An Extension of the Recursively Enumerable Turing Degrees

Stephen G. Simpson*

Pennsylvania State University
http://www.math.psu.edu/simpson/
simpson@math.psu.edu

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Background: the r.e. Turing degrees

For $X, Y \subseteq \omega = \{0, 1, 2, \ldots\}$,
X is *Turing reducible* to Y (i.e., $X \leq_T Y$)
iff X is computable using an oracle for Y.

The *Turing degrees* are the equivalence classes under $\leq_T$, ordered by $\leq_T$.

The l.u.b. of two Turing degrees is given by
$X \oplus Y = \{2n : n \in X\} \cup \{2n+1 : n \in Y\}$.

$X \subseteq \omega$ is *r.e.* (i.e., *recursively enumerable*)
iff it is the range of a recursive function.

An *r.e. Turing degree* is a Turing degree that contains an r.e. set.

$\mathcal{R}_T$ is the semilattice of r.e. Turing degrees.
This structure has been studied extensively by recursion theorists.

Background, continued:

$\mathcal{R}_T$ is the semilattice of r.e. Turing degrees.

Intensive study of lattice-theoretic properties of $\mathcal{R}_T$ has yielded nothing for f.o.m.

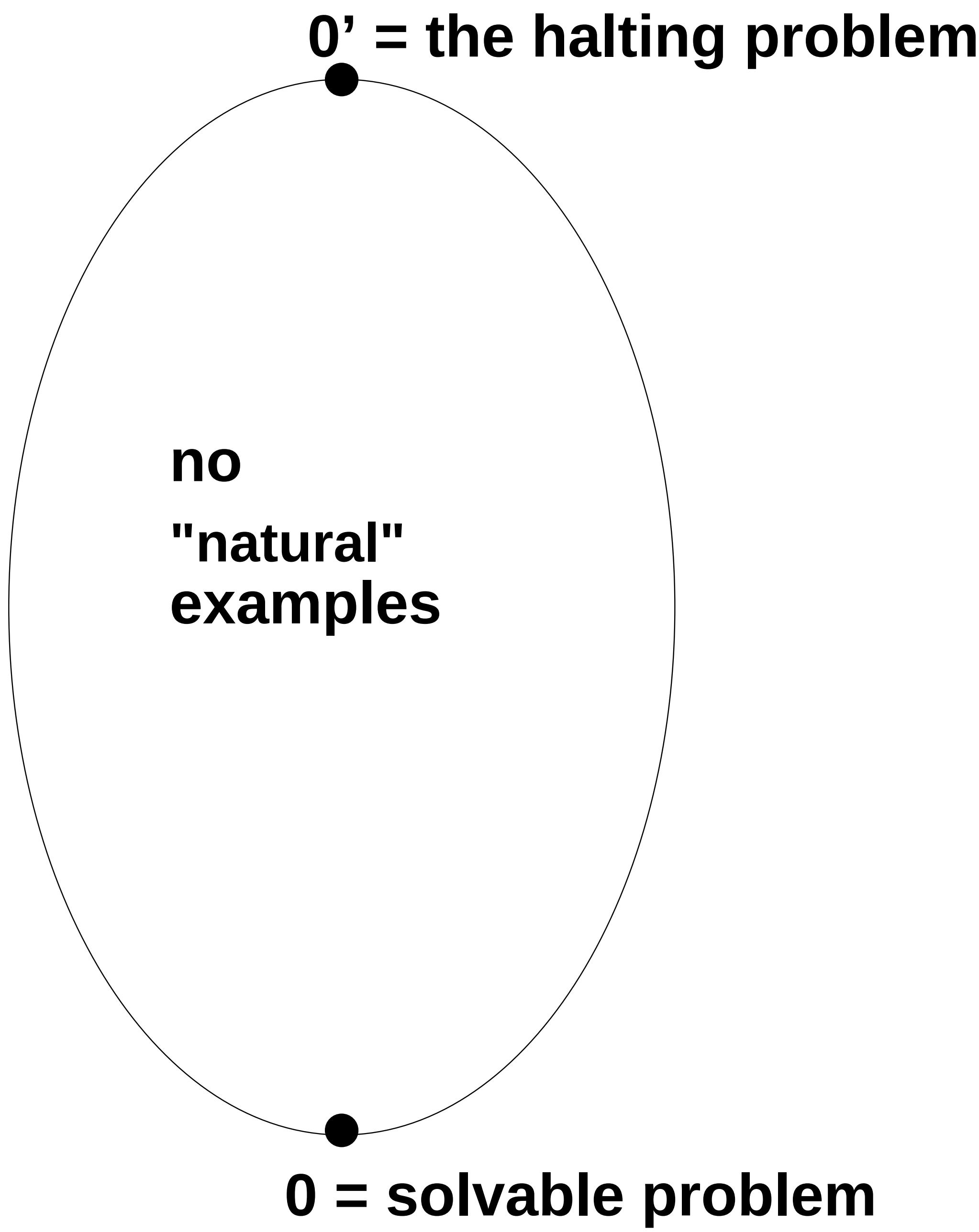
Moreover, after 50 years, the only known specific examples of r.e. Turing degrees are the bottom and top elements of $\mathcal{R}_T$.

