Subsystems of Second Order Arithmetic

Second Edition

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This is the second edition of my book on subsystems of second order arithmetic and reverse mathematics. It will be published by the Association for Symbolic Logic in their book series *Perspectives in Logic*.

PREFACE

Foundations of mathematics is the study of the most basic concepts and logical structure of mathematics, with an eye to the unity of human knowledge. Among the most basic mathematical concepts are: number, shape, set, function, algorithm, mathematical axiom, mathematical definition, mathematical proof. Typical questions in foundations of mathematics are: What is a number? What is a shape? What is a set? What is a function? What is an algorithm? What is a mathematical axiom? What is a mathematical definition? What is a mathematical proof? What are the most basic concepts of mathematics? What is the logical structure of mathematics? What are the appropriate axioms for numbers? What are the appropriate axioms for shapes? What are the appropriate axioms for sets? What are the appropriate axioms for functions? *Etc., etc.*

Obviously foundations of mathematics is a subject which is of the greatest mathematical and philosophical importance. Beyond this, foundations of mathematics is a rich subject with a long history, going back to Aristotle and Euclid and continuing in the hands of outstanding modern figures such as Descartes, Cauchy, Weierstraß, Dedekind, Peano, Frege, Russell, Cantor, Hilbert, Brouwer, Weyl, von Neumann, Skolem, Tarski, Heyting, and Gödel. An excellent reference for the modern era in foundations of mathematics is van Heijenoort [272].

In the late 19th and early 20th centuries, virtually all leading mathematicians were intensely interested in foundations of mathematics and spoke and wrote extensively on this subject. Today that is no longer the case. Regrettably, foundations of mathematics is now out of fashion. Today, most of the leading mathematicians are ignorant of foundations and focus mostly on structural questions. Today, foundations of mathematics is out of favor even among mathematical logicians, the majority of whom prefer to concentrate on methodological or other non-foundational issues.

This book is a contribution to foundations of mathematics. Almost all of the problems studied in this book are motivated by an overriding foundational question: *What are the appropriate axioms for mathematics?* We undertake a series of case studies to discover which are the appropriate axioms for proving particular theorems in core mathematical areas such as algebra, analysis, and topology. We focus on the language of second order arithmetic, because that language is the weakest one that is rich enough to express and develop the bulk of core mathematics. It turns out that, in many particular cases, if a mathematical theorem is proved from appropriately weak set existence axioms, then the axioms will be logically equivalent to the theorem. Furthermore, only a few specific set existence axioms arise repeatedly in this context: recursive comprehension, weak König's lemma, arithmetical comprehension, arithmetical transfinite recursion, Π^1_1 comprehension; corresponding to the formal systems RCA_0, WKL_0, ACA_0, ATR_0, Π^1_1-CA_0; which in turn correspond to classical foundational programs: constructivism, finitistic reductionism, predicativism, and predicative reductionism. This is the theme of Reverse Mathematics, which dominates part A of this book. Part B focuses on models of these and other subsystems of second order arithmetic. Additional results are presented in an appendix.

The formalization of mathematics within second order arithmetic goes back to Dedekind and was developed by Hilbert and Bernays in [115, supplement IV]. The present book may be viewed as a continuation of Hilbert/Bernays [115]. I hope that the present book will help to revive the study of foundations of mathematics and thereby earn for itself a permanent place in the history of the subject.

The first edition of this book [249] was published in January 1999. The second edition differs from the first only in that I have corrected some typographical errors and updated some bibliographical entries. Recent advances are in research papers by numerous authors, published in *Reverse Mathematics 2001* [228] and in scholarly journals. The web page for this book is

`http://www.math.psu.edu/simpson/sosoa/`.

I would like to develop this web page into a forum for research and scholarship, not only in subsystems of second order arithmetic, but in foundations of mathematics generally.

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Chapter I

INTRODUCTION

I.1. The Main Question

The purpose of this book is to use the tools of mathematical logic to study certain problems in foundations of mathematics. We are especially interested in the question of which set existence axioms are needed to prove the known theorems of mathematics.

The scope of this initial question is very broad, but we can narrow it down somewhat by dividing mathematics into two parts. On the one hand there is set-theoretic mathematics, and on the other hand there is what we call "non-set-theoretic" or "ordinary" mathematics. By *set-theoretic mathematics* we mean those branches of mathematics that were created by the set-theoretic revolution which took place approximately a century ago. We have in mind such branches as general topology, abstract functional analysis, the study of uncountable discrete algebraic structures, and of course abstract set theory itself.

We identify as *ordinary* or *non-set-theoretic* that body of mathematics which is prior to or independent of the introduction of abstract set-theoretic concepts. We have in mind such branches as geometry, number theory, calculus, differential equations, real and complex analysis, countable algebra, the topology of complete separable metric spaces, mathematical logic, and computability theory.

The distinction between set-theoretic and ordinary mathematics corresponds roughly to the distinction between "uncountable mathematics" and "countable mathematics". This formulation is valid if we stipulate that "countable mathematics" includes the study of possibly uncountable complete separable metric spaces. (A metric space is said to be separable if it has a countable dense subset.) Thus for instance the study of continuous functions of a real variable is certainly part of ordinary mathematics, even though it involves an uncountable algebraic structure, namely the real number system. The point is that in ordinary mathematics, the real line partakes of countability since it is always viewed as a separable metric space, never as being endowed with the discrete topology.

In this book we want to restrict our attention to ordinary, non-set-theoretic mathematics. The reason for this restriction is that the set existence axioms which are needed for set-theoretic mathematics are likely to be much stronger than those which are needed for ordinary mathematics. Thus our broad set existence question really consists of two subquestions which have little to do with each other. Furthermore, while nobody doubts the importance of strong set existence axioms in set theory itself and in set-theoretic mathematics generally, the role of set existence axioms in ordinary mathematics is much more problematical and interesting.

We therefore formulate our *Main Question* as follows: *Which set existence axioms are needed to prove the theorems of ordinary, non-set-theoretic mathematics?*

In any investigation of the Main Question, there arises the problem of choosing an appropriate language and appropriate set existence axioms. Since in ordinary mathematics the objects studied are almost always countable or separable, it would seem appropriate to consider a language in which countable objects occupy center stage. For this reason, we study the Main Question in the context of the language of second order arithmetic. This language is denoted L_2 and will be described in the next section. All of the set existence axioms which we consider in this book will be expressed as formulas of the language L_2.

I.2. Subsystems of Z_2

In this section we define Z_2, the formal system of second order arithmetic. We also introduce the concept of a subsystem of Z_2.

The *language of second order arithmetic* is a two-sorted language. This means that there are two distinct sorts of variables which are intended to range over two different kinds of object. Variables of the first sort are known as *number variables*, are denoted by $i, j, k, m, n, \dots$, and are intended to range over the set $\omega = \{0, 1, 2, \dots\}$ of all natural numbers. Variables of the second sort are known as *set variables*, are denoted by $X, Y, Z, \dots$, and are intended to range over all subsets of ω.

The terms and formulas of the language of second order arithmetic are as follows. *Numerical terms* are number variables, the constant symbols 0 and 1, and $t_1 + t_2$ and $t_1 \cdot t_2$ whenever t_1 and t_2 are numerical terms. Here $+$ and $\cdot$ are binary operation symbols intended to denote addition and multiplication of natural numbers. (Numerical terms are intended to denote natural numbers.) *Atomic formulas* are $t_1 = t_2$, $t_1 < t_2$, and $t_1 \in X$ where t_1 and t_2 are numerical terms and X is any set variable.

(The intended meanings of these respective atomic formulas are that t_1 equals t_2, t_1 is less than t_2, and t_1 is an element of X.) *Formulas* are built up from atomic formulas by means of propositional connectives $\wedge$, $\vee$, $\neg$, $\rightarrow$, $\leftrightarrow$ (and, or, not, implies, if and only if), *number quantifiers* $\forall n$, $\exists n$ (for all n, there exists n), and *set quantifiers* $\forall X$, $\exists X$ (for all X, there exists X). A *sentence* is a formula with no free variables.

DEFINITION I.2.1 (language of second order arithmetic). L_2 is defined to be the language of second order arithmetic as described above.

