A review of

Subsystems of Second Order Arithmetic
by Stephen G. Simpson

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This is the first book to appear on "reverse mathematics". The book provides an excellent introduction to the area and is packed with some interesting theorems and uses of mathematical logic.

The theme of reverse mathematics is to take a theorem of mathematics and determine the set existence axioms needed to prove the chosen theorem. The goal is to show that the theorem is actually equivalent to the needed axioms. In Simpson's book, this is done by formalizing theorems in the language of second order arithmetic and showing that they are equivalent to one of five subsystems of second order arithmetic.

The language of second order arithmetic is a two-sorted first order language. There are variables which range over numbers and variables which range over sets of numbers. The five subsystems under consideration are (in increasing order of strength) $RCA_0, WKL_0, ACA_0, ATR_0$ and Π^1_1-CA_0. The weakest system, RCA_0, consists of the following axioms: all the Peano axioms except for induction, induction restricted to Σ^0_1-formulas, $I\Sigma_1$, and comprehension restricted to Δ^0_1-formulas. WKL_0 is RCA_0 plus the statement "every infinite tree of sequences of 0's and 1's has an infinite path". ACA_0 is RCA_0 plus arithmetic comprehension. ATR_0 is ACA_0 plus the scheme "if X is a well order then you can do arithmetic recursion along X". Π^1_1-CA_0 is RCA_0 plus comprehension for Π^1_1 formulas.

The book has two distinct parts. The first part, consisting of the first six chapters, focuses on these five systems and shows various theorems of mathematics are equivalent to one of them. The work in Chapter 2 justifies Simpson's assertion that RCA_0 is a formal version of computable mathematics. In Chapter 3, Simpson discusses ACA_0 and proves it to be equivalent to many theorems of algebra, analysis, topology and combinatorics. For example, ACA_0 and "every countable field has a strong algebraic closure" are equivalent. Generally, a theorem implies ACA_0 if any model in the language of second order arithmetic in which the theorem is true is closed under the Turing jump.

The system WKL_0 is discussed in Chapter 4. In particular, it is shown that WKL_0 is equivalent to "every countable formally real field is orderable" and "there is a set separating every pair of disjoint Σ^0_1 sets" (Σ^0_1-separation).

In some cases, Simpson uses this last equivalence to show other equivalences. Some of these equivalences can be proven directly. For example, given an infinite computable tree T of sequences of 0's and 1's one can build a computable formally real field F such that there exists a computable homeomorphism between the orderings of F and the infinite paths through T (of course, one has to do this within RCA_0). Proofs of this nature are much more involved than just proving Σ^0_1-separation but they give more information. Given the focus and length of the book, it is understandable but regrettable that proofs of this nature were omitted. Other theorems of analysis, topology and combinatorics are discussed.

Chapters 5 and 6 focus on ATR_0 and Π^1_1-CA_0 respectively. Useful equivalences from logic include the following: ATR_0 is equivalent to Σ^1_1-separation, and Π^1_1-CA_0 is equivalent to "for every infinite sequence of infinite trees T_i, the collection of i such that T_i contains a path is a set". Some interesting equivalences from algebra are presented: ATR_0 is equivalent to "every countable reduced Abelian p-group has an Ulm resolution", and Π^1_1-CA_0 is equivalent to "every countable Abelian group is a direct sum of a divisible group and a reduced group". The bulk of these two chapters, however, is used to develop descriptive set theory within the two systems. Techniques of interest here are pseudohierarchies and the method of inner models.

The second part of the book is concerned with models of the five subsystems. Chapter 7 focuses on β-models and 8 focuses on ω-models. A structure $\mathcal{M}$ is an ω-model if its first order part is standard, and $\mathcal{M}$ is a β model if $\mathcal{M}$ is an ω-model and every true Σ^1_1 statement (with parameters) is true in $\mathcal{M}$. Other than the first two sections of Chapters 7 and 8, these chapters are very set and recursion theoretic. For example, in the third section of Chapter 8, hyperarithmetical theory is developed in ATR_0.

Chapter 9 is filled with various conservation results. The second section of Chapter 9, for example, has a proof of Harrington's result that any Π^1_1-sentence provable from WKL_0 is provable from RCA_0. Simpson ends with a summary of results and questions from measure theory, separable Banach spaces, and combinatorics as well as a very extensive bibliography.

The reviewer taught a graduate topics course at Notre Dame using this book. The course was a success. The book is readable and very carefully written. Students who have not seen the standard proofs had a difficult task at times but this is not unexpected. Working in these subsystems forces one be concerned with which axioms are used and as a result the proofs contain more details than the standard proofs. Often it is necessary to examine the standard proof or definition in order to motivate the one presented. Some proofs are particularly thorny. For example, the reviewer had difficulties with the proof of Silver's theorem about Borel equivalence relations in ATR_0 (Section 6.3) but a quick email reply from Simpson cleared up these difficulties. Simpson also maintains a web page with lots of information concerning the book; http://www.math.psu.edu/simpson/sosoa/.

How well did Simpson answer his "Main Question": "Which set existence axioms are needed to prove the theorems of *ordinary non-set-theoretic* mathematics?"[1] Simpson certainly makes a strong case that parts of ordinary non-set-theoretic mathematics can be done within RCA_0, WKL_0 and ACA_0. In the chapters on these subsystems, Simpson also develops a large collection of results from algebra, analysis, topology and combinatorics. Many of the non-set-theoretic theorems which are proven within these subsystems would appear early in the standard curriculum for mathematics graduate students.

The case is not as strong for the remaining systems, ATR_0 and Π^1_1-CA_0. There are fewer examples known and they are less varied than for the weaker systems. A large part of the book is concerned with developing descriptive set theory within these two subsystems. The only results, discussed in the book, which most mathematicians would agree are from "ordinary non-set-theoretic mathematics" developed in these subsystems are the two results on groups mentioned above and possibly the perfect set theorem and the Cantor-Bendixson theorem. However, there are a few new results for ATR_0 and Π^1_1-CA_0 not in the book that bolster Simpson's claim; Simpson does include references for many of these new results.

These five subsystems certainly provide an excellent tool for measuring the strength of a mathematical theorem. Reverse mathematics is an interesting and important area of mathematical logic, especially in combination with other interesting and important areas of mathematical logic including computability theory, computable (or effective) mathematics and proof theory.

For those interested in reverse mathematics or learning about reverse mathematics I highly recommend this book. I am glad I have a copy on my bookshelf.

[1] Page 2 Line 6. The reviewer added the emphasis.