Reverse Topology

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Two books on reverse mathematics (a status report):

1.

Stephen G. Simpson
Subsystems of Second Order Arithmetic
Perspectives in Mathematical Logic
Springer-Verlag, 1999
XIV + 445 pages

Out of print. 2nd edition to be published by the Association for Symbolic Logic.

2.

S. G. Simpson (editor)
Reverse Mathematics 2001

A volume of papers by various authors. Approximately 400 pages.

To be published in early 2005 by the Association for Symbolic Logic.

Remark:

This talk represents recent joint work with Carl Mummert, my advanced Ph.D. student at the Pennsylvania State University.

Background:

In my book *SOSOA*, a *complete separable metric space* is defined as the completion $X = \widehat{A}$ of a countable pseudometric space (A, d). Here $A \subseteq \mathbb{N}$ and $d : A \times A \to \mathbb{R}$.

Thus complete separable metric spaces are "coded" by countable objects. Using this coding, a great deal of analysis and geometry is developed in RCA_0, with many reverse mathematics results.

The Problem:

Until now there has been no reverse mathematics study of general topology.

The obstacle is that there is no way to discuss abstract topological spaces in the language of second order arithmetic.

This applies even to separable and second countable spaces.

The Solution:

To overcome this obstacle, we consider a restricted class of topological spaces, which we call *countably based poset spaces*.

This class includes all separable completely metrizable spaces, as well as many nonmetrizable spaces.

Furthermore, this class of spaces can be discussed in the language of second order arithmetic.

Let P be a *poset*, i.e., a partially ordered set.

Definition. A *filter* is a set $F \subseteq P$ such that

1. F is *upward closed*, i.e.,
   $(q \geq p \wedge p \in F) \Rightarrow q \in F$.

2. for all $p, q \in F$ there exists $r \in F$ such that $r \leq p$ and $r \leq q$.

Compare the treatment of forcing in Kunen's textbook of axiomatic set theory.

Definition. A *maximal filter* is a filter which is not properly included in any other filter.

By Zorn's Lemma, every filter is included in a maximal filter.

Definition.
MF$(P) = \{F \mid F$ is a maximal filter in $P\}$.

Definition.
$\mathrm{MF}(P) = \{F \mid F \text{ is a maximal filter in } P\}$.

MF(P) is a topological space with basic open sets

$$O_p = \{F \mid p \in F\}$$

for all $p \in P$.

Definition. A *poset space* is a space of the form MF(P) where P is a poset.

Definition. A *countably based poset space* is a space of the form MF(P) where P is a countable poset.

Thus, the second countable topological space MF(P) is "coded" by the countable poset P.

Therefore, countably based poset spaces can be discussed in the language of second order arithmetic.

Many topological spaces are homeomorphic to poset spaces:

- all complete separable metric spaces.
- the weak-star dual of any Banach space.
- any G_δ subset of any poset space.
- the Baire space ω^ω with the topology generated by the Σ^1_1 sets,
  i.e., the Gandy/Harrington topology.

The latter is a neat example of a countably based poset space which is Hausdorff but not metrizable. However, the dense open subset $\{f \in \omega^\omega \mid \omega_1^f = \omega_1^{\mathsf{CK}}\}$ is completely metrizable.

All poset spaces have the property of Baire: if $\forall n\,(U_n$ is dense open$)$ then $\bigcap_n U_n$ is dense.

All poset spaces have the strong Choquet property. All metrizable poset spaces are completely metrizable. See A. Kechris, *Classical Descriptive Set Theory*.

Remark:
The concept of maximal filter is absolute between β-models but not between ω-models. In moving from an ω-model to a larger one, maximal filters can be destroyed, i.e., points of MF(P) can disappear.

Example:
Let T be a recursive tree which has paths but no hyperarithmetical paths. Let $P = T \cup \{p_n \mid n \in \omega\}$ where $p_0 > p_1 > \cdots > p_n > \cdots$ and, for $t \in T$, $p_n > t$ if and only if $|t| \geq n$. Consider the filter $F = \{p_n \mid n \in \omega\}$. Let M be an ω-model of RCA_0. Then $M \models$ "F is a maximal filter" iff $M \models$ "T has no path", e.g., if $M \subseteq$ HYP. By passing to a larger ω-model M', we may create paths through T and thus destroy the maximality of the filter F.

