Mass Problems to the Rescue!

Stephen G. Simpson*

Pennsylvania State University
http://www.math.psu.edu/simpson/
simpson@math.psu.edu

American Mathematical Society

2004 Fall Central Sectional Meeting

Evanston, Illinois

October 23–24, 2004

Special Session on
Computability Theory and Applications

Background:

$\mathcal{R}_T$ is the upper semilattice of recursively enumerable Turing degrees.

Two basic, classical, unresolved issues concerning $\mathcal{R}_T$:

Issue 1. To find a specific, natural, r.e. Turing degree $\mathbf{a} \in \mathcal{R}_T$ which is $> \mathbf{0}$ and $< \mathbf{0}'$.

Issue 2. To find a "smallness property" of an infinite co-r.e. set $A \subseteq \omega$ which ensures that $\deg_T(A) = \mathbf{a} \in \mathcal{R}_T$ is $> \mathbf{0}$ and $< \mathbf{0}'$.

These unresolved issues go back to Post's 1944 paper, *Recursively enumerable sets of positive integers and their <u>decision problems</u>*.

Mass Problems to the Rescue!

We address Issues 1 and 2 by passing from *decision problems* to *mass problems*.

Outline of this talk:

We embed $\mathcal{R}_T$ into a slightly larger structure, $\mathcal{P}_w$, which is much better behaved. In the $\mathcal{P}_w$ context, we obtain satisfactory, positive answers to Issues 1 and 2.

What is this wonderful structure $\mathcal{P}_w$?

Briefly, $\mathcal{P}_w$ is the lattice of weak degrees of mass problems associated with nonempty Π^0_1 subsets of 2^ω.

In order to explain $\mathcal{P}_w$, we must first explain:

- mass problems,

- weak degrees, and

- nonempty Π^0_1 subsets of 2^ω.

Mass problems (informal discussion):

A "decision problem" is the problem of deciding whether a given $n \in \omega$ belongs to a fixed set $A \subseteq \omega$ or not. To compare decision problems, we use Turing reducibility. $A \leq_T B$ means that A can be computed using an oracle for B.

A "mass problem" is a problem with a not necessarily unique solution. (By contrast, a "decision problem" has only one solution.)

The "mass problem" associated with a set $P \subseteq 2^\omega$ is the "problem" of computing an element of P.

The "solutions" of P are the elements of P.

One mass problem is said to be "reducible" to another if, given any solution of the second problem, we can use it as an oracle to compute a solution of the first problem.

Rigorous definition:

Let P and Q be subsets of 2^ω.

We view P and Q as mass problems.

We say that P is *weakly reducible* to Q if

$$(\forall Y \in Q) \ (\exists X \in P) \ (X \leq_T Y) \,.$$

This is abbreviated $P \leq_w Q$.

Summary:

$P \leq_w Q$ means that, given any solution of Q, we can use it as an oracle to compute a solution of P.

Digression: weak vs. strong reducibility

Let P and Q be subsets of 2^ω.

1. P is *weakly reducible* to Q, $P \leq_w Q$, if for all $Y \in Q$ there exists e such that $\{e\}^Y \in P$.

2. P is *strongly reducible* to Q, $P \leq_s Q$, if there exists e such that $\{e\}^Y \in P$ for all $Y \in Q$.

Strong reducibility is a uniform variant of weak reducibility. By a result of Nerode, there is an analogy:

$$\frac{\text{weak reducibility}}{\text{Turing reducibility}} = \frac{\text{strong reducibility}}{\text{truth table reducibility}} \, .$$

In this talk we deal only with weak reducibility.

Historical note:

Weak reducibility is due to Muchnik 1963.

Strong reducibility is due to Medvedev 1955.

The lattice $\mathcal{P}_w$:

A *weak degree* is an equivalence class of subsets of 2^ω under the equivalence relation $P \leq_w Q$ and $Q \leq_w P$. The weak degrees have a partial ordering induced by $\leq_w$.

A set $P \subseteq 2^\omega$ is said to be Π_1^0 if it is the set of paths through a recursive subtree of the full binary tree of finite sequences of 0's and 1's.