$0 =$ Turing degree of recursive sets.

$0' =$ Turing degree of the Halting Problem.

There are infinitely many r.e. Turing degrees, but there are no known "natural" ones, other than 0 and $0'$.

A picture of the r.e. Turing degrees, $\mathcal{R}_T$:



Theme of this talk:

We embed the upper semilattice of r.e. Turing degrees, $\mathcal{R}_T$, into another structure, $\mathcal{P}_w$, which is

1. slightly larger,

2. somewhat better behaved,

3. much more relevant to f.o.m. (= foundations of mathematics).

An extension of the r.e. Turing degrees:

We define the Muchnik lattice $\mathcal{P}_w$.

The Cantor space is $2^\omega = \{X : \omega \to \{0,1\}\}$.

For $P, Q \subseteq 2^\omega$, P is *Muchnik reducible* to Q ($P \leq_w Q$) iff every member of Q computes a member of P, i.e., $\forall Y \in Q\ \exists X \in P\ X \leq_T Y$.

Muchnik degrees are equivalence classes of subsets of 2^ω under $\leq_w$, ordered by $\leq_w$.

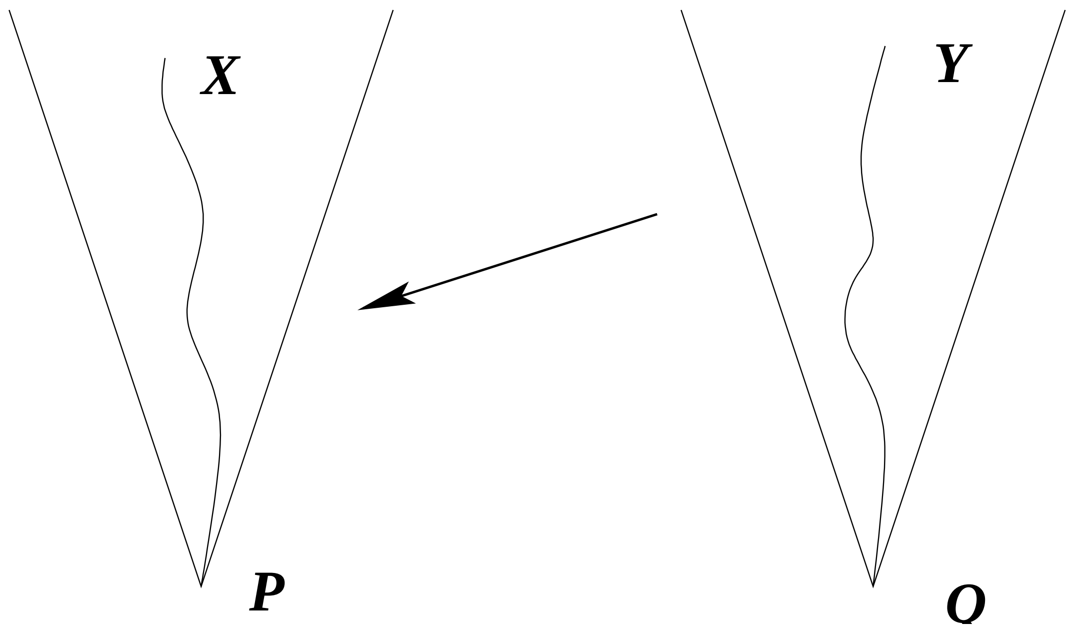
The l.u.b. of two Muchnik degrees is given by $P \times Q = \{X \oplus Y : X \in P \text{ and } Y \in Q\}$.
The g.l.b. is given by $P \cup Q$.

