Reverse Mathematics and Π^1_2 Comprehension

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NSF-DMS-0600823, NSF-DMS-0652637,
Grove Endowment, Templeton Foundation

AMS Special Session

Lexington, Kentucky

March 27–28, 2010

Two books on reverse mathematics:

1. RM2001

S. G. Simpson, editor
Reverse Mathematics 2001
(a volume of papers by various authors)
Volume 21, Lecture Notes in Logic
Association for Symbolic Logic
VIII + 401 pages, 2005

2. SOSOA

Stephen G. Simpson
Subsystems of Second Order Arithmetic
Second Edition
Perspectives in Logic
Association for Symbolic Logic
XVI + 444 pages, 2009

Reverse mathematics is a particular program within the foundations of mathematics.

The purpose of reverse mathematics is to discover which set existence axioms are needed in order to prove specific theorems of *ordinary* or *core* mathematics: real analysis, functional analysis, complex analysis, countable algebra, countable combinatorics, geometry, etc.

Often the theorems turn out to be equivalent to the axioms. Hence the slogan "reverse mathematics".

The language of second-order arithmetic is in some sense the most economical one for the logical, axiomatic development of the bulk of core mathematics. Therefore, in my book SOSOA, reverse mathematics is carried out in this context.

Some of the most important subsystems of second-order arithmetic are:

RCA_0 (= recursive comprehension), WKL_0,

ACA_0 (= arithmetical comprehension), ATR_0,

Π^1_1-CA_0 (= Π^1_1 comprehension).

These five systems collectively are known as "THE BIG FIVE".

In this talk we move beyond the big five, to Π^1_2-CA_0, i.e., Π^1_2 comprehension, which is known to be much, much stronger than Π^1_1 comprehension.

In recent years Rathjen and his colleagues have achieved *excellent proof-theoretical understanding* of Π^1_2-CA_0: ordinal notations, inductive definitions, μ-calculus.

Therefore, it seems safe to say:

In principle, any core mathematical theorem which is provable in Π^1_2-CA_0 can be given a constructive formulation which can then be proved constructively.

Remark:

This talk represents joint work with Carl Mummert, my Ph.D. student at the Pennsylvania State University.

Background:

In my book *SOSOA*, a *complete separable metric space* is defined as the completion $X = (\widehat{A}, \widehat{d})$ of a countable pseudometric space (A, d). Here $A \subseteq \mathbb{N}$ and $d : A \times A \to \mathbb{R}$.

Thus complete separable metric spaces are "coded" by countable objects. Using this coding, a great deal of analysis and geometry is developed in RCA_0, with many reverse mathematics results.

A conceptual difficulty:

Before Mummert/Simpson, there was no reverse mathematics study of general topology.

The obstacle was, there was no way to discuss abstract topological spaces in L_2, the language of second-order arithmetic. This was the case even for topological spaces which are separable or second countable.

The solution:

We overcome this obstacle by introducing a restricted class of topological spaces, called *countably based MF spaces*.

This class includes all complete separable metric spaces, as well as many nonmetrizable spaces.

Furthermore, this class of spaces can be discussed in L_2.

Let P be a *poset*, i.e., a partially ordered set.

Definition. A *filter* is a set $F \subseteq P$ such that

1. F is *upward closed*, i.e.,
   $(p \in F \wedge q \geq p) \Rightarrow q \in F$.

2. for all $p, q \in F$ there exists $r \in F$ such that
   $p \geq r \wedge q \geq r$.

Compare the treatment of forcing in Kunen's textbook of axiomatic set theory.

Definition. A *maximal filter* is a filter which is not properly included in any other filter.

By Zorn's Lemma, every filter is included in a maximal filter.

Definition.
MF(P) $= \{F \mid F$ is a maximal filter on $P\}$.

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MF(P) is a topological space with basic open sets
$$N_p = \{F \mid p \in F\}$$
for all $p \in P$.

Definition. An *MF space* is a space of the form MF(P) where P is a poset.

Definition. A *countably based MF space* is a space of the form MF(P) where P is a countable poset.

Thus, the second countable topological space MF(P) is "coded" by the countable poset P.

Therefore, countably based MF spaces can be defined and discussed in L_2. Thus we can do reverse mathematics in the usual setting, subsystems of second-order arithmetic.

Characterization problems:

1. To characterize those topological spaces which are homeomorphic to MF spaces.

2. To characterize those topological spaces which are homeomorphic to countably based MF spaces.

The first problem remains unsolved. The second problem has recently been solved by Carl Mummert and Frank Stephan.

