X)$ denotes *entropy*,

$$\mathsf{ent}(X) = \lim_{n\to\infty} \frac{\log_2 |\{x{\restriction}F_n \mid x \in X\}|}{|F_n|}.$$

This is known to be a conjugacy invariant.

Remark. The proof of our theorem involves ergodic theory (Shannon/McMillan/Breiman, the Variational Principle, etc.) plus a combinatorial argument which is similar to the proof of the Vitali Covering Lemma.

Remark. So far as I can tell, everything in the theorem is new, except the following old result due to Furstenberg 1967:

$$\text{dim}(X) = \text{ent}(X) \text{ provided } G = \mathbb{N}.$$

The proof of this special case is much easier.

Remark. The above theorem is an outcome of my discussions at Penn State during February–April 2010 with many people including John Clemens, Mike Hochman, Dan Mauldin, Jan Reimann, and Sasha Shen.

THE END.

THANK YOU!