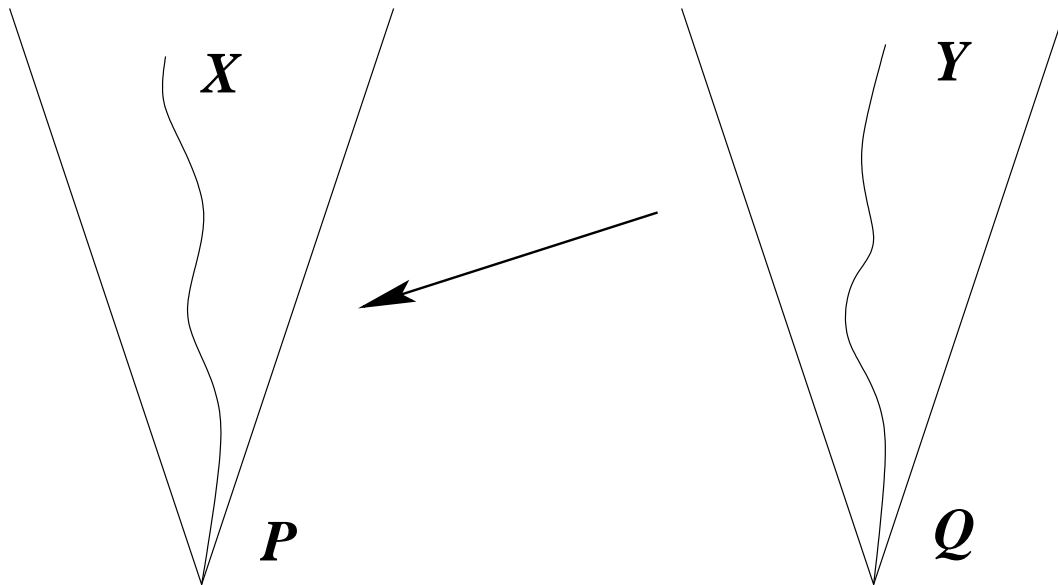$\mathcal{P}_w$ is the set of weak degrees of nonempty Π_1^0 subsets of 2^ω, ordered by weak reducibility.

$\mathcal{P}_w$ is a distributive lattice, with l.u.b. given by $P \times Q = \{X \oplus Y \mid X \in P, Y \in Q\}$, and g.l.b. given by $P \cup Q$.

The bottom element of $\mathcal{P}_w$ is the weak degree of 2^ω.

The top element of $\mathcal{P}_w$ is the weak degree of

$\quad$ PA = {completions of Peano Arithmetic} (Scott/Tennenbaum).

Weak reducibility of Π^0_1 subsets of 2^ω:

$P \leq_w Q$ means:

$(\forall Y \in Q)\ (\exists X \in P)\ (X \leq_T Y)$.

P, Q are given by infinite recursive subtrees of the full binary tree of finite sequences of 0's and 1's.

X, Y are infinite (nonrecursive) paths through P, Q respectively.

The lattice $\mathcal{P}_w$:

A *weak degree* is an equivalence class of subsets of 2^ω under the equivalence relation $P \leq_w Q$ and $Q \leq_w P$. The weak degrees have a partial ordering induced by $\leq_w$.

A set $P \subseteq 2^\omega$ is said to be Π^0_1 if it is the set of paths through a recursive subtree of the full binary tree of finite sequences of 0's and 1's.

$\mathcal{P}_w$ is the set of weak degrees of nonempty Π^0_1 subsets of 2^ω, ordered by weak reducibility.

$\mathcal{P}_w$ is a distributive lattice, with l.u.b. given by $P \times Q = \{X \oplus Y \mid X \in P, Y \in Q\}$, and g.l.b. given by $P \cup Q$.

The bottom element of $\mathcal{P}_w$ is the weak degree of 2^ω.

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$\quad$ PA $= \{$completions of Peano Arithmetic$\}$

(Scott/Tennenbaum).

Embedding $\mathcal{R}_T$ into $\mathcal{P}_w$:

Theorem (Simpson 2002):

There is a natural embedding $\phi : \mathcal{R}_T \to \mathcal{P}_w$.