$P \subseteq 2^\omega$ is Π^0_1 iff P is the set of paths through a recursive subtree of the full binary tree of finite sequences of 0's and 1's.

$\mathcal{P}_w$ is the lattice of Muchnik degrees of nonempty Π^0_1 subsets of 2^ω.

(It is important here that 2^ω is compact.)

Muchnik reducibility:



$P \leq_w Q$ means:

$\forall Y \in Q \;\; \exists X \in P \;\; X \leq_T Y$.

P, Q are given by recursive subtrees of the full binary tree of finite sequences of 0's and 1's.

X, Y are infinite (nonrecursive) paths through P, Q respectively.

An extension, continued:

Properties of $\mathcal{P}_w$, the lattice of Muchnik degrees of nonempty Π^0_1 subsets of 2^ω:

(a) $\mathcal{P}_w$ is a distributive lattice. Thus, its structure is more regular than that of $\mathcal{R}_T$.

$\mathcal{P}_w$ has a bottom and a top element:

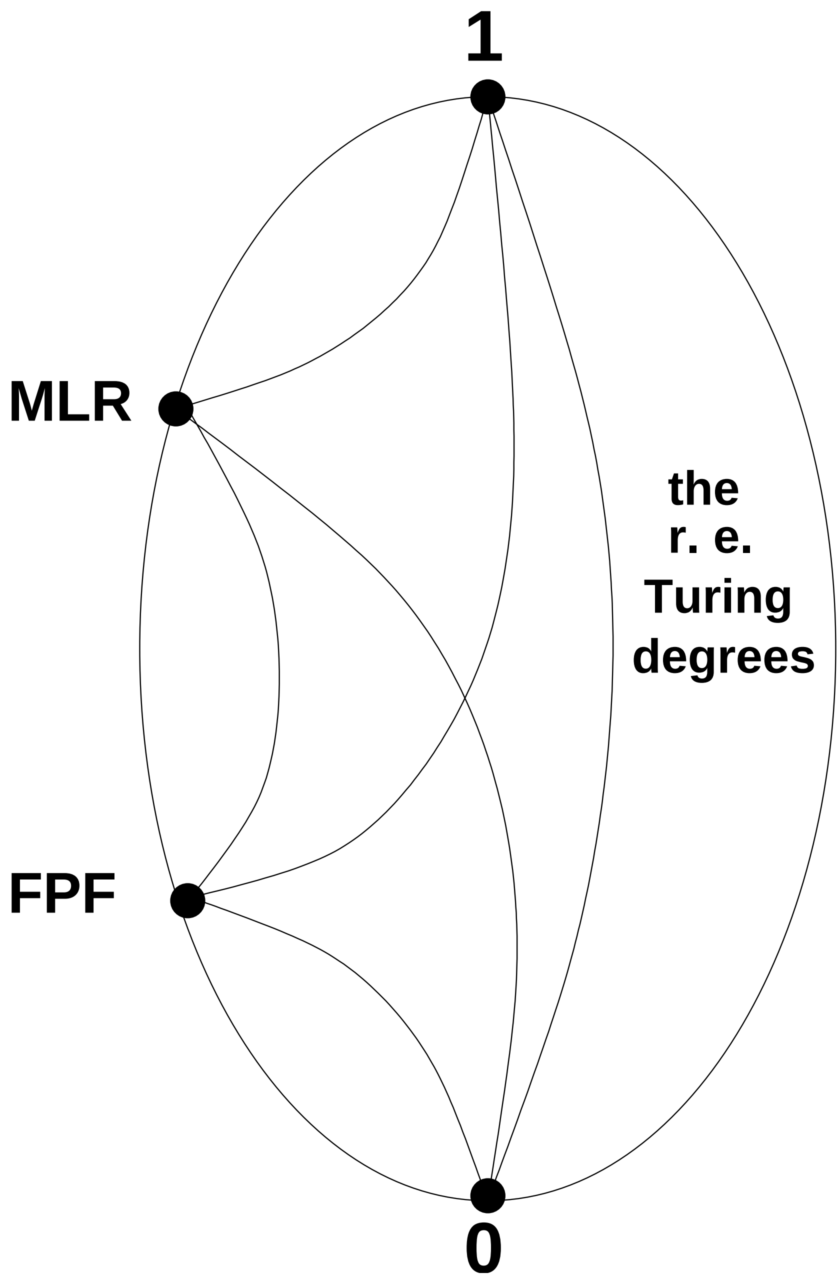
$0 =$ the Muchnik degree of 2^ω,

$1 =$ the Muchnik degree of the set of completions of theories that are *sufficiently strong*, in the sense of the Gödel/Rosser Theorem: EFA, PA, Z_2, ZFC, ...

(b) There are at least two other "natural" Muchnik degrees in $\mathcal{P}_w$. See below.

In these important senses, the Muchnik lattice $\mathcal{P}_w$ **is better than** $\mathcal{R}_T$, the semilattice of r.e. Turing degrees. It overcomes some of the well known deficiencies of $\mathcal{R}_T$.
(Simpson, August 1999, on FOM)

A picture of the Muchnik lattice $\mathcal{P}_w$:



An extension, continued:

Two "natural" Muchnik degrees in $\mathcal{P}_w$.

MLR = the Muchnik degree of the set of Martin-Löf random sequences of 0's and 1's. (essentially due to Kučera 1985)

FPF = the Muchnik degree of the set of fixed-point-free functions, in the sense of the Arslanov Completeness Criterion. (Simpson 2002)

In $\mathcal{P}_w$ we have $0 <$ FPF $<$ MLR < 1.

F.o.m. connection: The Muchnik degrees MLR and FPF correspond to subsystems of WKL_0 which arise in the Reverse Mathematics of measure theory (Yu/Simpson 1990) and continuous functions (Giusto/Simpson 2000), respectively. The Muchnik degree 1 corresponds to WKL_0 itself.

Problem: Find additional "natural" Muchnik degrees in $\mathcal{P}_w$.

MLR = the Muchnik degree of the set of
Martin-Löf random (1-random) reals

= the maximum Muchnik degree of a Π^0_1 subset of 2^ω of positive measure.

(implicit in Kučera 1985)

FPF = the Muchnik degree of the set of
fixed-point-free functions

= the Muchnik degree of the set of diagonally non-recursive functions

= the Muchnik degree of the set of effectively immune sets

= the Muchnik degree of the set of effectively biimmune sets

(implicit in Jockusch 1989)

Some additional "natural" Muchnik degrees in $\mathcal{P}_w$:

$\mathsf{MLR}_2 =$ the Muchnik degree of $R_2 \cup \mathsf{PA}$, where R_2 is the set of 2-random reals, and PA is the set of complete extensions of Peano Arithmetic.