In writing terms and formulas of L_2, we shall use parentheses and brackets to indicate grouping, as is customary in mathematical logic textbooks. We shall also use some obvious abbreviations. For instance, $2+2=4$ stands for $(1+1)+(1+1)=((1+1)+1)+1$, $(m+n)^2 \notin X$ stands for $\neg((m+n)\cdot(m+n) \in X)$, $s \leq t$ stands for $s < t \vee s = t$, and $\varphi \wedge \psi \wedge \theta$ stands for $(\varphi \wedge \psi) \wedge \theta$.

The semantics of the language L_2 are given by the following definition.

DEFINITION I.2.2 (L_2-structures). A *model for* L_2, also called a *structure for* L_2 or an L_2-*structure*, is an ordered 7-tuple

$$M = (|M|, \mathcal{S}_M, +_M, \cdot_M, 0_M, 1_M, <_M) ,$$

where $|M|$ is a set which serves as the range of the number variables, $\mathcal{S}_M$ is a set of subsets of $|M|$ serving as the range of the set variables, $+_M$ and $\cdot_M$ are binary operations on $|M|$, 0_M and 1_M are distinguished elements of $|M|$, and $<_M$ is a binary relation on $|M|$. We always assume that the sets $|M|$ and $\mathcal{S}_M$ are disjoint and nonempty. Formulas of L_2 are interpreted in M in the obvious way.

In discussing a particular model M as above, it is useful to consider formulas with parameters from $|M| \cup \mathcal{S}_M$. We make the following slightly more general definition.

DEFINITION I.2.3 (parameters). Let $\mathcal{B}$ be any subset of $|M| \cup \mathcal{S}_M$. By a *formula with parameters from* $\mathcal{B}$ we mean a formula of the extended language $\mathrm{L}_2(\mathcal{B})$. Here $\mathrm{L}_2(\mathcal{B})$ consists of L_2 augmented by new constant symbols corresponding to the elements of $\mathcal{B}$. By a *sentence with parameters from* $\mathcal{B}$ we mean a sentence of $\mathrm{L}_2(\mathcal{B})$, *i.e.*, a formula of $\mathrm{L}_2(\mathcal{B})$ which has no free variables.

In the language $\mathrm{L}_2(|M| \cup \mathcal{S}_M)$, constant symbols corresponding to elements of $\mathcal{S}_M$ (respectively $|M|$) are treated syntactically as unquantified set variables (respectively unquantified number variables). Sentences and formulas with parameters from $|M| \cup \mathcal{S}_M$ are interpreted in M in the obvious way. A set $A \subseteq |M|$ is said to be *definable over M allowing parameters from* $\mathcal{B}$ if there exists a formula $\varphi(n)$ with parameters from $\mathcal{B}$ and no free variables other than n such that

$$A = \{a \in |M| : M \models \varphi(a)\} \ .$$

Here $M \models \varphi(a)$ means that M *satisfies* $\varphi(a)$, *i.e.*, $\varphi(a)$ is true in M.

We now discuss some specific L_2-structures. The *intended model* for L_2 is of course the model

$$(\omega, P(\omega), +, \cdot, 0, 1, <)$$

where ω is the set of natural numbers, $P(\omega)$ is the set of all subsets of ω, and $+, \cdot, 0, 1, <$ are as usual. By an *ω-model* we mean an L_2-structure of the form

$$(\omega, \mathcal{S}, +, \cdot, 0, 1, <)$$

where $\emptyset \neq \mathcal{S} \subseteq P(\omega)$. Thus an ω-model differs from the intended model only by having a possibly smaller collection $\mathcal{S}$ of sets to serve as the range of the set variables. We sometimes speak of the *ω-model $\mathcal{S}$* when we really mean the ω-model $(\omega, \mathcal{S}, +, \cdot, 0, 1, <)$. In some parts of this book we shall be concerned with a special class of ω-models known as β-models. This class will be defined in §I.5.

We now present the formal system of second order arithmetic.

Definition I.2.4 (second order arithmetic). The *axioms of second order arithmetic* consist of the universal closures of the following L_2-formulas:

(i) basic axioms:

$$\begin{array}{l} n+1 \neq 0 \\ m+1 = n+1 \rightarrow m = n \\ m+0 = m \\ m+(n+1) = (m+n)+1 \\ m \cdot 0 = 0 \\ m \cdot (n+1) = (m \cdot n)+m \\ \neg\, m < 0 \\ m < n+1 \leftrightarrow (m < n \vee m = n) \end{array}$$

(ii) induction axiom:

$$(0 \in X \wedge \forall n\, (n \in X \rightarrow n+1 \in X)) \rightarrow \forall n\, (n \in X)$$

(iii) comprehension scheme:

$$\exists X\, \forall n\, (n \in X \leftrightarrow \varphi(n))$$

where $\varphi(n)$ is any formula of L_2 in which X does not occur freely.

Intuitively, the given instance of the comprehension scheme says that there exists a set $X = \{n : \varphi(n)\}$ = the set of all n such that $\varphi(n)$ holds. This set is said to be *defined by* the given formula $\varphi(n)$. For example, if $\varphi(n)$

is the formula $\exists m\,(m+m=n)$, then this instance of the comprehension scheme asserts the existence of the set of even numbers.

In the comprehension scheme, $\varphi(n)$ may contain free variables in addition to n. These free variables may be referred to as *parameters* of this instance of the comprehension scheme. Such terminology is in harmony with definition I.2.3 and the discussion following it. For example, taking $\varphi(n)$ to be the formula $n \notin Y$, we have an instance of comprehension,

$$\forall Y\,\exists X\,\forall n\,(n \in X \leftrightarrow n \notin Y)\ ,$$

asserting that for any given set Y there exists a set $X =$ the complement of Y. Here the variable Y plays the role of a parameter.

Note that an L_2-structure M satisfies I.2.4(iii), the comprehension scheme, if and only if $\mathcal{S}_M$ contains all subsets of $|M|$ which are definable over M allowing parameters from $|M| \cup \mathcal{S}_M$. In particular, the comprehension scheme is valid in the intended model. Note also that the basic axioms I.2.4(i) and the induction axiom I.2.4(ii) are valid in any ω-model. In fact, any ω-model satisfies the full *second order induction scheme*, *i.e.*, the universal closure of

$$(\varphi(0) \wedge \forall n\,(\varphi(n) \rightarrow \varphi(n+1))) \rightarrow \forall n\,\varphi(n)\ ,$$

where $\varphi(n)$ is any formula of L_2. In addition, the second order induction scheme is valid in any model of I.2.4(ii) plus I.2.4(iii).

By *second order arithmetic* we mean the formal system in the language L_2 consisting of the axioms of second order arithmetic, together with all formulas of L_2 which are deducible from those axioms by means of the usual logical axioms and rules of inference. The formal system of second order arithmetic is also known as Z_2, for obvious reasons, or Π^1_∞-CA_0, for reasons which will become clear in §I.5.

In general, a *formal system* is defined by specifying a language and some axioms. Any formula of the given language which is logically deducible from the given axioms is said to be a *theorem* of the given formal system. At all times we assume the usual logical rules and axioms, including equality axioms and the law of the excluded middle.

This book will be largely concerned with certain specific subsystems of second order arithmetic and the formalization of ordinary mathematics within those systems. By a *subsystem of* Z_2 we mean of course a formal system in the language L_2 each of whose axioms is a theorem of Z_2. When introducing a new subsystem of Z_2, we shall specify the axioms of the system by writing down some formulas of L_2. The axioms are then taken to be the universal closures of those formulas.

If T is any subsystem of Z_2, a *model of* T is any L_2-structure satisfying the axioms of T. By Gödel's completeness theorem applied to the two-sorted language L_2, we have the following important principle: A given

L_2-sentence σ is a theorem of T if and only if all models of T satisfy σ. An *ω-model of T* is of course any ω-model which satisfies the axioms of T, and similarly a *β-model of T* is any β-model satisfying the axioms of T. Chapters VII, VIII, and IX of this book constitute a thorough study of models of subsystems of Z_2. Chapter VII is concerned with β-models, chapter VIII is concerned with ω-models other than β-models, and chapter IX is concerned with models other than ω-models.

All of the subsystems of Z_2 which we shall consider consist of the basic axioms I.2.4(i), the induction axiom I.2.4(ii), and some set existence axioms. The various subsystems will differ from each other only with respect to their set existence axioms. Recall from §I.1 that our Main Question concerns the role of set existence axioms in ordinary mathematics. Thus, a principal theme of this book will be the formal development of specific portions of ordinary mathematics within specific subsystems of Z_2. We shall see that subsystems of Z_2 provide a setting in which the Main Question can be investigated in a precise and fruitful way. Although Z_2 has infinitely many subsystems, it will turn out that only a handful of them are useful in our study of the Main Question.