($\mathcal{R}_T =$ the semilattice of Turing degrees of r.e. subsets of ω. $\mathcal{P}_w =$ the lattice of weak degrees of nonempty Π_1^0 subsets of 2^ω.)

The embedding ϕ is given by

$$\phi : \deg_T(A) \mapsto \deg_w(\mathsf{PA} \cup \{A\}).$$

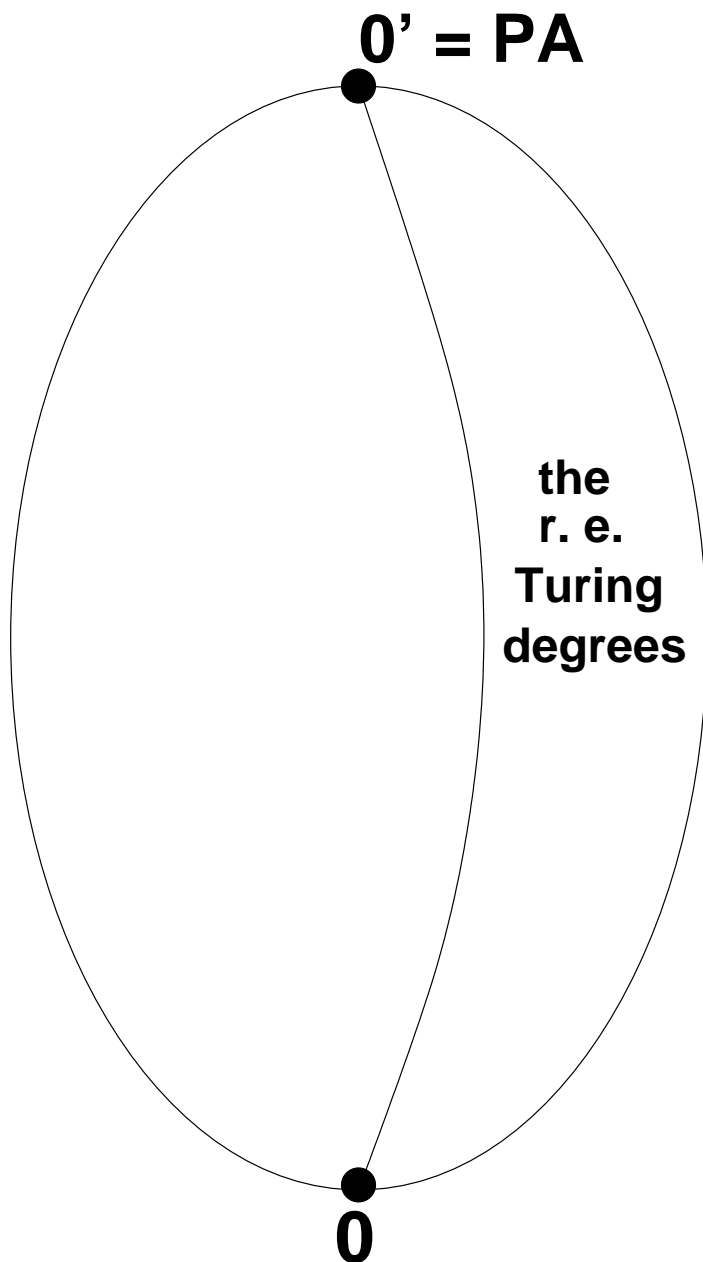Note: $\mathsf{PA} \cup \{A\}$ is not a Π_1^0 set. However, it is of the same weak degree as a Π_1^0 set.

The embedding ϕ is one-to-one and preserves $\leq$, l.u.b., and the top and bottom elements.

Convention:
We identify $\mathcal{R}_T$ with its image in $\mathcal{P}_w$ under ϕ.

In particular, we identify $\mathbf{0}', \mathbf{0} \in \mathcal{R}_T$ with the top and bottom elements of $\mathcal{P}_w$.

A picture of the lattice $\mathcal{P}_w$:

0' = PA

the
r. e.
Turing
degrees

0

$\mathcal{R}_T$ is embedded in $\mathcal{P}_w$. $\mathbf{0'}$ and $\mathbf{0}$ are the top and bottom elements of both $\mathcal{R}_T$ and $\mathcal{P}_w$.

