Reverse Mathematics, Mass Problems, and Effective Randomness

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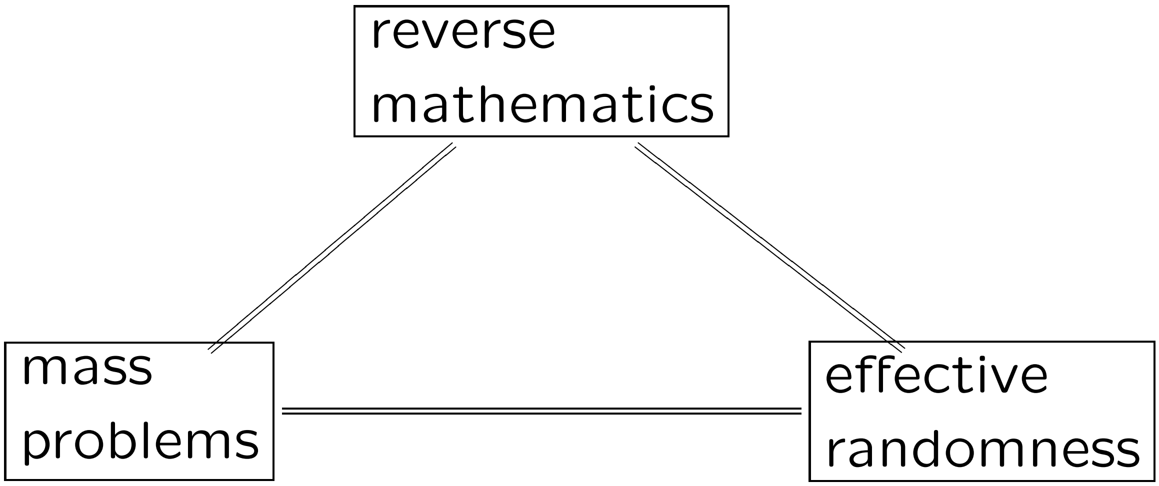


Foundations of mathematics is the study of the most basic concepts and logical structure of mathematics as a whole.

Reverse mathematics is a particular research program in the foundations of mathematics.

The goal of reverse mathematics is to classify core mathematical theorems up to logical equivalence, according to which set-existence axioms are needed to prove them.

This is carried out in the context of subsystems of second order arithmetic.

This leads to a remarkably regular structure. A large number of theorems fall into a small number of equivalence classes.

Books on reverse mathematics:

Stephen G. Simpson
Subsystems of Second Order Arithmetic
Perspectives in Mathematical Logic
Springer-Verlag, 1999, XIV + 445 pages
(out of print)

S. G. Simpson (editor)
Reverse Mathematics 2001
(a volume of papers by various authors)
Lecture Notes in Logic
Association for Symbolic Logic
2005, X + 401 pages

Stephen G. Simpson
Subsystems of Second Order Arithmetic
Second Edition
Perspectives in Logic
Association for Symbolic Logic
approximately 460 pages, in press

Reverse mathematics of measure theory.

The first wave:

In 1987 Simpson and X. Yu introduced a subsystem of second order arithmetic known as WWKL_0. The principal axiom of WWKL_0 is equivalent to

$$\forall X\, \exists Y\, (Y \text{ is random relative to } X).$$

Many theorems of measure theory are equivalent to this axiom.

Example: the Vitali Covering Theorem.

See Brown/Giusto/Simpson, Archive for Mathematical Logic, 41, 2003, 191–206.

The second wave:

N. Dobrinen and S. Simpson, Almost everywhere domination, Journal of Symbolic Logic, 69, 2004, 914–922, considered the reverse mathematics of measure-theoretic regularity statements:

1. Every G_δ set includes an F_σ set of the same measure.

2. Every G_δ set includes a closed set of measure within an arbitrarily small epsilon.

3. Every G_δ set of positive measure includes a closed set of positive measure.

By Dobrinen/Simpson, the corresponding set-existence axioms are:

1. For all A there exists B such that B is uniformly almost everywhere dominating relative to A.

2. For all A there exists B such that B is almost everywhere dominating relative to A.

3. For all A there exists B such that B is positive measure dominating relative to A.

Definition. B is said to be *almost everywhere dominating* if, for measure one many X, each X-computable function is dominated by some B-computable function.

Here B-computable means: computable using B as a Turing oracle.

There is a close relationship between a. e. domination and effective randomness.

Definition (Nies 2002).
We say that A is LR*-reducible to* B if

$$\forall X\,(X \text{ is } B\text{-random} \Rightarrow X \text{ is } A\text{-random}).$$

Theorem 1 (Kjos-Hanssen 2005).
B is positive measure dominating
$\iff 0' \leq_{LR} B$.