Theorem (Mummert/Stephan).
Let X be a topological space.
The following are equivalent.

1. X is homeomorphic to a countably based MF space.

2. X is second countable and has the strong Choquet property, and each point of X is a closed set.

Examples of MF spaces.

Many topological spaces are homeomorphic to MF spaces:

- all complete metric spaces.
- all locally compact Hausdorff spaces.
- the weak-star dual of any Banach space.
- any G_δ subset of any MF space.
- the Baire space ω^ω with the topology generated by the Σ^1_1 sets, i.e., the Gandy/Harrington topology.

The latter is a neat example of a countably based MF space which is Hausdorff but not metrizable. However, the dense open subset $\{f \in \omega^\omega \mid \omega_1^f = \omega_1^{\mathsf{CK}}\}$ is completely metrizable.

Theorem. Every complete (separable) metric space is homeomorphic to a (countably based) MF space.

Proof (sketch). Let $\widehat{A}$ be a complete metric space with dense subset A. Let $P = A \times \mathbb{Q}^+$ ordered by $(a, r) < (b, s)$ if and only if $d(a, b) + r < s$. We argue that MF(P) is homeomorphic to X. Given a maximal filter F on P, we claim that $\inf\{r \mid (a, r) \in F\} = 0$. Suppose the inf is $h > 0$. Let $(a, r) \in F$ be such that $h \leq r < 4h/3$. We show that $(a, r/3) < (b, s)$ for all $(b, s) \in F$, contradicting maximality. Given $(b, s) \in F$, let $(c, t) \in F$ be such that $(c, t) < (a, r)$ and $(c, t) < (b, s)$. We have $h \leq t < r < 4h/3$ and $d(a, c) + h \leq d(a, c) + t < r < 4h/3$, hence $d(a, c) < h/3$, hence $d(a, c) + r/3 < d(a, c) + 4h/9 \leq h/3 + 4h/9 = 7h/9 < t$ so $(a, r/3) < (c, t) < (b, s)$, proving the claim. Hence F is generated by $(a_0, r_0) > (a_1, r_1) > \cdots > (a_n, r_n) > \cdots$ with $\lim_n r_n = 0$, giving a point of $\widehat{A}$.

Metrization theorems:

Urysohn Metrization Theorem. A second countable topological space is metrizable if and only if it is regular. (A topological space is said to be *regular* if, for every open set U and point $x \in U$, there exists an open set V such that $x \in V$ and the closure of V is included in U. See Kelley, *General Topology*.)

Choquet Metrization Theorem. A topological space is completely metrizable if and only if it is metrizable and has the *strong Choquet property*. (This is a game-theoretic property which is similar to, but stronger than, the property of Baire. See Kechris, *Classical Descriptive Set Theory*.)

Theorem. All MF spaces have the strong Choquet property. (See Mummert's Ph.D. thesis, 2005.)

Metrization theorems, continued.

Combining the above results, we have the following metrization theorem for countably based MF spaces.

> MFMT: A countably based MF space is completely metrizable if and only if it is regular.

Note that the statement MFMT can be formalized as a sentence in the language of second-order arithmetic.

We study the reverse mathematics of MFMT.

We consider the following subsystems of second-order arithmetic.

ACA_0 = arithmetical comprehension.

$\Pi^1_1\text{-}\mathsf{CA}_0 = \Pi^1_1$ comprehension.

$\Pi^1_2\text{-}\mathsf{CA}_0 = \Pi^1_2$ comprehension.

Remark. The fundamental concepts of the theory of MF spaces can be formalized in ACA_0. In particular, it is provable in ACA_0 that every complete separable metric space is homeomorphic to a countably based MF space.

Theorem (Mummert/Simpson).
MFMT is equivalent to $\Pi^1_2\text{-}\mathsf{CA}_0$.
The equivalence is provable in $\Pi^1_1\text{-}\mathsf{CA}_0$.

We outline the proof of this theorem.

Lemma 1. MFMT is provable in Π^1_2-CA_0.

Proof. Part 1. Assume MF(P) is regular. Use Π^1_2 comprehension to form the set $\{(p,q) \in P \times P \mid N_p \supseteq \text{closure of } N_q\}$. Use this set as a parameter. Follow Matthias Schröder's effective adaptation of the original Urysohn argument, to find a metric d_1 for MF(P). Thus MF(P) is metrizable.

