An extension of the recursively enumerable Turing degrees

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Abstract:

We embed the upper semilattice of r.e. Turing degrees into a slightly larger structure which is better behaved and more foundationally relevant. For $P, Q \subseteq 2^\omega$, we say P is Muchnik reducible to Q if for all $Y \in Q$ there exists $X \in P$ such that X is Turing reducible to Y. We let $\mathcal{P}_w$ denote the lattice of Muchnik degrees of nonempty Π^0_1 subsets of 2^ω. $\mathcal{P}_w$ is a countable distributive lattice with 0 and 1. Every countable distributive lattice is lattice embeddable in every nontrivial initial segment of $\mathcal{P}_w$ (Binns/Simpson). Every nonzero Muchnik degree in $\mathcal{P}_w$ is join reducible in $\mathcal{P}_w$ (Binns). We construct a natural upper semilattice embedding i of the r.e. Turing degrees into $\mathcal{P}_w$. We have $i(0) = 0$ and $i(0') = 1$. We show that $\mathcal{P}_w$ contains at least two specific, natural Muchnik degrees other than 0 and 1, viz., the Muchnik degree MLR of Martin-Lof random reals, and the Muchnik degree FPF of fixed point free functions. In $\mathcal{P}_w$ we have $0 < \mathrm{FPF} < \mathrm{MLR} < 1$, and these Muchnik degrees correspond to subsystems of Z_2 which have arisen in Reverse Mathematics. Moreover, FPF and MLR are incomparable with $i(a)$ for all r.e. Turing degrees $0 < a < 0'$.