Here $0'$ is a Turing oracle for the Halting Problem.

Theorem 2
(Binns/Kjos-Hanssen/Miller/Solomon 2006).
B is uniformly almost everywhere dominating
$\iff B$ is almost everywhere dominating
$\iff B$ is positive measure dominating.

Thus, it seems likely that all of the measure-theoretic regularity statments considered by Dobrinen/Simpson fall into the same reverse mathematics classification.

Because of this work by Kjos-Hanssen and Binns/Kjos-Hanssen/Miller/Solomon, it seems foundationally desirable to improve our understanding of the binary relation $A \leq_{LR} B$, and especially of the set $\{B \mid 0' \leq_{LR} B\}$.

Here is a recent characterization of $\leq_{LR}$ in terms of Kolmogorov complexity.

Definition (Nies 2002).

We say that A is *LK-reducible to B* if

$$K^B(\tau) \leq K^A(\tau) + O(1).$$

Here K^B denotes prefix-free Kolmogorov complexity relative to the Turing oracle B.

Theorem 3 (B/K-H/M/S 2006).

$A \leq_{LR} B \iff A \leq_{LK} B$.

This is an improvement of some earlier results due to Nies 2002. In particular, Nies had proved that $A \leq_{LR} 0 \iff A \leq_{LK} 0$.

Another recent result:

Theorem 4 (Simpson 2006).

If $A \leq_{LR} B$ and A is recursively enumerable, then A' is truth-table computable from B'.

Here B' denotes the Turing jump of B.

Corollary (Simpson 2006).

If $0' \leq_{LR} B$ then B is *superhigh*, i.e., $0''$ is truth-table computable from B'.

Again, these results improve on some earlier results due to Nies 2002.

The corollary seems especially interesting, because $0' \leq_{LR} B \iff$
B is almost everywhere dominating.

Remark. Nies/Hirschfeldt/Stephan have shown that four concepts coincide:

1. A is low-for-random, i.e., $A \leq_{LR} 0$.

2. A is basic-for-random, i.e., $A \leq_T X$ for some A-random X.

3. A is low-for-K, i.e., $K(\tau) \leq K^A(\tau) + O(1)$.

4. A is K-trivial, i.e., $K(A \upharpoonright n) \leq K(n) + O(1)$.

Question. How does this play out in the context of LR-reducibility? Specifically, can we characterize LR-reducibility in terms of relative K-triviality?

Note. We can characterize relative K-triviality in terms of LR-reducibility. Namely, A is K-trivial relative to B $\Longleftrightarrow A \oplus B \leq_{LR} B$.

Caution. $A \leq_{LR} 0 \iff A$ is low-for-random.

However, $A \leq_{LR} B$ is not equivalent to A being low-for-random relative to B, even in the special case $A = 0'$.

What actually holds is:

A is low-for-random relative to B $\iff A \oplus B \leq_{LR} B$.

This binary relation is not transitive!

Caution. If $A \leq_{LR} 0$ and $B \leq_{LR} 0$ then $A \oplus B \leq_{LR} 0$. This follows from results of Nies, Advances in Mathematics, and the Downey/Hirschfeldt/Nies/Stephan paper, "Trivial reals".

However, $A \leq_{LR} C$ and $B \leq_{LR} C$ do not imply $A \oplus B \leq_{LR} C$.

In fact, we can find a C such that $0' \leq_{LR} C$ (i.e., C is almost everywhere dominating), but $0' \oplus C \not\leq_{LR} C$
(i.e., $0'$ is not low-for-random relative to C).

Question. If $A \leq_{LR} X$ and X is A-random, does it follow that $A \leq_{LR} 0$?

This would be an improvement of the Hungry Sets Theorem, due to Hirschfeldt/Nies/Stephan. This theorem has $\leq_T$ instead of $\leq_{LR}$.

Question. If A is random and $A \leq_{LR} B$ and B is C-random, does it follow that A is C-random?

This would be an improvement of a theorem of Miller/Yu 2004, which has $\leq_T$ instead of $\leq_{LR}$.

Reverse mathematics of general topology.

Background:

In my book Subsystems of Second Order Arithmetic, a *complete separable metric space* is defined as the completion $X = (\widehat{A}, \widehat{d})$ of a countable pseudometric space (A, d). Here $A \subseteq \mathbb{N}$ and $d : A \times A \to \mathbb{R}$.

Thus complete separable metric spaces are "coded" by countable objects. Using this coding, a great deal of analysis and geometry is developed in RCA_0, with many reverse mathematics results.

However, until recently, there was no reverse mathematics study of general topology.