Digression: structural properties of $\mathcal{P}_w$

1. $\mathcal{P}_w$ is a countable distributive lattice. Every countable distributive lattice is lattice embeddable in every initial segment of $\mathcal{P}_w$. (Binns/Simpson 2001)

2. The $\mathcal{P}_w$ analog of the Sacks Splittting Theorem holds. (Stephen Binns, 2002)

3. We conjecture that the $\mathcal{P}_w$ analog of the Sacks Density Theorem holds.

These structural results for $\mathcal{P}_w$ are proved by means of priority arguments, just as for $\mathcal{R}_T$.

4. Within $\mathcal{P}_w$ the degrees $\mathbf{r}_1$ and $\inf(\mathbf{r}_2, \mathbf{0}')$ are meet irreducible and do not join to $\mathbf{0}'$. (Simpson 2002, 2004)

5. $\mathbf{0}$ is meet irreducible. (This is trivial.)

Response to Issue 1:

Issue 1 was:

To find a specific, natural example of a recursively enumerable Turing degree which is $> \mathbf{0}$ and $< \mathbf{0}'$.

We do not know how to do this.

However, in the $\mathcal{P}_w$ context, we have discovered many specific, natural degrees which are $> \mathbf{0}$ and $< \mathbf{0}'$.

The specific, natural degrees in $\mathcal{P}_w$ which we have discovered are related to foundationally interesting topics:

- algorithmic randomness,
- diagonal nonrecursiveness,
- reverse mathematics,
- subrecursive hierarchies,
- computational complexity.

0' = PA

inf(r_2, 0')

r_1

the r. e. Turing degrees

d_REC

d

0
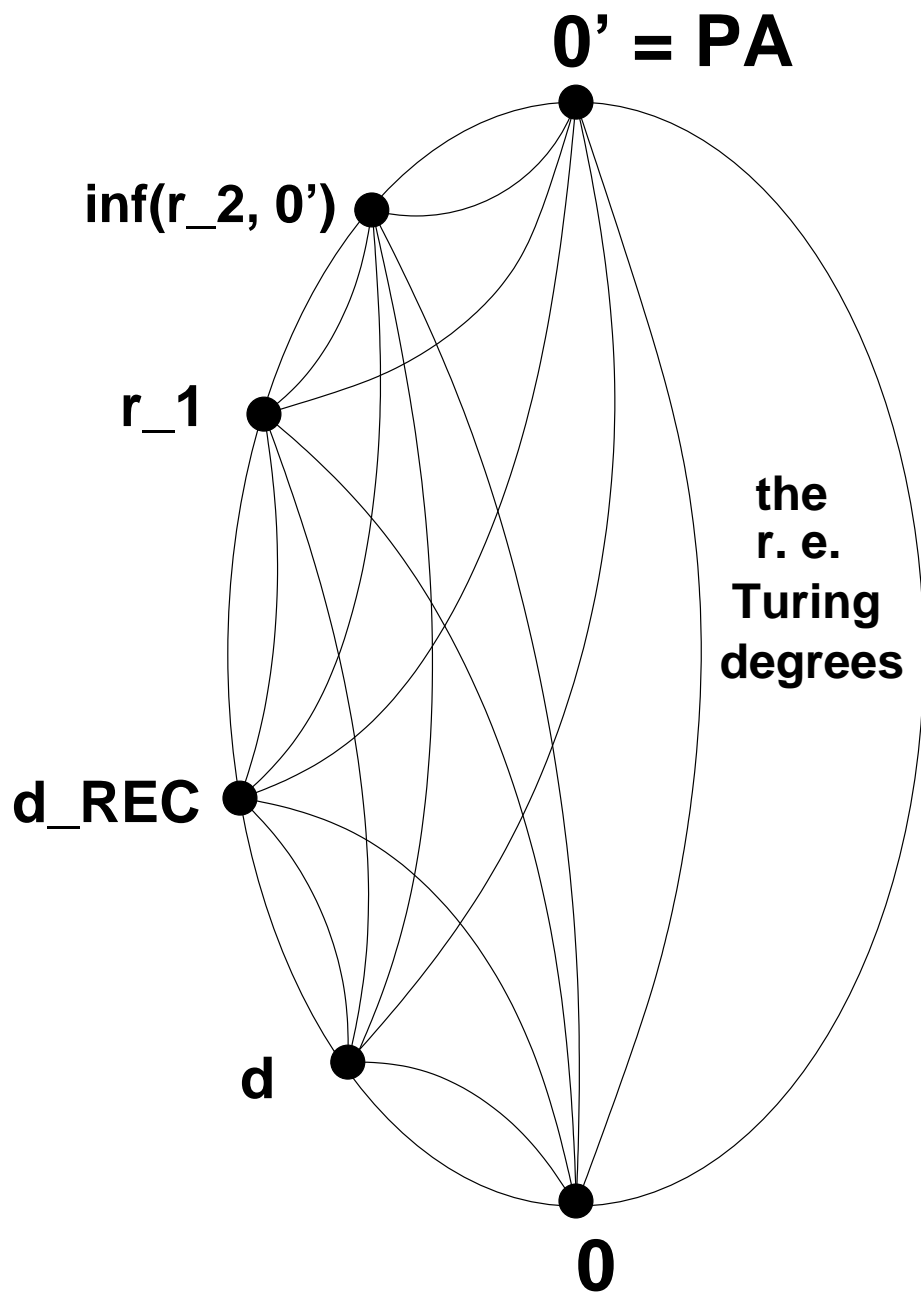
Note: Except for $0'$ and 0, the r.e. Turing degrees are incomparable with all of the known, specific, natural degrees in $\mathcal{P}_w$.