Notes for §I.2. The formal system Z_2 of second order arithmetic was introduced in Hilbert/Bernays [115] (in an equivalent form, using a somewhat different language and axioms). The development of a portion of ordinary mathematics within Z_2 is outlined in Supplement IV of Hilbert/Bernays [115]. The present book may be regarded as a continuation of the research begun by Hilbert and Bernays.

I.3. The System ACA_0

The previous section contained generalities about subsystems of Z_2. The purpose of this section is to introduce a particular subsystem of Z_2 which is of central importance, namely ACA_0.

In our designation ACA_0, the acronym ACA stands for arithmetical comprehension axiom. This is because ACA_0 contains axioms asserting the existence of any set which is arithmetically definable from given sets (in a sense to be made precise below). The subscript 0 denotes restricted induction. This means that ACA_0 does not include the full second order induction scheme (as defined in §I.2). We assume only the induction axiom I.2.4(ii).

We now proceed to the definition of ACA_0.

DEFINITION I.3.1 (arithmetical formulas). A formula of L_2, or more generally a formula of $L_2(|M| \cup \mathcal{S}_M)$ where M is any L_2-structure, is said to be *arithmetical* if it contains no set quantifiers, *i.e.*, all of the quantifiers appearing in the formula are number quantifiers.

Note that arithmetical formulas of L_2 may contain free set variables, as well as free and bound number variables and number quantifiers. Arithmetical formulas of $\mathrm{L}_2(|M| \cup \mathcal{S}_M)$ may additionally contain set parameters and number parameters, *i.e.*, constant symbols denoting fixed elements of $\mathcal{S}_M$ and $|M|$ respectively.

Examples of arithmetical formulas of L_2 are

$$\forall n\,(n \in X \to \exists m\,(m + m = n))\ ,$$

asserting that all elements of the set X are even, and

$$\forall m\,\forall k\,(n = m \cdot k \to (m = 1 \vee k = 1)) \wedge n > 1 \wedge n \in X\ ,$$

asserting that n is a prime number and is an element of X. An example of a non-arithmetical formula is

$$\exists Y\,\forall n\,(n \in X \leftrightarrow \exists i\,\exists j\,(i \in Y \wedge j \in Y \wedge i + n = j))$$

asserting that X is the set of differences of elements of some set Y.

Definition I.3.2 (arithmetical comprehension). The *arithmetical comprehension scheme* is the restriction of the comprehension scheme I.2.4(iii) to arithmetical formulas $\varphi(n)$. Thus we have the universal closure of

$$\exists X\,\forall n\,(n \in X \leftrightarrow \varphi(n))$$

whenever $\varphi(n)$ is a formula of L_2 which is arithmetical and in which X does not occur freely. ACA_0 is the subsystem of Z_2 whose axioms are the arithmetical comprehension scheme, the induction axiom I.2.4(ii), and the basic axioms I.2.4(i).

Note that an L_2-structure

$$M \;=\; (|M|, \mathcal{S}_M, +_M, \cdot_M, 0_M, 1_M, <_M)$$

satisfies the arithmetical comprehension scheme if and only if $\mathcal{S}_M$ contains all subsets of $|M|$ which are definable over M by arithmetical formulas with parameters from $|M| \cup \mathcal{S}_M$. Thus, a model of ACA_0 is any such L_2-structure which in addition satisfies the induction axiom and the basic axioms.

An easy consequence of the arithmetical comprehension scheme and the induction axiom is the *arithmetical induction scheme*:

$$(\varphi(0) \wedge \forall n\,(\varphi(n) \to \varphi(n+1))) \to \forall n\,\varphi(n)$$

for all L_2-formulas $\varphi(n)$ which are arithmetical. Thus any model of ACA_0 is also a model of the arithmetical induction scheme. (Note however that ACA_0 does not include the second order induction scheme, as defined in §I.2.)

REMARK I.3.3 (first order arithmetic). We wish to remark that there is a close relationship between ACA_0 and first order arithmetic. Let L_1 be the *language of first order arithmetic*, *i.e.*, L_1 is just L_2 with the set variables omitted. *First order arithmetic* is the formal system Z_1 whose language is L_1 and whose axioms are the basic axioms I.2.4(i) plus the *first order induction scheme*:

$$(\varphi(0) \wedge \forall n\,(\varphi(n) \rightarrow \varphi(n+1))) \rightarrow \forall n\,\varphi(n)$$

for all L_1-formulas $\varphi(n)$. In the literature of mathematical logic, first order arithmetic is sometimes known as *Peano arithmetic*, PA. By the previous paragraph, every theorem of Z_1 is a theorem of ACA_0. In model-theoretic terms, this means that for any model $(|M|, \mathcal{S}_M, +_M, \cdot_M, 0_M, 1_M, <_M)$ of ACA_0, its first order part $(|M|, +_M, \cdot_M, 0_M, 1_M, <_M)$ is a model of Z_1. In §IX.1 we shall prove a converse to this result: Given a model

$$(|M|, +_M, \cdot_M, 0_M, 1_M, <_M) \tag{1}$$

of first order arithmetic, we can find $\mathcal{S}_M \subseteq P(|M|)$ such that

$$(|M|, \mathcal{S}_M, +_M, \cdot_M, 0_M, 1_M, <_M)$$

is a model of ACA_0. (Namely, we can take $\mathcal{S}_M = \mathrm{Def}(M) =$ the set of all $A \subseteq |M|$ such that A is definable over (1) allowing parameters from $|M|$.) It follows that, for any L_1-sentence σ, σ is a theorem of ACA_0 if and only if σ is a theorem of Z_1. In other words, ACA_0 is a *conservative extension* of first order arithmetic. This may also be expressed by saying that Z_1, or equivalently PA, is the *first order part* of ACA_0. For details, see §IX.1.

REMARK I.3.4 (ω-models of ACA_0). Assuming familiarity with some basic concepts of recursive function theory, we can characterize the ω-models of ACA_0 as follows. $\mathcal{S} \subseteq P(\omega)$ is an ω-model of ACA_0 if and only if

(i) $\mathcal{S} \neq \emptyset$;
(ii) $A \in \mathcal{S}$ and $B \in \mathcal{S}$ imply $A \oplus B \in \mathcal{S}$;
(iii) $A \in \mathcal{S}$ and $B \leq_{\mathrm{T}} A$ imply $B \in \mathcal{S}$;
(iv) $A \in \mathcal{S}$ implies $\mathrm{TJ}(A) \in \mathcal{S}$.

(This result is proved in §VIII.1.)

Here $A \oplus B$ is the *recursive join* of A and B, defined by

$$A \oplus B \;=\; \{2n : n \in A\} \cup \{2n+1 : n \in B\}\ .$$

$B \leq_{\mathrm{T}} A$ means that B is *Turing reducible* to A, *i.e.*, B is *recursive in* A, *i.e.*, the characteristic function of B is computable assuming an oracle for the characteristic function of A. $\mathrm{TJ}(A)$ denotes the *Turing jump* of A, *i.e.*, the complete recursively enumerable set relative to A.

In particular, ACA_0 has a minimum (*i.e.*, unique smallest) ω-model, namely

$$\mathrm{ARITH} = \{A \in P(\omega) : \exists n \in \omega\,(A \leq_{\mathrm{T}} \mathrm{TJ}(n, \emptyset))\}\ ,$$

where $\mathrm{TJ}(n, X)$ is defined inductively by $\mathrm{TJ}(0, X) = X$, $\mathrm{TJ}(n+1, X) = \mathrm{TJ}(\mathrm{TJ}(n, X))$. More generally, given a set $B \in P(\omega)$, there is a unique smallest ω-model of ACA_0 containing B, consisting of all sets which are arithmetical in B. (For $A, B \in P(\omega)$, we say that A is *arithmetical in* B if $A \leq_{\mathrm{T}} \mathrm{TJ}(n, B)$ for some $n \in \omega$. This is equivalent to saying that A is definable in some or any ω-model $(\omega, \mathcal{S}, +, \cdot, 0, 1, <)$, $B \in \mathcal{S} \subseteq P(\omega)$, by an arithmetical formula with B as a parameter.)

Models of ACA_0 are discussed further in §§VIII.1, IX.1, and IX.4. The development of ordinary mathematics within ACA_0 is discussed in §I.4 and in chapters II, III, and IV.