The obstacle was, there was no way to discuss abstract topological spaces in L_2, the language of second order arithmetic. This was the case even for topological spaces which are separable or second countable.

To overcome this conceptual difficulty, Mummert and Simpson introduced a restricted class of topological spaces, called the *countably based MF spaces*.

This class includes all complete separable metric spaces, as well as many nonmetrizable spaces.

Furthermore, this class of spaces can be discussed in L_2.

Details:

Let P be a *poset*, i.e., a partially ordered set.

Definition. A *filter* is a set $F \subseteq P$ such that

1. for all $p, q \in F$ there exists $r \in F$ such that $r \leq p$ and $r \leq q$.

2. F is *upward closed*, i.e., $(q \geq p \wedge p \in F) \Rightarrow q \in F$.

Compare the treatment of forcing in Kunen's textbook of axiomatic set theory.

Definition. A *maximal filter* is a filter which is not properly included in any other filter.

By Zorn's Lemma, every filter is included in a maximal filter.

Definition.
MF(P) $= \{F \mid F$ is a maximal filter on $P\}$.

Definition.
$\mathsf{MF}(P) = \{F \mid F$ is a maximal filter on $P\}$.

$\mathsf{MF}(P)$ is a topological space with basic open sets
$$N_p = \{F \mid p \in F\}$$
for all $p \in P$.

Definition. An *MF space* is a space of the form $\mathsf{MF}(P)$ where P is a poset.

Definition. A *countably based MF space* is a space of the form $\mathsf{MF}(P)$ where P is a countable poset.

Thus, the second countable topological space $\mathsf{MF}(P)$ is "coded" by the countable poset P.

Therefore, countably based MF spaces can be defined and discussed in L_2. Thus we can do reverse mathematics in the usual setting, subsystems of second order arithmetic.

Examples:

Theorem (Mummert/Simpson).
Every complete (separable) metric space is homeomorphic to a (countably based) MF space.

Many of the topological spaces which arise in analysis and geometry are complete separable metric spaces. Therefore, they may be viewed as countably based MF spaces.

On the other hand, there are many other (countably based) MF spaces which are not metrizable.

An example is the Baire space ω^ω with the topology generated by the Σ^1_1 sets, i.e., the Gandy/Harrington topology. This space plays a key role in modern descriptive set theory (Kechris, Hjorth, et al).

Recently, Carl Mummert and Frank Stephan have characterized the countably based MF spaces up to homeomorphism as the second countable T_1 spaces with the strong Choquet property.

References:

Carl Mummert and Stephen G. Simpson, Reverse Mathematics and Π^1_2 Comprehension, Bulletin of Symbolic Logic, 11, 2005, pages 526–533.

Carl Mummert, Ph.D. thesis, *On the Reverse Mathematics of General Topology*, 2005, Pennsylvania State University, VI + 102 pages.

Forthcoming papers of Mummert, Mummert/Stephan, etc.

A new research direction:

the reverse mathematics of topological measure theory.

By means of countably based MF spaces, one can formulate many interesting reverse mathematics problems in the area of topological measure theory. For example, one can consider the reverse mathematics of weak convergence of measures on general topological spaces (Billingsley, Topsøe, et al).

Mass problems (informal discussion):

A "decision problem" is the problem of deciding whether a given $n \in \omega$ belongs to a fixed set $A \subseteq \omega$ or not. To compare decision problems, we use Turing reducibility. $A \leq_T B$ means that A can be computed using an oracle for B.

A "mass problem" is a problem with a not necessarily unique solution. By contrast, a "decision problem" has only one solution.

The "mass problem" associated with a set $P \subseteq \omega^\omega$ is the "problem" of computing an element of P.

The "solutions" of P are the elements of P.

One mass problem is said to be "reducible" to another if, given *any* solution of the second problem, we can use it as an oracle to compute *some* solution of the first problem.

Mass problems (rigorous definition):

Let P and Q be subsets of ω^ω.

We view P and Q as mass problems.

We say that P is *weakly reducible* to Q if

$$(\forall Y \in Q)\ (\exists X \in P)\ (X \leq_T Y)\ .$$

This is abbreviated $P \leq_w Q$.

Summary:

$P \leq_w Q$ means that, given any solution of the mass problem Q, we can use it as a Turing oracle to compute a solution of the mass problem P.

The lattice $\mathcal{P}_w$:

We focus on Π^0_1 subsets of 2^ω, i.e., $P = \{\text{paths through } T\}$ where T is a recursive subtree of $2^{<\omega}$, the full binary tree of finite sequences of 0's and 1's.

We define $\mathcal{P}_w$ to be the set of weak degrees of nonempty Π^0_1 subsets of 2^ω, ordered by weak reducibility.