FPF^n, $n = 2, 3, \ldots$, where $\mathsf{FPF}^1 = \mathsf{FPF}$ and

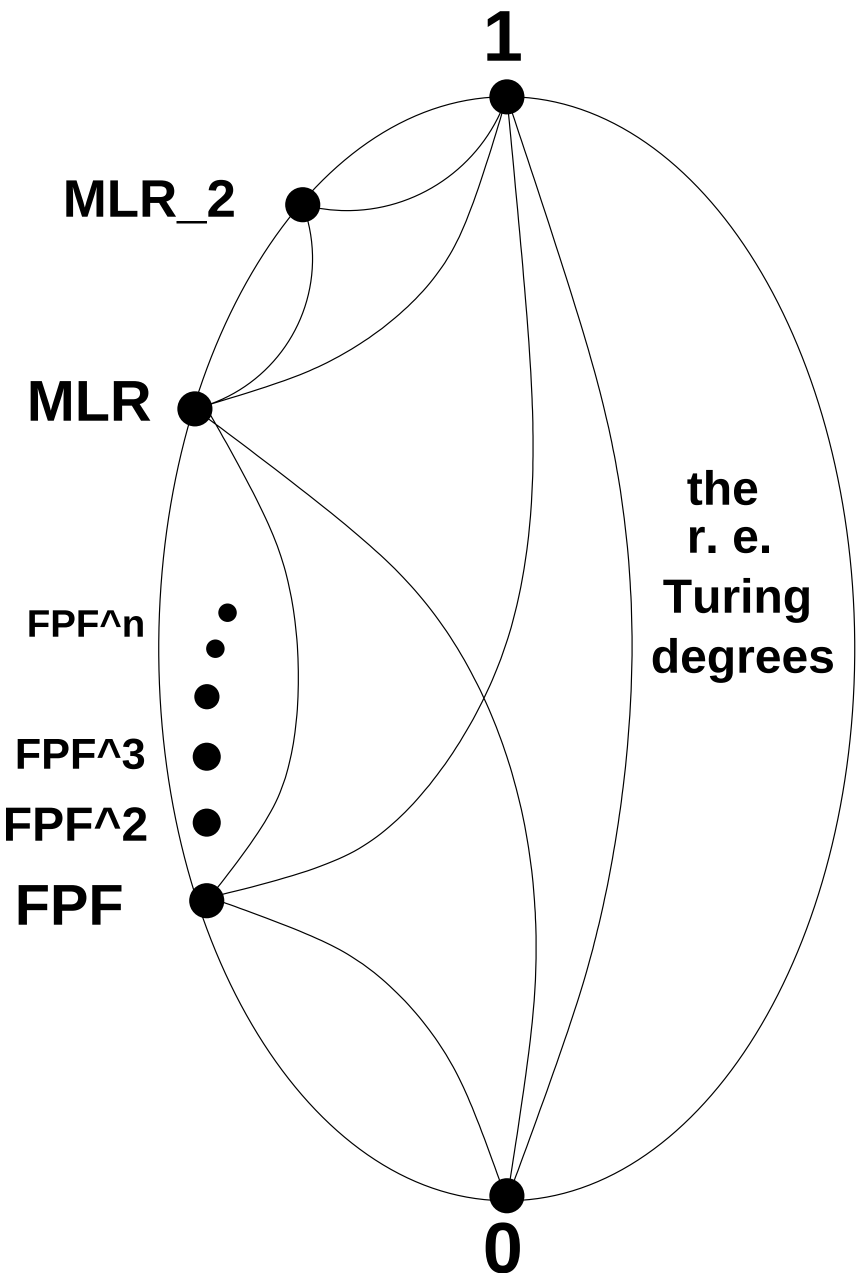
$$\mathsf{FPF}^{n+1} = \{f \oplus g : f \in \mathsf{FPF}^n, g \in \mathsf{FPF}^f\}.$$

In $\mathcal{P}_w$ we have:

$$\mathsf{MLR} < \mathsf{MLR}_2 < 1, \text{ and}$$

$$\mathsf{FPF} < \mathsf{FPF}^2 < \mathsf{FPF}^3 < \cdots < \mathsf{FPF}^n < \cdots.$$

A picture of the Muchnik lattice $\mathcal{P}_w$:



An extension, continued:

Further properties of the Muchnik lattice $\mathcal{P}_w$.