In order to overcome this difficulty, we consider the following variant of MF(P).

A variant definition.

A filter $F \subseteq P$ is *unbounded* if there is no $q \in P$ such that $q < p$ for all $p \in F$. We put

UF(P) $= \{F \mid F$ is an unbounded filter in $P\}$.

UF(P) is a topological space with basic open sets as before, $O_p = \{F \mid p \in F\}$ for all $p \in P$.

Trivially, every maximal filter is unbounded. Hence MF(P) $\subseteq$ UF(P).

The spaces MF(P) and UF(P) are "coded" by the poset P.

Definition. A *poset space* is a space of the form MF(P) or UF(P) where P is a poset.

Definition. A *countably based poset space* is a space of the form MF(P) or UF(P) where P is a countable poset.

Separation properties:

Every MF-space is T_1, i.e., points are closed.

There are UF-spaces which are not T_1.

Every T_1 UF-space is an MF-space.

In fact, if UF(P) is T_1, then UF(P) = MF(P).

There are MF-spaces which are not Hausdorff. In fact, there are UF-spaces which are T_1 but not Hausdorff.

Example:

Let P consist of $p_0 > p_1 > \cdots > p_n > \cdots$ and $q_0 > q_1 > \cdots > q_n > \cdots$ and $r_n < p_n$ and $r_n < q_n$ for all n. The points of UF(P) = MF(P) are $F = \{p_n \mid n \in \omega\}$ and $G = \{q_n \mid n \in \omega\}$ and H_n generated by r_n for each n. The sequence H_n, $n \in \omega$ converges to both F and G, so this is not Hausdorff.

Theorem. Every complete (separable) metric space is homeomorphic to a (countably based) poset space.

Proof. If X is a complete metric space, let A be a dense subset, and let $P = A \times \mathbb{Q}^+$ ordered by $(a, r) < (b, s)$ iff $d(a, b) + r < s$. We argue that $\mathrm{UF}(P) = \mathrm{MF}(P)$ is homeomorphic to X. Given an unbounded filter F, we claim that $\inf\{r \mid (a, r) \in F\} = 0$. Suppose the inf is $h > 0$. Let $(a, r) \in F$ be such that $h \leq r < 4h/3$. We show that $(a, r/3) < (b, s)$ for all $(b, s) \in F$, contradicting unboundedness. Given $(b, s) \in F$, let $(c, t) \in F$ be such that $(c, t) < (a, r)$ and $(c, t) < (b, s)$. We have $h \leq t < r < 4h/3$ and $d(a, c) + h \leq d(a, c) + t < r < 4h/3$, hence $d(a, c) < h/3$, hence $d(a, c) + r/3 < d(a, c) + 4h/9 \leq h/3 + 4h/9 = 7h/9 < t$ so $(a, r/3) < (c, t) < (b, s)$, proving the claim. Hence F is generated by $(a_0, r_0) > (a_1, r_1) > \cdots > (a_n, r_n) > \cdots$ with $\lim_n r_n = 0$ giving a point of X.

Reverse metrization problems:

Let X be a countably based poset space.

Each of the following metrization theorems can be formalized in RCA_0. One can then seek a reverse mathematics classification.

1. If X is regular and Hausdorff, then X is metrizable.

2. If X is regular and Hausdorff, then X is completely metrizable.

3. If X is compact and Hausdorff, then X is metrizable.

4. If X is compact and Hausdorff, then X is completely metrizable.

Characterization problems:

We know that poset spaces have the strong Choquet property, and countably based poset spaces are second countable.

Among all topological spaces with these properties, it seems difficult to characterize those which are homeomorphic to (countably based) poset spaces.

I will now yield the balance of my time to Carl Mummert. Carl will report on his results on the reverse mathematics of countably based poset spaces, etc.

THE END