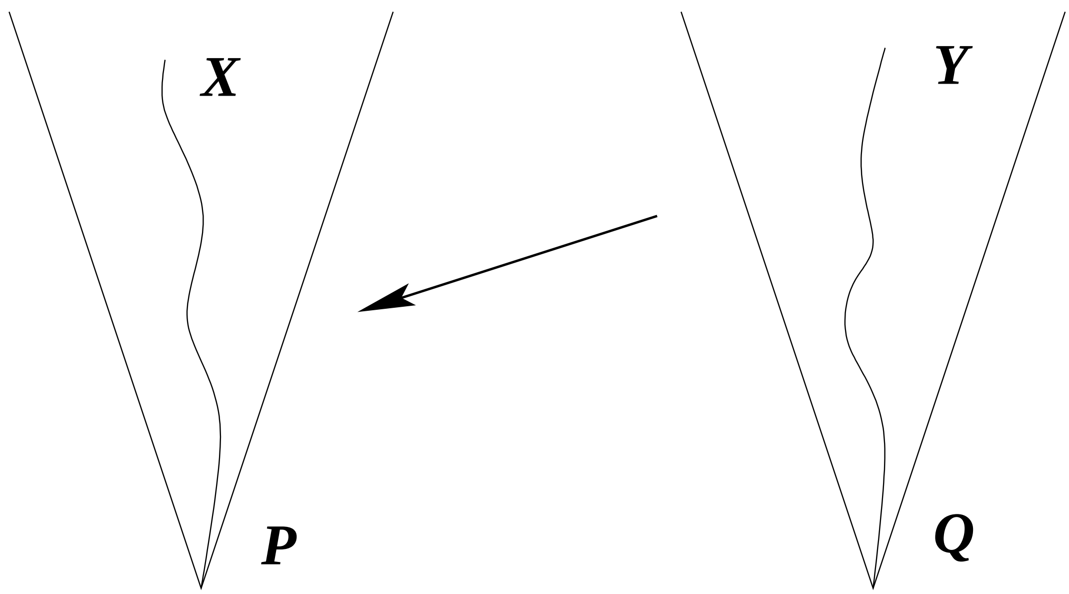
Basic facts about $\mathcal{P}_w$:

1. $\mathcal{P}_w$ is a distributive lattice, with l.u.b. given by $P \times Q = \{X \oplus Y \mid X \in P, Y \in Q\}$, and g.l.b. given by $P \cup Q$.

2. The bottom element of $\mathcal{P}_w$ is the weak degree of 2^ω.

3. The top element of $\mathcal{P}_w$ is the weak degree of PA $= \{\text{completions of Peano Arithmetic}\}$. (Scott/Tennenbaum).

Weak reducibility of Π^0_1 subsets of 2^ω:



$P \leq_w Q$ means:
$(\forall Y \in Q)\ (\exists X \in P)\ (X \leq_T Y)$.

P, Q are given by infinite recursive subtrees of the full binary tree of finite sequences of 0's and 1's.

X, Y are infinite (nonrecursive) paths through P, Q respectively.

Embedding $\mathcal{R}_T$ into $\mathcal{P}_w$:

Let $\mathcal{R}_T$ be the upper semilattice of recursively enumerable Turing degrees.

Theorem (Simpson 2002):

There is a natural embedding $\phi : \mathcal{R}_T \to \mathcal{P}_w$.

The embedding ϕ is given by

$$\phi : \text{deg}_T(A) \mapsto \text{deg}_w(\text{PA} \cup \{A\}).$$

Note: $\text{PA} \cup \{A\}$ is not a Π^0_1 set. However, it is of the same weak degree as a Π^0_1 set. This is a non-obvious fact.