1. $\mathcal{P}_w$ is a countable distributive lattice. Every countable distributive lattice is lattice embeddable in every initial segment of $\mathcal{P}_w$. (Binns/Simpson 2001)

2. For all $P > 0$ there exist $P_1, P_2 < P$ such that $P =$ l.u.b.(P_1, P_2). (Stephen Binns, 2002)

3. There does not exist $P < 1$ such that l.u.b.$(P, \mathrm{MLR}) = 1$. (Simpson 2001)

4. There do not exist $P_1, P_2 > \mathrm{MLR}$ such that g.l.b.$(P_1, P_2) = \mathrm{MLR}$. (Simpson 2001)

5. If $P > 0$ is thin, then P is Muchnik incomparable with MLR. (Simpson 2001)

6. There do not exist $P_1, P_2 > 0$ such that g.l.b.$(P_1, P_2) = 0$. (trivial)

7. If $S \subseteq 2^\omega$ is Σ^0_3, then for all Π^0_1 $P \subseteq 2^\omega$ there exists Π^0_1 $Q \subseteq 2^\omega$ such that Q is Muchnik equivalent to $S \cup P$. (Simpson 2002)

8. Given $P \geq_w \mathsf{FPF}$, we have a semilattice embedding of the r.e. Turing degrees into $\mathcal{P}_w$, given by $X \mapsto \{X\} \cup P$. (Simpson 2002)

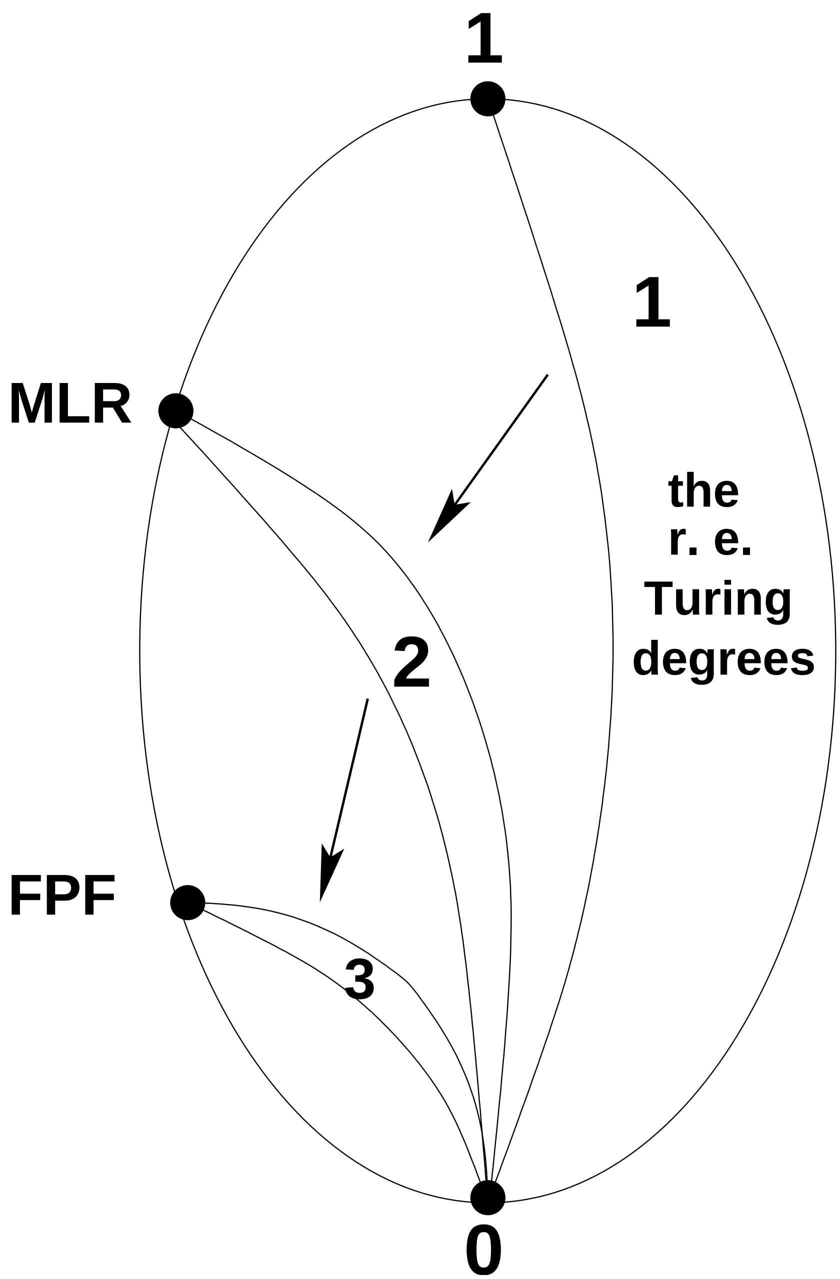
In particular, we have these three "natural" embeddings:

The *Gödel/Rosser embedding*, $X \mapsto \{X\} \cup 1$.