Some specific, natural degrees in $\mathcal{P}_w$:

$\mathbf{r}_n =$ the weak degree of the set of n-random reals.

$\mathbf{d} =$ the weak degree of the set of diagonally nonrecursive functions.

$\mathbf{d}_{\mathsf{REC}} =$ the weak degree of the set of diagonally nonrecursive functions which are recursively bounded.

Theorem (Simpson 2002, Ambos et al 2004):

In $\mathcal{P}_w$ we have

$$0 < \mathbf{d} < \mathbf{d}_{\mathsf{REC}} < \mathbf{r}_1 < \inf(\mathbf{r}_2, \mathbf{0}') < \mathbf{0}'.$$

Theorem (Simpson 2004):

1. $\mathbf{r}_1$ is the maximum weak degree of a Π^0_1 subset of 2^ω which is of positive measure.

2. $\inf(\mathbf{r}_2, \mathbf{0}')$ is the maximum weak degree of a Π^0_1 subset of 2^ω whose Turing upward closure is of positive measure.

Digression: structural properties of $\mathcal{P}_w$

1. $\mathcal{P}_w$ is a countable distributive lattice. Every countable distributive lattice is lattice embeddable in every initial segment of $\mathcal{P}_w$. (Binns/Simpson 2001)

2. The $\mathcal{P}_w$ analog of the Sacks Splittting Theorem holds. (Stephen Binns, 2002)

3. We conjecture that the $\mathcal{P}_w$ analog of the Sacks Density Theorem holds.

These structural results for $\mathcal{P}_w$ are proved by means of priority arguments, just as for $\mathcal{R}_T$.

4. Within $\mathcal{P}_w$ the degrees $\mathbf{r}_1$ and $\inf(\mathbf{r}_2, \mathbf{0}')$ are meet irreducible and do not join to $\mathbf{0}'$. (Simpson 2002, 2004)

5. $\mathbf{0}$ is meet irreducible. (This is trivial.)

Some additional, specific degrees in $\mathcal{P}_w$:

$\mathbf{d}_\alpha =$ the weak degree of the set of diagonally nonrecursive functions which are bounded by a recursive function at level α of the Wainer hierarchy.

$\mathbf{d}^2 =$ the weak degree of the set of $f \oplus g$ such that f is diagonally nonrecursive, and g is diagonally nonrecursive relative to f. More generally, define $\mathbf{d}^n$ for all $n \geq 1$.