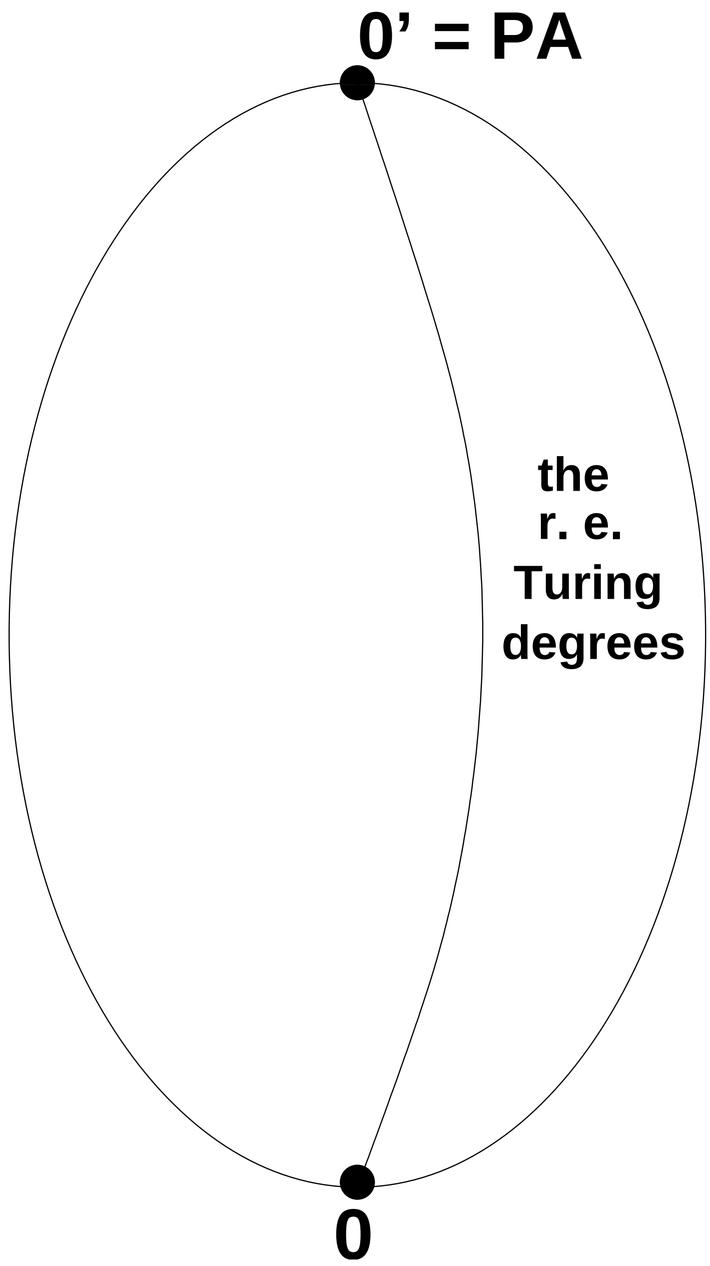
The embedding ϕ is one-to-one and preserves $\leq$, l.u.b., and the top and bottom elements. The one-to-oneness is not obvious.

Convention:

We identify $\mathcal{R}_T$ with its image in $\mathcal{P}_w$ under ϕ.

In particular, we identify $\mathbf{0}', \mathbf{0} \in \mathcal{R}_T$ with the top and bottom elements of $\mathcal{P}_w$.

A picture of the lattice $\mathcal{P}_w$:



$\mathcal{R}_T$ is embedded in $\mathcal{P}_w$. $\mathbf{0}'$ and $\mathbf{0}$ are the top and bottom elements of both $\mathcal{R}_T$ and $\mathcal{P}_w$.