Notes for §I.3. By remark I.3.3, the system ACA_0 is closely related to first order arithmetic. First order arithmetic is one of the best known and most studied formal systems in the literature of mathematical logic. See for instance Hilbert/Bernays [115], Mendelson [185, chapter 3], Takeuti [261, chapter 2], Shoenfield [222, chapter 8], Hájek/Pudlák [100], and Kaye [137]. By remark I.3.4, ω-models of ACA_0 are closely related to basic concepts of recursion theory such as relative recursiveness, the Turing jump operator, and the arithmetical hierarchy. For an introduction to these concepts, see for instance Rogers [208, chapters 13–15], Shoenfield [222, chapter 7], Cutland [43], or Lerman [161, chapters I–III].

I.4. Mathematics Within ACA_0

The formal system ACA_0 was introduced in the previous section. We now outline the development of certain portions of ordinary mathematics within ACA_0. The material presented in this section will be restated and greatly refined and extended in chapters II, III, and IV. The present discussion is intended as a partial preview of those chapters.

If X and Y are set variables, we use $X = Y$ and $X \subseteq Y$ as abbreviations for the formulas $\forall n\,(n \in X \leftrightarrow n \in Y)$ and $\forall n\,(n \in X \rightarrow n \in Y)$ respectively.

Within ACA_0, we define $\mathbb{N}$ to be the unique set X such that $\forall n\,(n \in X)$. (The existence of this set follows from arithmetical comprehension applied to the formula $\varphi(n) \equiv n = n$.) Thus, in any model

$$M = (|M|, \mathcal{S}_M, +_M, \cdot_M, 0_M, 1_M, <_M)$$

of ACA_0, $\mathbb{N}$ denotes $|M|$, the set of natural numbers in the sense of M, and we have $|M| \in \mathcal{S}_M$. We shall distinguish between $\mathbb{N}$ and ω, reserving ω to denote the set of natural numbers in the sense of "the real world," *i.e.*, the metatheory in which we are working, whatever that metatheory might be.

Within ACA_0, we define a *numerical pairing function* by

$$(m, n) = (m + n)^2 + m .$$

Within ACA_0 we can prove that, for all $m, n, i, j \in \mathbb{N}$, $(m, n) = (i, j)$ if and only if $m = i$ and $n = j$. Moreover, using arithmetical comprehension, we can prove that for all sets $X, Y \subseteq \mathbb{N}$, there exists a set $X \times Y \subseteq \mathbb{N}$ consisting of all (m, n) such that $m \in X$ and $n \in Y$. In particular we have $\mathbb{N} \times \mathbb{N} \subseteq \mathbb{N}$.

For $X, Y \subseteq \mathbb{N}$, a *function* $f : X \to Y$ is defined to be a set $f \subseteq X \times Y$ such that for all $m \in X$ there is exactly one $n \in Y$ such that $(m, n) \in f$. For $m \in X$, $f(m)$ is defined to be the unique n such that $(m, n) \in f$. The usual properties of such functions can be proved in ACA_0. In particular, we have *primitive recursion*. This means that, given $f : X \to Y$ and $g : \mathbb{N} \times X \times Y \to Y$, there is a unique $h : \mathbb{N} \times X \to Y$ defined by $h(0, m) = f(m)$, $h(n+1, m) = g(n, m, h(n, m))$ for all $n \in \mathbb{N}$ and $m \in X$. The existence of h is proved by arithmetical comprehension, and the uniqueness of h is proved by arithmetical induction. (For details, see §II.3.) In particular, we have the *exponential function* $\exp(m, n) = m^n$, defined by $m^0 = 1$, $m^{n+1} = m^n \cdot m$ for all $m, n \in \mathbb{N}$. The usual properties of the exponential function can be proved in ACA_0.

In developing ordinary mathematics within ACA_0, our first major task is to set up the *number systems*, *i.e.*, the natural numbers, the integers, the rational number system, and the real number system.

The natural number system is essentially already given to us by the language and axioms of ACA_0. Thus, within ACA_0, a *natural number* is defined to be an element of $\mathbb{N}$, and the *natural number system* is defined to be the structure $\mathbb{N}, +_\mathbb{N}, \cdot_\mathbb{N}, 0_\mathbb{N}, 1_\mathbb{N}, <_\mathbb{N}, =_\mathbb{N}$, where $+_\mathbb{N} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is defined by $m +_\mathbb{N} n = m + n$, *etc.* (Thus for instance $+_\mathbb{N}$ is the set of triples $((m, n), k) \in (\mathbb{N} \times \mathbb{N}) \times \mathbb{N}$ such that $m + n = k$. The existence of this set follows from arithmetical comprehension.) This means that, when we are working within any particular model $M = (|M|, \mathcal{S}_M, +_M, \cdot_M, 0_M, 1_M, <_M)$ of ACA_0, a natural number is any element of $|M|$, and the role of the natural number system is played by $|M|, +_M, \cdot_M, 0_M, 1_M, <_M, =_M$. (Here $=_M$ is the identity relation on $|M|$.)

Basic properties of the natural number system, such as uniqueness of prime power decomposition, can be proved in ACA_0 using arithmetical induction. (Here one can follow the usual development within first order arithmetic, as presented in textbooks of mathematical logic. Alternatively, see chapter II.)

In order to define the set $\mathbb{Z}$ of *integers* within (any model of) ACA_0, we first use arithmetical comprehension to prove the existence of an equivalence relation $\equiv_{\mathbb{Z}} \subseteq (\mathbb{N} \times \mathbb{N}) \times (\mathbb{N} \times \mathbb{N})$ defined by $(m,n) \equiv_{\mathbb{Z}} (i,j)$ if and only if $m + j = n + i$. We then use arithmetical comprehension again, this time with $\equiv_{\mathbb{Z}}$ as a parameter, to prove the existence of the set $\mathbb{Z}$ consisting of all $(m,n) \in \mathbb{N} \times \mathbb{N}$ such that that (m,n) is the minimum element of its equivalence class with respect to $\equiv_{\mathbb{Z}}$. (Here minimality is taken with respect to $<_{\mathbb{N}}$, using the fact that $\mathbb{N} \times \mathbb{N}$ is a subset of $\mathbb{N}$. Thus $\mathbb{Z}$ consists of one element of each $\equiv_{\mathbb{Z}}$-equivalence class.) Define $+_{\mathbb{Z}} : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ by letting $(m,n) +_{\mathbb{Z}} (i,j)$ be the unique element of $\mathbb{Z}$ such that $(m,n) +_{\mathbb{Z}} (i,j) \equiv_{\mathbb{Z}} (m+i, n+j)$. Here again arithmetical comprehension is used to prove the existence of $+_{\mathbb{Z}}$. Similarly, define $-_{\mathbb{Z}} : \mathbb{Z} \to \mathbb{Z}$ by $-_{\mathbb{Z}}(m,n) \equiv_{\mathbb{Z}} (n,m)$, and define $\cdot_{\mathbb{Z}} : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ by $(m,n) \cdot_{\mathbb{Z}} (i,j) \equiv_{\mathbb{Z}} (mi + nj, mj + ni)$. Let $0_{\mathbb{Z}} = (0,0)$ and $1_{\mathbb{Z}} = (1,0)$. Define a relation $<_{\mathbb{Z}} \subseteq \mathbb{Z} \times \mathbb{Z}$ by letting $(m,n) <_{\mathbb{Z}} (i,j)$ if and only if $m+j < n+i$. Finally, let $=_{\mathbb{Z}}$ be the identity relation on $\mathbb{Z}$. This completes our definition of the system of integers within ACA_0. We can prove within ACA_0 that the system $\mathbb{Z}, +_{\mathbb{Z}}, -_{\mathbb{Z}}, \cdot_{\mathbb{Z}}, 0_{\mathbb{Z}}, 1_{\mathbb{Z}}, <_{\mathbb{Z}}, =_{\mathbb{Z}}$ has the usual properties of an ordered integral domain, the Euclidean property, *etc.*