Theorem (Simpson 2004, Ambos et al 2004): In $\mathcal{P}_w$ we have

$$\mathbf{r}_1 > \mathbf{d}_0 > \mathbf{d}_1 > \cdots > \mathbf{d}_\alpha > \cdots > \mathbf{d}_{\mathsf{REC}}$$

and

$$\mathbf{d} = \mathbf{d}^1 < \mathbf{d}^2 < \cdots < \mathbf{d}^n < \cdots < \mathbf{r}_1 \,.$$

We conjecture that $\mathbf{d}^n$ is incomparable with $\mathbf{d}_\alpha$ and with $\mathbf{d}_{\mathsf{REC}}$. This would be the first example of specific, natural degrees in $\mathcal{P}_w$ which are incomparable with each other.

Response to Issue 2:

Issue 2 was:

To find a "smallness property" of an infinite Π^0_1 (i.e., co-r.e.) set $A \subseteq \omega$ which ensures that the Turing degree of A is $> \mathbf{0}$ and $< \mathbf{0}'$.

We do not know how to do this.

However, in the $\mathcal{P}_w$ context, we have identified several "smallness properties" of a Π^0_1 set $P \subseteq 2^\omega$ which ensure that the weak degree of P is $> \mathbf{0}$ and $< \mathbf{0}'$.

One result of this type:

Theorem (Simpson 2002):

Let $\mathbf{p}$ be the weak degree of a Π^0_1 set $P \subseteq 2^\omega$ which is thin and perfect. Then $\mathbf{p}$ is incomparable with $\mathbf{r}_1$. Hence $\mathbf{0} < \mathbf{p} < \mathbf{0}'$.

Background on thin Π_1^0 sets:

Definition:

A Π_1^0 set $P \subseteq 2^\omega$ is said to be *thin* if, for all Π_1^0 sets $Q \subseteq P$, $P \setminus Q$ is Π_1^0.

Thin perfect Π_1^0 subsets of 2^ω have been constructed by means of priority arguments. Much is known about them. For example, any two such sets are automorphic in the lattice of Π_1^0 subsets of 2^ω under inclusion.

(Martin/Pour-El 1970,
Downey/Jockusch/Stob 1990, 1996,
Cholak et al 2001)

Some additional "smallness properties":

Let P be a Π_1^0 subset of 2^ω.

1. P is *small* if there is no recursive function f such that for all n there exist n members of P which differ at level $f(n)$ in the binary tree. (Binns 2003)

Example: Let $A \subseteq \omega$ be hypersimple, and let $A = A_1 \cup A_2$ where A_1, A_2 are r.e. Then $P = \{X \in 2^\omega \mid X \text{ separates } A_1, A_2\}$ is small.

Theorem (Binns):

If P is small, the weak degree of P is $< \mathbf{0}'$.

2. P is *h-small* if there is no recursive, canonically indexed sequence of pairwise disjoint clopen sets D_n, $n \in \omega$, such that $P \cap D_n \neq \emptyset$ for all n. (Simpson 2003)

Theorem (Simpson):

If P is h-small, the weak degree of P is $< \mathbf{0}'$.

Summary of this talk:

There are basic, unresolved issues concerning $\mathcal{R}_T$, the semilattice of recursively enumerable Turing degrees. One of the issues is the lack of specific, natural, r.e. degrees.

We embed $\mathcal{R}_T$ into $\mathcal{P}_w$, the lattice of weak degrees of nonempty Π_1^0 subsets of 2^ω. We identify $\mathcal{R}_T$ with its image in $\mathcal{P}_w$.

In the $\mathcal{P}_w$ context, some of the unresolved issues can be satisfactorily addressed.

In particular, $\mathcal{P}_w$ contains many specific, natural degrees which are related to foundationally interesting topics:

- algorithmic randomness,
- reverse mathematics,
- computational complexity.

References:

Binns, Small Π_1^0 classes, 22 pages,
Archive for Mathematical Logic, to appear.

Simpson, Mass problems and randomness,
29 pages, Bulletin of Symbolic Logic, to
appear.

Simpson, An extension of the recursively
enumerable Turing degrees, 15 pages,
submitted for publication.

Some of my papers are available at

`http://www.math.psu.edu/simpson/papers/`.

Transparencies for my talks are available at

`http://www.math.psu.edu/simpson/talks/`.

THE END