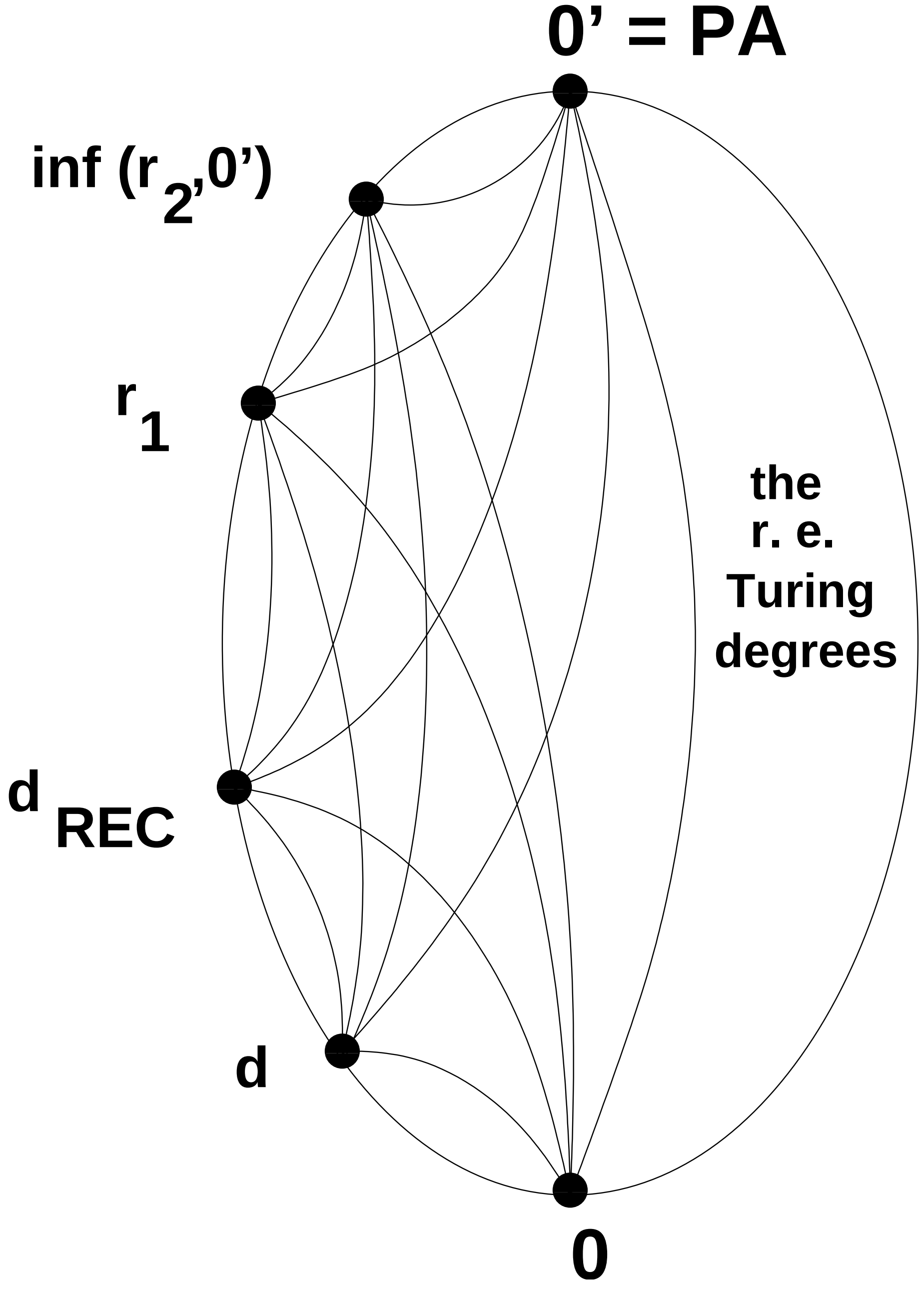
Specific, natural degrees in $\mathcal{P}_w$:

A fundamental open problem concerning the recursively enumerable Turing degrees is to find a specific, natural example of such a degree, other than $\mathbf{0}$ and $\mathbf{0}'$.

In the $\mathcal{P}_w$ context, we have discovered many specific, natural degrees which are $> \mathbf{0}$ and $< \mathbf{0}'$.

The specific, natural degrees in $\mathcal{P}_w$ which we have discovered are related to foundationally interesting topics:

- effective randomness,
- diagonal nonrecursiveness,
- reverse mathematics,
- subrecursive hierarchies,
- computational complexity.



Note. Except for $\mathbf{0}'$ and $\mathbf{0}$, the r.e. Turing degrees are incomparable with all of these specific, natural degrees in $\mathcal{P}_w$.

Some specific, natural degrees in $\mathcal{P}_w$.

$\mathbf{r}_n =$ the weak degree of the set of n-random reals.

$\mathbf{d} =$ the weak degree of the set of diagonally nonrecursive functions.

$\mathbf{d}_{\mathsf{REC}} =$ the weak degree of the set of diagonally nonrecursive functions which are recursively bounded.

Theorem
(Simpson 2002, Ambos-Spies et al 2004)

In $\mathcal{P}_w$ we have

$$\mathbf{0} < \mathbf{d} < \mathbf{d}_{\mathsf{REC}} < \mathbf{r}_1 < \inf(\mathbf{r}_2, \mathbf{0}') < \mathbf{0}'.$$

Theorem (Simpson 2002).

1. $\mathbf{r}_1$ is the maximum weak degree of a Π^0_1 subset of 2^ω which is of positive measure.