The *Martin-Löf embedding*, $X \mapsto \{X\} \cup \mathsf{MLR}$.

The *Arslanov embedding*, $X \mapsto \{X\} \cup \mathsf{FPF}$.

Three "natural" embeddings of the r.e. Turing degrees into the Muchnik lattice $\mathcal{P}_w$:



An extension, continued:

As we have seen,
the r.e. Turing degrees are embedded in $\mathcal{P}_w$.

Technical Note: Using a generalized Arslanov criterion, we can embed a wider class of Turing degrees: those that are $\leq 0'$ and n-REA for some $n \in \omega$.

Summary: The intensively studied semilattice of r.e. Turing degrees, $\mathcal{R}_T$, is included in the mathematically more natural, but less studied, Muchnik lattice, $\mathcal{P}_w$.

Moral: By studying the Muchnik lattice $\mathcal{P}_w$ instead of the r.e. Turing degrees, recursion theorists could connect better to f.o.m.

Foundations of mathematics (f.o.m.):

Foundations of mathematics is the study of the most basic concepts and logical structure of mathematics as a whole.

Among the most basic mathematical concepts are:

number, set, function, algorithm, mathematical definition, mathematical proof, mathematical theorem, mathematical axiom.

Some big names in f.o.m. are:

Aristotle, Euclid, Descartes, Leibniz, . . . , Dedekind, Cantor, Frege, Russell, Zermelo, Hilbert, Weyl, Brouwer, Skolem, Gödel, Church, Turing, Post, Kleene, . . .

F.o.m. and Reverse Mathematics:

A key f.o.m. question:

What are the appropriate axioms for mathematics?

Reverse Mathematics examines a specific case of this question:

What axioms are needed to prove specific, known theorems of ordinary mathematics?

We examine this question in the context of subsystems of Z_2.

Z_2 = second order arithmetic.

Reverse Mathematics (continued):

In reverse mathematics, we develop a table indicating precisely which mathematical theorems can be proved in which subsystems of Z_2.

	RCA_0	WKL_0	ACA_0	ATR_0	Π^1_1-CA_0
analysis (separable):					
differential equations	X	X			
continuous functions	X, X	X, X	X		
completeness, etc.	X	X	X		
Banach spaces	X	X, X			X
open and closed sets	X	X		X, X	X
Borel and analytic sets	X			X, X	X, X
algebra (countable):					
countable fields	X	X, X	X		
commutative rings	X	X	X		
vector spaces	X		X		
Abelian groups	X		X	X	X
miscellaneous:					
mathematical logic	X	X			
countable ordinals	X		X	X, X	
infinite matchings		X	X	X	
the Ramsey property			X	X	X
infinite games			X	X	X

Foundational consequences of reverse mathematics:

By means of reverse mathematics, we identify five particular subsystems of Z_2 as being mathematically natural. We correlate these systems to traditional f.o.m. programs.

RCA_0	constructivism	Bishop
WKL_0	finitistic reductionism	Hilbert
ACA_0	predicativism	Weyl, Feferman
ATR_0	predicative reductionism	Friedman, Simpson
Π^1_1-CA_0	impredicativity	Feferman et al.

We analyze these foundational proposals in terms of their consequences for mathematical practice. Under the various proposals, which mathematical theorems are "lost"?

Reverse mathematics provides precise answers to such questions.

Two books on reverse mathematics:

1.

Stephen G. Simpson
Subsystems of Second Order Arithmetic
Perspectives in Mathematical Logic
Springer-Verlag, 1999
XIV + 445 pages

`http://www.math.psu.edu/simpson/sosoa/`

2.

S. G. Simpson (editor)
Reverse Mathematics 2001

A volume of papers by various authors,
to appear,
approximately 400 pages.