In a similar manner, we can define within ACA_0 the set of *rational numbers*, $\mathbb{Q}$. Let $\mathbb{Z}^+ = \{a \in \mathbb{Z} : 0 <_{\mathbb{Z}} a\}$ be the set of positive integers, and let $\equiv_{\mathbb{Q}}$ be the equivalence relation on $\mathbb{Z} \times \mathbb{Z}^+$ defined by $(a,b) \equiv_{\mathbb{Q}} (c,d)$ if and only if $a \cdot_{\mathbb{Z}} d = b \cdot_{\mathbb{Z}} c$. Then $\mathbb{Q}$ is defined to be the set of all $(a,b) \in \mathbb{Z} \times \mathbb{Z}^+$ such that (a,b) is the $<_{\mathbb{N}}$-minimum element of its $\equiv_{\mathbb{Q}}$-equivalence class. Operations $+_{\mathbb{Q}}, -_{\mathbb{Q}}, \cdot_{\mathbb{Q}}$ on $\mathbb{Q}$ are defined by $(a,b) +_{\mathbb{Q}} (c,d) \equiv_{\mathbb{Q}} (a \cdot_{\mathbb{Z}} d +_{\mathbb{Z}} b \cdot_{\mathbb{Z}} c, b \cdot_{\mathbb{Z}} d)$, $-_{\mathbb{Q}}(a,b) \equiv_{\mathbb{Q}} (-_{\mathbb{Z}} a, b)$, and $(a,b) \cdot_{\mathbb{Q}} (c,d) \equiv_{\mathbb{Q}} (a \cdot_{\mathbb{Z}} c, b \cdot_{\mathbb{Z}} d)$. We let $0_{\mathbb{Q}} \equiv_{\mathbb{Q}} (0_{\mathbb{Z}}, 1_{\mathbb{Z}})$ and $1_{\mathbb{Q}} \equiv_{\mathbb{Q}} (1_{\mathbb{Z}}, 1_{\mathbb{Z}})$, and we define a binary relation $<_{\mathbb{Q}}$ on $\mathbb{Q}$ by letting $(a,b) <_{\mathbb{Q}} (c,d)$ if and only if $a \cdot_{\mathbb{Z}} d <_{\mathbb{Z}} b \cdot_{\mathbb{Z}} c$. Finally $=_{\mathbb{Q}}$ is the identity relation on $\mathbb{Q}$. We can then prove within ACA_0 that the rational number system $\mathbb{Q}$, $+_{\mathbb{Q}}$, $-_{\mathbb{Q}}$, $\cdot_{\mathbb{Q}}$, $0_{\mathbb{Q}}$, $1_{\mathbb{Q}}$, $<_{\mathbb{Q}}$, $=_{\mathbb{Q}}$ has the usual properties of an ordered field, *etc.*

We make the usual identifications whereby $\mathbb{N}$ is regarded as a subset of $\mathbb{Z}$ and $\mathbb{Z}$ is regarded as a subset of $\mathbb{Q}$. (Namely $m \in \mathbb{N}$ is identified with $(m,0) \in \mathbb{Z}$, and $a \in \mathbb{Z}$ is identified with $(a, 1_{\mathbb{Z}}) \in \mathbb{Q}$.) We use $+$ ambiguously to denote $+_{\mathbb{N}}$, $+_{\mathbb{Z}}$, or $+_{\mathbb{Q}}$ and similarly for $-, \cdot, 0, 1, <$. For $q, r \in \mathbb{Q}$ we write $q - r = q + (-r)$, and if $r \neq 0$, $q/r =$ the unique $q' \in \mathbb{Q}$ such that $q = q' \cdot r$. The function $\exp(q,a) = q^a$ for $q \in \mathbb{Q} \setminus \{0\}$ and $a \in \mathbb{Z}$ is obtained by primitive recursion in the obvious way. The *absolute value* function $|| : \mathbb{Q} \to \mathbb{Q}$ is defined by $|q| = q$ if $q \geq 0$, $-q$ otherwise.

REMARK I.4.1. The idea behind our definitions of $\mathbb{Z}$ and $\mathbb{Q}$ within ACA_0 is that $(m,n) \in \mathbb{N} \times \mathbb{N}$ corresponds to the integer $m - n$, while $(a,b) \in \mathbb{Z} \times \mathbb{Z}^+$ corresponds to the rational number a/b. Our treatment of $\mathbb{Z}$ and $\mathbb{Q}$ is similar to the classical Dedekind construction. The major difference is that we define $\mathbb{Z}$ and $\mathbb{Q}$ to be sets of representatives of the equivalence classes of $\equiv_{\mathbb{Z}}$ and $\equiv_{\mathbb{Q}}$ respectively, while Dedekind uses the equivalence classes themselves. Our reason for using representatives is that we are limited to the language of second order arithmetic, while Dedekind was working in a richer set-theoretic context.

A *sequence of rational numbers* is defined to be a function $f : \mathbb{N} \to \mathbb{Q}$. We denote such a sequence as $\langle q_n : n \in \mathbb{N} \rangle$, or simply $\langle q_n \rangle$, where $q_n = f(n)$. Similarly, a *double sequence of rational numbers* is a function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}$, denoted $\langle q_{mn} : m, n \in \mathbb{N} \rangle$ or simply $\langle q_{mn} \rangle$, where $q_{mn} = f(m, n)$.

DEFINITION I.4.2 (real numbers). Within ACA_0, a *real number* is defined to be a Cauchy sequence of rational numbers, *i.e.*, a sequence of rational numbers $x = \langle q_n : n \in \mathbb{N} \rangle$ such that

$$\forall \epsilon \, (\epsilon > 0 \to \exists m \, \forall n \, (m < n \to |q_m - q_n| < \epsilon)) \ .$$

(But see remark I.4.4 below.) Here ϵ ranges over $\mathbb{Q}$. If $x = \langle q_n \rangle$ and $y = \langle q'_n \rangle$ are real numbers, we write $x =_{\mathbb{R}} y$ to mean that $\lim_n |q_n - q'_n| = 0$, *i.e.*,

$$\forall \epsilon \, (\epsilon > 0 \to \exists m \, \forall n \, (m < n \to |q_n - q'_n| < \epsilon)) \ ,$$

and we write $x <_{\mathbb{R}} y$ to mean that

$$\exists \epsilon \, (\epsilon > 0 \land \exists m \, \forall n \, (m < n \to q_n + \epsilon < q'_n)) \ .$$

Also $x +_{\mathbb{R}} y = \langle q_n + q'_n \rangle$, $x \cdot_{\mathbb{R}} y = \langle q_n \cdot q'_n \rangle$, $-_{\mathbb{R}} x = \langle -q_n \rangle$, $0_{\mathbb{R}} = \langle 0 \rangle$, $1_{\mathbb{R}} = \langle 1 \rangle$.

Informally, we use $\mathbb{R}$ to denote the set of all real numbers. Thus $x \in \mathbb{R}$ means that x is a real number. (Formally, we cannot speak of the set $\mathbb{R}$ within the language of second order arithmetic, since it is a set of sets.) We shall usually omit the subscript $\mathbb{R}$ in $+_{\mathbb{R}}, -_{\mathbb{R}}, \cdot_{\mathbb{R}}, 0_{\mathbb{R}}, 1_{\mathbb{R}}, <_{\mathbb{R}}, =_{\mathbb{R}}$. Thus the *real number system* consists of $\mathbb{R}, +, -, \cdot, 0, 1, <, =$. We shall sometimes identify a rational number $q \in \mathbb{Q}$ with the corresponding real number $x_q = \langle q \rangle$.

REMARK I.4.3. Note that we have not attempted to select elements of the $=_{\mathbb{R}}$-equivalence classes. The reason is that there is no convenient way to do so in ACA_0. Instead, we must accustom ourselves to the fact that $=$ on $\mathbb{R}$ (*i.e.*, $=_{\mathbb{R}}$) is an equivalence relation other than the identity relation. This will not cause any serious difficulties.

REMARK I.4.4. The above definition of the real number system is similar but not identical to the one which we shall actually use in our detailed discussion of ordinary mathematics within ACA_0, chapters II through IV. The reason for the discrepancy is that the above definition, while suitable for use in ACA_0 and intuitively appealing, is not suitable for use in weaker systems such as RCA_0. (RCA_0 will be introduced in §§I.7 and I.8 below.)

The definition used for the detailed development is slightly less natural, but it has the advantage of working smoothly in weaker systems. In any case, the two definitions are equivalent over ACA_0, equivalent in the sense that the two versions of the real number system which they define can be proved in ACA_0 to be isomorphic.

Within ACA_0 one can prove that the real number system has the usual properties of an Archimedean ordered field, *etc.* The *complex numbers* can be introduced as usual as pairs of real numbers. Within ACA_0, it is straightforward to carry out the proofs of all the basic results in real and complex linear and polynomial algebra. For example, the fundamental theorem of algebra can be proved in ACA_0.