Part 2. Fix a countable dense set $A \subseteq$ MF(P). Use Π^1_2 comprehension to form the sets $\{(a,r,p) \in A \times \mathbb{Q}^+ \times P \mid B(a,r) \subseteq N_p\}$ and $\{(a,r,p) \in A \times \mathbb{Q}^+ \times P \mid N_p \subseteq B(a,r)\}$, where $B(a,r) = \{x \in \text{MF}(P) \mid d_1(a,x) < r\}$. Using these sets as parameters, construct a G_δ set in $(\widehat{A}, \widehat{d}_1)$ which has the same points as MF(P) and is homeomorphic to MF(P). It follows that MF(P) is homeomorphic to a complete separable metric space $(\widehat{A}, \widehat{d}_2)$.

Note: Choquet's game-theoretic argument is not formalizable in second-order arithmetic. Instead, we argue directly within Π^1_2-CA_0.

Lemma 2. Over Π^1_1-CA_0, MFMT implies Π^1_2 comprehension.

Proof. Let $\psi(n, X)$ be a Π^1_1 formula. Assuming MFMT, we prove the existence of the Σ^1_2 set $S = \{n \mid \exists X\, \psi(n, X)\}$.

We write $\psi(n, X) \equiv \neg\, \exists f\, \forall m\, R(n, X[m], f[m])$ where $X[m] = \langle X(0), \ldots, X(m-1)\rangle$ and $f[m] = \langle f(0), \ldots, f(m-1)\rangle$. Let P be the countable poset consisting of all $(n, X[k], f[k])$ such that $(\forall m \leq k)\, R(n, X[m], f[m])$, plus all $(n, X[k])$, partially ordered by:

1. $(n, X[k], f[k]) < (n', X'[k'], f'[k'])$ iff $n = n'$ and $X[k] \supset X'[k']$ and $f[k] \supset f'[k']$.

2. $(n, X[k]) < (n', X'[k'])$ iff $n = n'$ and $X[k] \supset X'[k']$.

3. $(n, X[k], f[k]) < (n', X'[k'])$ iff $n = n'$ and $X[k] \supset X'[k']$.

4. $(n, X[k]) < (n', X'[k'], f'[k'])$ never.

The maximal filters on P are of three types:

1. $F = \{p \in P \mid q \leq p\}$,
where q is a minimal element of P.

2. $F = \{(n, X[k], f[k]), (n, X[k]) \mid k \in \mathbb{N}\}$,
where n, X, f are such that
$\forall m\, R(n, X[m], f[m])$ holds.

3. $F = \{(n, X[k]) \mid k \in \mathbb{N}\}$,
where n, X are such that $\psi(n, X)$ holds.

Let C be the closed set in MF(P) consisting of all F of type 3. The complement of C is the open set $\bigcup_{n \in \mathbb{N}} N_{(n,\langle\rangle,\langle\rangle)}$.

By Kondo's Π^1_1 Uniformization Theorem (provable in Π^1_1-CA_0, SOSOA §VI.2), we may assume that $\forall n\, (\exists \text{ at most one } X)\, \psi(n, X)$. Thus, for each n, $C \cap N_{(n,\langle\rangle)}$ contains at most one point.

Under this assumption, it is straightforward to show that MF(P) is regular.

By MFMT, there is a homeomorphism $\Phi : \mathrm{MF}(P) \cong \widehat{A}$, where $\widehat{A}$ is a complete separable metric space. In particular, $\Phi(C) \subseteq \widehat{A}$ is closed, and the open sets $\Phi(N_{(n,\langle\rangle)}) \subseteq \widehat{A}$ are arithmetical uniformly in n, using a code of Φ^{-1} as a parameter. Hence by Π^1_1 comprehension we may form the set

$$\begin{aligned} S &= \{n \mid \Phi(C) \cap \Phi(N_{(n,\langle\rangle)}) \neq \emptyset\} \\ &= \{n \mid C \cap N_{(n,\langle\rangle)} \neq \emptyset\} \\ &= \{n \mid \exists X\, \psi(n, X)\}\,. \end{aligned}$$

This completes the proof.

Remark. This is the first instance of a core mathematical theorem equivalent to Π^1_2 comprehension. Previous reverse mathematics results within second-order arithmetic have involved only weaker set existence axioms.

(However, Heinatsch and Möllerfeld have shown that Π^1_2-CA_0 proves the same Π^1_1 sentences as $\mathrm{ACA}_0 + {<}\omega$-$\Sigma^0_2$ determinacy.)

Another result:

Theorem (Mummert). The following are equivalent over ATR_0.

1. In any countably based MF space, any uncountable closed set contains a perfect set.

2. $\forall X\,(\aleph_1^{L(X)}$ is countable).

References:

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