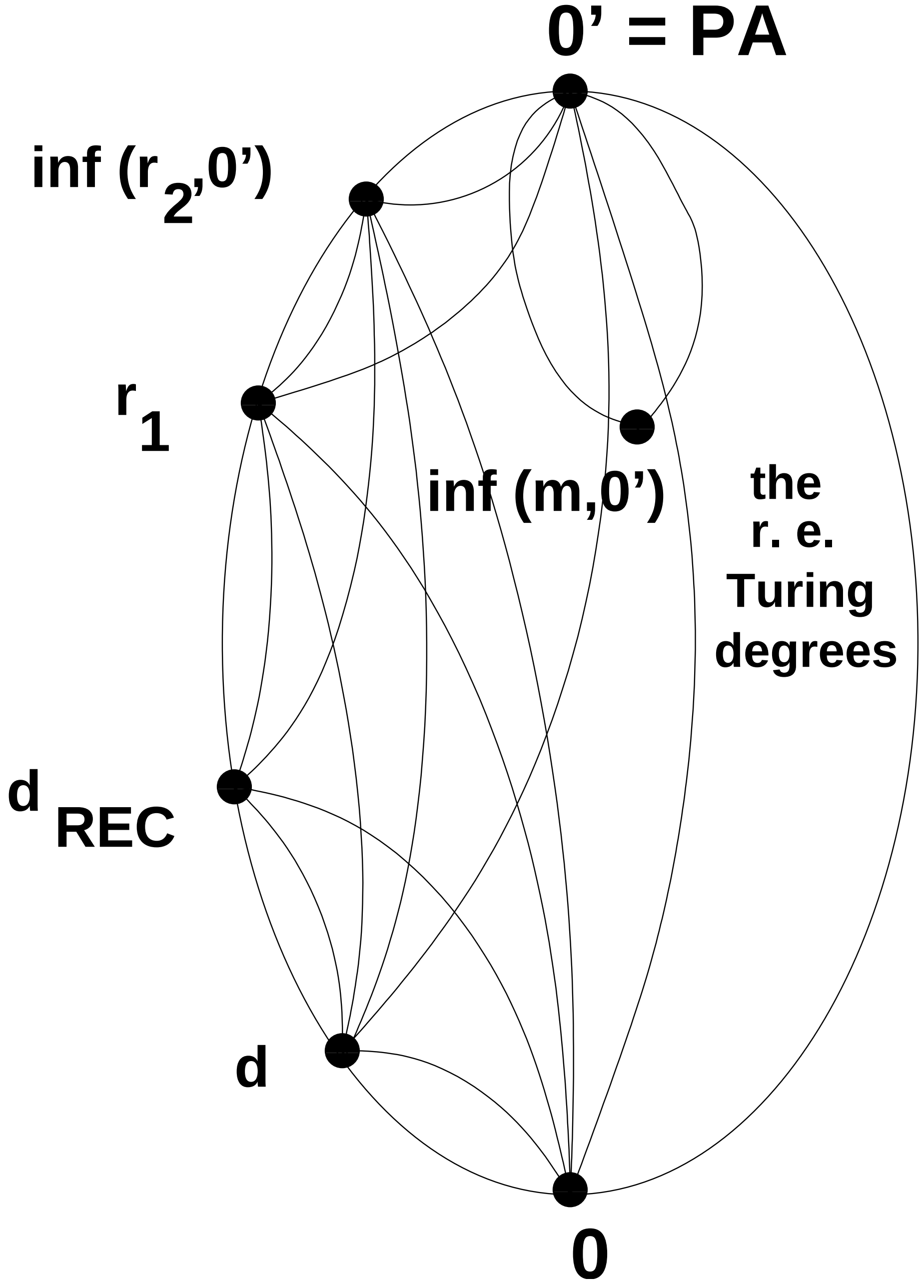
2. $\inf(\mathbf{r}_2, \mathbf{0}')$ is the maximum weak degree of a Π^0_1 subset of 2^ω whose Turing upward closure is of positive measure.

Another specific, natural degree in $\mathcal{P}_w$ is provided by the work of Kjos-Hanssen and Binns/Kjos-Hanssen/Miller/Solomon on almost everywhere domination.

Definition. Let $\mathbf{m} = \deg_w(\text{AED})$ where $\text{AED} = \{B \mid B$ is almost everywhere dominating$\}$.

It can be shown that $\inf(\mathbf{m}, \mathbf{0}')$ belongs to $\mathcal{P}_w$. Again, this is not obvious, because $\text{AED} \cup \text{PA}$ is not Π^0_1.

Interestingly, $\inf(\mathbf{m}, \mathbf{0}')$ lies below some recursively enumerable Turing degrees which are strictly below $\mathbf{0}'$. This is in contrast to the behavior of $\mathbf{r}_1$, $\inf(\mathbf{r}_2, \mathbf{0}')$, $\mathbf{d}$, and $\mathbf{d}_{\text{REC}}$.



Note how the behavior of inf$(\mathbf{m}, \mathbf{0}')$ contrasts with that of inf$(\mathbf{r}_2, \mathbf{0}')$, $\mathbf{r}_1$, $\mathbf{d}_{\text{REC}}$, and $\mathbf{d}$.

Questions????

Some additional examples ?

It seems reasonable to think that additional examples of specific, natural degrees in $\mathcal{P}_w$ could be obtained by replacing measure by Hausdorff dimension.

For rational s with $0 \leq s \leq 1$, let $Q_s = \{X \in 2^\omega \mid \mathrm{dim}(X) = s\}$. Here dim denotes *effective Hausdorff dimension* as defined by Jack Lutz.