`http://www.math.psu.edu/simpson/revmath/`

An introduction to WKL_0**:**

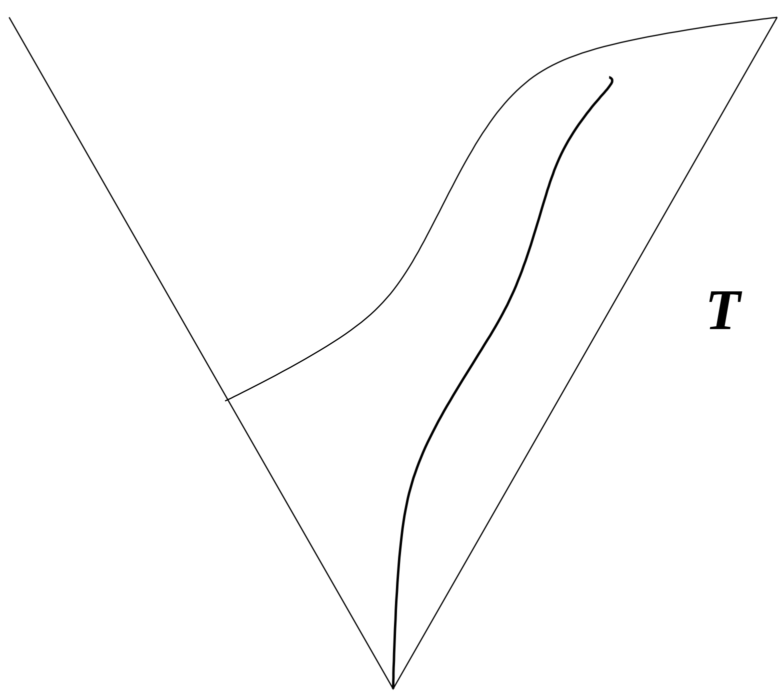
$\mathsf{RCA}_0 = \Sigma^0_1$ induction

$+ \Delta^0_1$ (i.e., recursive) comprehension

("formalized recursive mathematics")

$\mathsf{WKL}_0 = \mathsf{RCA}_0 +$ Weak König's Lemma:

Every infinite subtree of the full binary tree of finite sequences of 0's and 1's has an infinite path. (a "formalized compactness principle")



Reverse Mathematics for WKL_0:

WKL_0 is equivalent over RCA_0 to each of the following mathematical statements:

1. The Heine/Borel Covering Lemma: Every covering of $[0,1]$ by a sequence of open intervals has a finite subcovering.

2. Every covering of a compact metric space by a sequence of open sets has a finite subcovering.

3. Every continuous real-valued function on $[0,1]$ (or on any compact metric space) is bounded (uniformly continuous, Riemann integrable).

6. The Maximum Principle: Every continuous real-valued function on $[0,1]$ (or on any compact metric space) has (or attains) a supremum.

R. M. for WKL_0 (continued):

7. The local existence theorem for solutions of (finite systems of) ordinary differential equations.

8. Gödel's Completeness Theorem: every finite (or countable) set of sentences in the predicate calculus has a countable model.

9. Every countable commutative ring has a prime ideal.

10. Every countable field (of characteristic 0) has a unique algebraic closure.

11. Every countable formally real field is orderable.

12. Every countable formally real field has a (unique) real closure.

R. M. for WKL_0 (continued):

13. Brouwer's Fixed Point Theorem: Every (uniformly) continuous function $\phi : [0,1]^n \to [0,1]^n$ has a fixed point.

14. The Separable Hahn/Banach Theorem: If f is a bounded linear functional on a subspace of a separable Banach space, and if $\|f\| \leq 1$, then f has an extension $\widetilde{f}$ to the whole space such that $\|\widetilde{f}\| \leq 1$.

15. Banach's Theorem: In a separable Banach space, given two disjoint convex open sets A and B, there exists a closed hyperplane H such that A is on one side of H and B is on the other.

16. Every countable k-regular bipartite graph has a perfect matching.

Some of my papers are available at
`http://www.math.psu.edu/simpson/papers/`.

Transparencies for my talks are available at
`http://www.math.psu.edu/simpson/talks/`.

THE END