A *sequence of real numbers* is defined to be a double sequence of rational numbers $\langle q_{mn} : m, n \in \mathbb{N}\rangle$ such that for each m, $\langle q_{mn} : n \in \mathbb{N}\rangle$ is a real number. Such a sequence of real numbers is denoted $\langle x_m : m \in \mathbb{N}\rangle$, where $x_m = \langle q_{mn} : n \in \mathbb{N}\rangle$. Within ACA_0 we can prove that every bounded sequence of real numbers has a least upper bound. This is a very useful completeness property of the real number system. For instance, it implies that an infinite series of positive terms is convergent if and only if the finite partial sums are bounded. (Stronger completeness properties for the most part cannot be proved in ACA_0.)

We now turn to abstract algebra within ACA_0. Because of the restriction to the language of second order arithmetic, we cannot expect to obtain a good general theory of arbitrary (countable and uncountable) algebraic structures. However, we can develop *countable algebra*, *i.e.*, the theory of countable algebraic structures, within ACA_0.

For instance, a *countable commutative ring* is defined within ACA_0 to be a structure $R, +_R, -_R, \cdot_R, 0_R, 1_R$, where $R \subseteq \mathbb{N}$, $+_R : R \times R \to R$, *etc.*, and the usual commutative ring axioms are assumed. (We include $0 \neq 1$ among those axioms.) The subscript R is usually omitted. (An example is the ring of integers, $\mathbb{Z}, +_{\mathbb{Z}}, -_{\mathbb{Z}}, \cdot_{\mathbb{Z}}, 0_{\mathbb{Z}}, 1_{\mathbb{Z}}$, which was introduced above.) An *ideal* in R is a set $I \subseteq R$ such that $a \in I$ and $b \in I$ imply $a + b \in I$, $a \in I$ and $r \in R$ imply $a \cdot r \in I$, and $0 \in I$ and $1 \notin I$. We define an equivalence relation $=_I$ on R by $r =_I s$ if and only if $r - s \in I$. We let R/I be the set of $r \in R$ such that r is the $<_{\mathbb{N}}$-minimum element of its equivalence class under $=_I$. Thus R/I consists of one element of each $=_I$-equivalence class of elements of R. With the appropriate operations, R/I becomes a countable commutative ring, the quotient ring of R by I. The ideal I is said to be *prime* if R/I is an integral domain, and *maximal* if R/I is a field. With these definitions, the countable case of many basic results of commutative algebra can be proved in ACA_0. See §§III.5 and IV.6.

Other countable algebraic structures, *e.g.*, countable groups, can be defined and discussed in a similar manner, within ACA_0. Countable fields are discussed in §§II.9, IV.4 and IV.5, and countable vector spaces are discussed in §III.4. It turns out that part of the theory of countable Abelian groups can be developed in ACA_0, but other parts of the theory require stronger systems. See §§III.6, V.7 and VI.4.

Next we indicate how some basic concepts and results of analysis and topology can be developed within ACA_0.

Definition I.4.5 (complete separable metric spaces). Within ACA_0, a (code for a) *complete separable metric space* is a nonempty set $A \subseteq \mathbb{N}$ together with a function $d : A \times A \to \mathbb{R}$ satisfying $d(a,a) = 0$, $d(a,b) = d(b,a) \geq 0$, and $d(a,c) \leq d(a,b) + d(b,c)$ for all $a,b,c \in A$. (Formally, d is a sequence of real numbers, indexed by $A \times A$.) We define a *point of the complete separable metric space* $\widehat{A}$ to be a sequence $x = \langle a_n : n \in \mathbb{N}\rangle$, $a_n \in A$, satisfying

$$\forall \epsilon\,(\epsilon > 0 \to \exists m\,\forall n\,(m < n \to d(a_m, a_n) < \epsilon))\ .$$

The pseudometric d is extended from A to $\widehat{A}$ by

$$d(x,y) = \lim_n d(a_n, b_n)$$

where $x = \langle a_n : n \in \mathbb{N}\rangle$ and $y = \langle b_n : n \in \mathbb{N}\rangle$. We write $x = y$ if and only if $d(x,y) = 0$.

For example, $\mathbb{R} = \widehat{\mathbb{Q}}$ under the metric $d(q,q') = |q - q'|$.

The idea of the above definition is that a complete separable metric space $\widehat{A}$ is presented by specifying a countable dense set A together with the restriction of the metric to A. Then $\widehat{A}$ is defined as the completion of A under the restricted metric. Just as in the case of the real number system, several difficulties arise from the circumstance that ACA_0 is formalized in the language of second order arithmetic. First, there is no variable or term that can denote the set of all points in $\widehat{A}$ (although we can use notations such as $x \in \widehat{A}$, meaning that x is a point of $\widehat{A}$). Second, equality for points of $\widehat{A}$ is an equivalence relation other than the identity relation. These difficulties are minor and do not seriously affect the content of the mathematical development concerning complete separable metric spaces within ACA_0. They only affect the outward form of that development. A more important limitation is that, in the language of second order arithmetic, we cannot speak at all about nonseparable metric spaces. This remark is related to our remarks in §I.1 about set-theoretic versus "ordinary" or non-set-theoretic mathematics.

Definition I.4.6 (continuous functions). Within ACA_0, if $\widehat{A}$ and $\widehat{B}$ are complete separable metric spaces, a (code for a) *continuous function* $\phi : \widehat{A} \to \widehat{B}$ is a set $\Phi \subseteq A \times \mathbb{Q}^+ \times B \times \mathbb{Q}^+$ satisfying the following coherence conditions:

1. $(a,r,b,s) \in \Phi$ and $(a,r,b',s') \in \Phi$ imply $d(b,b') < s+s'$;
2. $(a,r,b,s) \in \Phi$ and $d(b,b')+s < s'$ imply $(a,r,b',s') \in \Phi$;
3. $(a,r,b,s) \in \Phi$ and $d(a,a')+r' < r$ imply $(a',r',b,s) \in \Phi$.

Here a' ranges over A, b' ranges over B, and r' and s' range over

$$\mathbb{Q}^+ = \{q \in \mathbb{Q} : q > 0\},$$

the positive rational numbers. In addition we require: for all $x \in \widehat{A}$ and $\epsilon > 0$ there exists $(a,r,b,s) \in \Phi$ such that $d(a,x) < r$ and $s < \epsilon$.

We can prove in ACA_0 that for all $x \in \widehat{A}$ there exists $y \in \widehat{B}$ such that $d(b,y) \leq s$ for all $(a,r,b,s) \in \Phi$ such that $d(a,x) < r$. This y is unique up to equality of points in $\widehat{B}$, and we define $\phi(x) = y$. It can be shown that $x = x'$ implies $\phi(x) = \phi(x')$.

The idea of the above definition is that $(a,r,b,s) \in \Phi$ is a *neighborhood condition* giving us a piece of information about the continuous function $\phi : \widehat{A} \to \widehat{B}$. Namely, $(a,r,b,s) \in \Phi$ tells us that for all $x \in \widehat{A}$, $d(x,a) < r$ implies $d(\phi(x),b) \leq s$. The code Φ consists of sufficiently many neighborhood conditions so as to determine $\phi(x) \in \widehat{B}$ for all $x \in \widehat{A}$.

Taking $\widehat{A} = \mathbb{R}^n$ and $\widehat{B} = \mathbb{R}$ in the above definition, we obtain a concept of continuous real-valued function of n real variables. Using this, the theory of differential and integral equations, calculus of variations, *etc.*, can be developed as usual, within ACA_0. For instance, the Ascoli lemma can be proved in ACA_0 and then used to obtain the Peano existence theorem for solutions of ordinary differential equations (see §§III.2 and IV.8).

DEFINITION I.4.7 (open sets). Within ACA_0, let $\widehat{A}$ be a complete separable metric space. A (code for an) *open set* in $\widehat{A}$ is any set $U \subseteq A \times \mathbb{Q}^+$. For $x \in \widehat{A}$ we write $x \in U$ if and only if $d(x,a) < r$ for some $(a,r) \in U$.

The idea of definition I.4.7 is that $(a,r) \in A \times \mathbb{Q}^+$ is a code for a *neighborhood* or *basic open set* $\mathrm{B}(a,r)$ in $\widehat{A}$. Here $x \in \mathrm{B}(a,r)$ if and only if $d(a,x) < r$. An open set U is then defined as a union of basic open sets.