The Q_s's are uniformly Σ^0_3, so by the Embedding Lemma we have
$\mathbf{q}_s = \mathrm{deg}_w(Q_s) \in \mathcal{P}_w$ and
$\mathbf{q}_{>s} = \inf_{t>s} \mathbf{q}_t \in \mathcal{P}_w$. By "dilution" we have
$\mathbf{q}_s \leq \mathbf{q}_t \leq \mathbf{r}_1$ for all $s < t$.

Question. What other relationships hold among the $\mathbf{q}_s$'s?

Question. What relationships hold among the $\mathbf{q}_s$'s?

Conceivably $\mathbf{q}_s < \mathbf{q}_t < \mathbf{r}_1$ for all $s < t \leq 1$. At the other extreme, it is possible that $\mathbf{q}_s = \mathbf{r}_1$ for all $s > 0$.

This is essentially just Reimann's "dimension extraction problem". The problem is, does $\dim(X) > 0$ imply existence of $Y \leq_T X$ such that Y is random? Does $0 < \dim(X) < 1$ imply existence of $Y \leq_T X$ such that $\dim(Y) > \dim(X)$?

Question. What relationships hold among the $\mathbf{q}_s$'s and other specific, natural degrees in $\mathcal{P}_w$ such as $\mathbf{r}_1$, $\mathbf{d}_{\text{REC}}$, $\mathbf{d}$, etc.?

Question. Can we find specific, natural degrees in $\mathcal{P}_w$ analogous to $\inf(\mathbf{m}, \mathbf{0}')$, replacing positive measure domination by positive Hausdorff dimension domination, positive effective Hausdorff dimension domination, etc.?

Smallness properties of Π^0_1 subsets of 2^ω.

There are many "smallness properties" of Π^0_1 sets $P \subseteq 2^\omega$ which insure that the weak degree of P is $> \mathbf{0}$ and $< \mathbf{0}'$. Here is one result of this type.

Definition.

A Π^0_1 set $P \subseteq 2^\omega$ is said to be *thin* if, for all Π^0_1 sets $Q \subseteq P$, $P \setminus Q$ is Π^0_1.

Thin perfect Π^0_1 subsets of 2^ω have been constructed by means of priority arguments. Much is known about them. For example, any two such sets are automorphic in the lattice of Π^0_1 subsets of 2^ω under inclusion. See Martin/Pour-El 1970, Downey/Jockusch/Stob 1990, 1996, Cholak et al 2001.

Theorem (Simpson 2002).

Let $\mathbf{p}$ be the weak degree of a Π^0_1 set $P \subseteq 2^\omega$ which is thin and perfect. Then $\mathbf{p}$ is incomparable with $\mathbf{r}_1$. Hence $\mathbf{0} < \mathbf{p} < \mathbf{0}'$.

Relationship to measure and dimension.

Theorem (Simpson 2002). If $P \subseteq 2^\omega$ is thin and perfect, then P is of measure 0.

Theorem (Binns 2006). If $P \subseteq 2^\omega$ is thin and perfect, then P is of Hausdorff dimension 0.

Note (Hitchcock 2000). For any Π^0_1 set $P \subseteq 2^\omega$, the effective Hausdorff dimension of P is equal to the Hausdorff dimension of P.

Question (Simpson 2002). Does there exist a thin perfect Π^0_1 set $P \subseteq 2^\omega$ such that the Turing upward closure of P is of measure > 0?

Note. This is equivalent to asking whether the weak degree of such a set can be $\leq \inf(\mathbf{r}_2, \mathbf{0}')$.

Note (Reimann). By a theorem in Reimann's thesis, all Turing cones are of Hausdorff dimension 1.

Some additional "smallness properties":

Let P be a Π^0_1 subset of 2^ω.

Definition. P is *small* if there is no recursive function f such that for all n there exist n members of P which differ at level $f(n)$ in the binary tree. (Binns 2003)

Example. Let $A \subseteq \omega$ be hypersimple, and let $A = B_1 \cup B_2$ where B_1, B_2 are r.e. Then $P = \{X \in 2^\omega \mid X \text{ separates } B_1, B_2\}$ is small.

Definition. P is *h-small* if there is no recursive, canonically indexed sequence of pairwise disjoint clopen sets D_n, $n \in \omega$, such that $P \cap D_n \neq \emptyset$ for all n. (Simpson 2003)

For many of these smallness properties, there are results and questions similar to the ones which we formulated above for thin perfect Π^0_1 sets. One can ask about the measure and dimension of P, and about the measure of the Turing upward closure of P.

Additional references:

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Some of my papers are available at

`http://www.math.psu.edu/simpson/papers/`.

Transparencies for my talks are available at

`http://www.math.psu.edu/simpson/talks/`.

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