With definitions I.4.6 and I.4.7, the usual proofs of fundamental topological results can be carried out within ACA_0, for the case of complete separable metric spaces. For instance, the Baire category theorem and the Tietze extension theorem go through in this setting (see §§II.5, II.6, and II.7).

A *separable Banach space* is defined within ACA_0 to be a complete separable metric space $\widehat{A}$ arising from a countable pseudonormed vector space A over the rational field $\mathbb{Q}$. For example, let $A = \mathbb{Q}[x]$ be the ring of polynomials in one variable x over $\mathbb{Q}$. With the metric

$$d(f,g) = \left[\int_0^1 |f(x) - g(x)|^p \, dx\right]^{1/p},$$

$1 \leq p < \infty$, we have $\widehat{A} = \mathrm{L}_p[0,1]$. Similarly, with the metric

$$d(f,g) = \sup_{0 \leq x \leq 1} |f(x) - g(x)| ,$$

we have $\widehat{A} = \mathrm{C}[0,1]$. As suggested by these examples, the basic theory of separable Banach and Frechet spaces can be developed formally within ACA_0. In particular, the Hahn/Banach theorem, the open mapping theorem, and the Banach/Steinhaus uniform boundedness principle can be proved in this setting (see §§II.10, IV.9, X.2).

REMARK I.4.8. As in remark I.4.4, the above definitions of complete separable metric space, continuous function, open set, and separable Banach space are not the ones which we shall actually use in our detailed development in chapters II, III, and IV. However, the two sets of definitions are equivalent in ACA_0.

Notes for §I.4. The observation that a great deal of ordinary mathematics can be developed formally within a system something like ACA_0 goes back to Weyl [274]; see also definition X.3.2. See also Takeuti [260] and Zahn [281].

I.5. Π^1_1-CA_0 and Stronger Systems

In this section we introduce Π^1_1-CA_0 and some other subsystems of Z_2. These systems are much stronger than ACA_0.

DEFINITION I.5.1 (Π^1_1 formulas). A formula φ is said to be Π^1_1 if it is of the form $\forall X\, \theta$, where X is a set variable and θ is an arithmetical formula. A formula φ is said to be Σ^1_1 if it is of the form $\exists X\, \theta$, where X is a set variable and θ is an arithmetical formula.

More generally, for $0 \leq k \in \omega$, a formula φ is said to be Π^1_k if it is of the form

$$\forall X_1\, \exists X_2\, \forall X_3 \cdots X_k\ \theta ,$$

where $X_1, \ldots, X_k$ are set variables and θ is an arithmetical formula. A formula φ is said to be Σ^1_k if it is of the form

$$\exists X_1\, \forall X_2\, \exists X_3 \cdots X_k\ \theta ,$$

where $X_1, \ldots, X_k$ are set variables and θ is an arithmetical formula. In both cases, φ consists of k alternating set quantifiers followed by a formula with no set quantifiers. In the Π^1_k case, the first set quantifier is universal, while in the Σ^1_k case it is existential (assuming $k \geq 1$). Thus for instance a Π^1_2 formula is of the form $\forall X\, \exists Y\, \theta$, and a Σ^1_2 formula is of the form

$\exists X\,\forall Y\,\theta$, where θ is arithmetical. A Π^1_0 or Σ^1_0 formula is the same thing as an arithmetical formula.

The equivalences $\neg\,\forall X\,\varphi \equiv \exists X\,\neg\,\varphi$, $\neg\,\exists X\,\varphi \equiv \forall X\,\neg\,\varphi$, and $\neg\,\neg\,\varphi \equiv \varphi$ imply that any Π^1_k formula is logically equivalent to the negation of a Σ^1_k formula, and vice versa. Moreover, using Π^1_k (respectively Σ^1_k) to denote the class of formulas logically equivalent to a Π^1_k formula (respectively a Σ^1_k formula), we have

$$\Pi^1_k \cup \Sigma^1_k \ \subseteq\ \Pi^1_{k+1} \cap \Sigma^1_{k+1}$$

for all $k \in \omega$. (This is proved by introducing dummy quantifiers.)

The hierarchy of L_2-formulas Π^1_k, $k \in \omega$, is closely related to the projective hierarchy in descriptive set theory.

DEFINITION I.5.2 (Π^1_1 and Π^1_k comprehension). Π^1_1-CA_0 is the subsystem of Z_2 whose axioms are the basic axioms I.2.4(i), the induction axiom I.2.4(ii), and the comprehension scheme I.2.4(iii) restricted to L_2-formulas $\varphi(n)$ which are Π^1_1. Thus we have the universal closure of

$$\exists X\,\forall n\,(n \in X \leftrightarrow \varphi(n))$$

for all Π^1_1 formulas $\varphi(n)$ in which X does not occur freely.

The systems Π^1_k-CA_0, $k \in \omega$, are defined similarly, with Π^1_k replacing Π^1_1. In particular Π^1_0-CA_0 is just ACA_0, and for all $k \in \omega$ we have

$$\Pi^1_k\text{-}\mathsf{CA}_0 \ \subseteq\ \Pi^1_{k+1}\text{-}\mathsf{CA}_0 \ .$$

It is also clear that

$$\mathsf{Z}_2 \ = \ \bigcup_{k\in\omega} \Pi^1_k\text{-}\mathsf{CA}_0 \ .$$

For this reason, Z_2 is sometimes denoted Π^1_∞-CA_0.

It would be possible to introduce systems Σ^1_k-CA_0, $k \in \omega$, but they would be superfluous, because a simple argument shows that Σ^1_k-CA_0 and Π^1_k-CA_0 are equivalent, *i.e.*, they have the same theorems.

[Namely, given a Σ^1_k formula $\varphi(n)$, there is a logically equivalent formula $\neg\,\psi(n)$ where $\psi(n)$ is Π^1_k. Reasoning within Π^1_k-CA_0 and applying Π^1_k comprehension, we see that there exists a set Y such that

$$\forall n\,(n \in Y \leftrightarrow \psi(n)) \ .$$

Applying arithmetical comprehension with Y as a parameter, there exists a set X such that

$$\forall n\,(n \in X \leftrightarrow n \notin Y) \ .$$

Then clearly

$$\forall n\,(n \in X \leftrightarrow \varphi(n)) \ .$$

This shows that all the axioms of Σ^1_k-CA_0 are theorems of Π^1_k-CA_0. The converse is proved similarly.]

We now discuss models of Π^1_k-CA_0, $1 \le k \le \infty$.

As explained in §I.3 above, ACA_0 has a minimum ω-model, and this model is very natural from both the recursion-theoretic and the model-theoretic points of view. It is therefore reasonable to ask about minimum ω-models of Π^1_k-CA_0. It turns out that, for $1 \le k \le \infty$, there is no minimum (or even minimal) ω-model of Π^1_k-CA_0. These negative results will be proved in §VIII.6. However, we can obtain a positive result by considering β-models instead of ω-models. The relevant definition is as follows.

DEFINITION I.5.3 (β-models). A *β-model* is an ω-model $\mathcal{S} \subseteq P(\omega)$ with the following property. If σ is any Π^1_1 or Σ^1_1 sentence with parameters from $\mathcal{S}$, then $(\omega, \mathcal{S}, +, \cdot, 0, 1, <)$ satisfies σ if and only if the intended model

$$(\omega, P(\omega), +, \cdot, 0, 1, <)$$

satisfies σ.

If T is any subsystem of Z_2, a *β-model of T* is any β-model satisfying the axioms of T. Chapter VII is a thorough study of β-models of subsystems of Z_2.

REMARK I.5.4 (β-models of Π^1_1-CA_0). For readers who are familiar with some basic concepts of hyperarithmetical theory, the β-models of Π^1_1-CA_0 can be characterized as follows. $\mathcal{S} \subseteq P(\omega)$ is a β-model of Π^1_1-CA_0 if and only if

(i) $\mathcal{S} \neq \emptyset$;
(ii) $A \in \mathcal{S}$ and $B \in \mathcal{S}$ imply $A \oplus B \in \mathcal{S}$;
(iii) $A \in \mathcal{S}$ and $B \le_{\mathrm{H}} A$ imply $B \in \mathcal{S}$;
(iv) $A \in \mathcal{S}$ implies $\mathrm{HJ}(A) \in \mathcal{S}$.

Here $B \le_{\mathrm{H}} A$ means that B is hyperarithmetical in A, and $\mathrm{HJ}(A)$ denotes the hyperjump of A. In particular, there is a minimum (*i.e.*, unique smallest) β-model of Π^1_1-CA_0, namely

$$\{A \in P(\omega) : \exists n \in \omega\, A \le_{\mathrm{H}} \mathrm{HJ}(n, \emptyset)\}$$

where $\mathrm{HJ}(0, X) = X$, $\mathrm{HJ}(n+1, X) = \mathrm{HJ}(\mathrm{HJ}(n, X))$. These results will be proved in §VII.1.

REMARK I.5.5 (minimum β-models of Π^1_k-CA_0). More generally, for each k in the range $1 \le k \le \infty$, it can be shown that there exists a minimum β-model of Π^1_k-CA_0. These models can be described in terms of Gödel's theory of constructible sets. For any ordinal number α, let L_α be the αth level of the constructible hierarchy. Then the minimum β-model of Π^1_k-CA_0 is of the form $\mathrm{L}_\alpha \cap P(\omega)$, where $\alpha = \alpha_k$ is a countable ordinal number depending on k. Moreover, $\alpha_1 < \alpha_2 < \cdots < \alpha_\infty$, and the β-models $\mathrm{L}_{\alpha_k} \cap P(\omega)$,

$1 \le k \le \infty$, are all distinct. (These results are proved in §§VII.5 and VII.7.) It follows that, for each k, Π^1_{k+1}-CA_0 is properly stronger than Π^1_k-CA_0.

The development of ordinary mathematics within Π^1_1-CA_0 and stronger systems is discussed in §I.6 and in chapters V and VI. Models of Π^1_1-CA_0 and some stronger systems, including but not limited to Π^1_k-CA_0 for $k \ge 2$, are discussed in §§VII.1, VII.5, VII.6, VII.7, VIII.6, and IX.4. Our treatment of constructible sets is in §VII.4. Our treatment of hyperarithmetical theory is in §VIII.3.

Notes for §I.5. For an exposition of Gödel's theory of constructible sets, see any good textbook of axiomatic set theory, *e.g.* Jech [130].

I.6. Mathematics Within Π^1_1-CA_0

The system Π^1_1-CA_0 was introduced in the previous section. We now discuss the development of ordinary mathematics within Π^1_1-CA_0. The material presented here will be restated and greatly refined and expanded in chapters V and VI.

We have seen in §I.4 that a large part of ordinary mathematics can already be developed in ACA_0, a subsystem of Z_2 which is much weaker than Π^1_1-CA_0. However, there are certain exceptional theorems of ordinary mathematics which can be proved in Π^1_1-CA_0 but cannot be proved in ACA_0. The exceptional theorems come from several branches of mathematics including countable algebra, the topology of the real line, countable combinatorics, and classical descriptive set theory.

What many of these exceptional theorems have in common is that they directly or indirectly involve countable ordinal numbers. The relevant definition is as follows.

DEFINITION I.6.1 (countable ordinal numbers). Within ACA_0 we define a *countable linear ordering* to be a structure $A, <_A$, where $A \subseteq \mathbb{N}$ and $<_A \subseteq A \times A$ is an irreflexive linear ordering of A, *i.e.*, $<_A$ is transitive and, for all $a, b \in A$, exactly one of $a = b$ or $a <_A b$ or $b <_A a$ holds. The countable linear ordering $A, <_A$ is called a *countable well ordering* if there is no sequence $\langle a_n : n \in \mathbb{N}\rangle$ of elements of A such that $a_{n+1} <_A a_n$ for all $n \in \mathbb{N}$. We view a countable well ordering $A, <_A$ as a code for a countable ordinal number, α, which is intuitively just the order type of $A, <_A$. Two countable well orderings $A, <_A$ and $B, <_B$ are said to encode the same countable ordinal number if and only if they are isomorphic. Two countable well orderings $A, <_A$ and $B, <_B$ are said to be *comparable* if they are isomorphic or if one of them is isomorphic to a proper initial segment of the other. (Letting α and β be the corresponding countable ordinal numbers, this means that either $\alpha = \beta$ or $\alpha < \beta$ or $\beta < \alpha$.)

REMARK I.6.2. The fact that any two countable well orderings are comparable turns out to be provable in Π^1_1-CA_0 but not in ACA_0 (see theorem I.11.5.1 and §V.6). Thus Π^1_1-CA_0, but not ACA_0, is strong enough to develop a good theory of countable ordinal numbers. Because of this, Π^1_1-CA_0 is strong enough to prove several important theorems of ordinary mathematics which are not provable in ACA_0. We now present several examples of this phenomenon.

EXAMPLE I.6.3 (Ulm's theorem). Consider the well known structure theory for countable Abelian groups. Let $G, +_G, -_G, 0_G$ be a countable Abelian group. We say that G is *divisible* if for all $a \in G$ and $n > 0$ there exists $b \in G$ such that $nb = a$. We say that G is *reduced* if G has no nontrivial divisible subgroup. Within Π^1_1-CA_0, but not within ACA_0, one can prove that every countable Abelian group is the direct sum of a divisible group and a reduced group. Now assume that G is a countable Abelian p-group. (This means that for every $a \in G$ there exists $n \in \mathbb{N}$ such that $p^n a = 0$. Here p is a fixed prime number.) One defines a transfinite sequence of subgroups $G_0 = G$, $G_{\alpha+1} = pG_\alpha$, and for limit ordinals δ, $G_\delta = \bigcap_{\alpha<\delta} G_\alpha$. Thus G is reduced if and only if $G_\infty = 0$. The *Ulm invariants* of G are the numbers $\dim(P_\alpha/P_{\alpha+1})$, where $P_\alpha = \{a \in G_\alpha : pa = 0\}$ and the dimension is taken over the integers modulo p. Each Ulm invariant is either a natural number or ∞. *Ulm's theorem* states that two countable reduced Abelian p-groups are isomorphic if and only if their Ulm invariants are the same. Using the theory of countable ordinal numbers which is available in Π^1_1-CA_0, one can carry out the construction of the Ulm invariants and the usual proof of Ulm's theorem within Π^1_1-CA_0. Thus Ulm's theorem is a result of classical algebra which can be proved in Π^1_1-CA_0 but not in ACA_0. More on this topic is in §§V.7 and VI.4.

EXAMPLE I.6.4 (the Cantor/Bendixson theorem). Next we consider a theorem concerning closed sets in n-dimensional Euclidean space. A *closed set* in $\mathbb{R}^n$ is defined to be the complement of an open set. (Open sets were discussed in definition I.4.7.)

If C is a closed set in $\mathbb{R}^n$, an *isolated point of* C is a point $x \in C$ such that $\{x\} = C \cap U$ for some open set U. Clearly C has at most countably many isolated points. We say that C is *perfect* if C has no isolated points. For any closed set C, the *derived set* of C is a closed set C' consisting of all points of C which are not isolated. Thus $C \setminus C'$ is countable, and $C' = C$ if and only if C is perfect. Given a closed set C, the derived sequence of C is a transfinite sequence of closed subsets of C, defined by $C_0 = C$, $C_{\alpha+1} =$ the derived set of C_α, and for limit ordinals δ, $C_\delta = \bigcap_{\alpha<\delta} C_\alpha$. Within Π^1_1-CA_0 we can prove that for all countable ordinal numbers α, the closed set C_α exists. Furthermore $C_{\beta+1} = C_\beta$ for some countable ordinal number β. In this case we clearly have $C_\beta = C_\alpha$ for all $\alpha > \beta$, so we write $C_\beta = C_\infty$.

Clearly C_∞ is a perfect closed set. In fact, C_∞ can be characterized as the largest perfect closed subset of C, and C_∞ is therefore known as the *perfect kernel* of C.

In summary, for any closed set C we have $C = K \cup S$ where K is a perfect closed set (namely $K = C_\infty$) and S is a countable set (namely $S =$ the union of the sets $C_\alpha \setminus C_{\alpha+1}$ for all countable ordinal numbers α). If K happens to be the empty set, then C is itself countable.

The fact that every closed set in $\mathbb{R}^n$ is the union of a perfect closed set and a countable set is known as the *Cantor/Bendixson theorem*. It can be shown that the Cantor/Bendixson theorem is provable in Π^1_1-CA_0 but not in weaker systems such as ACA_0. This example is particularly striking because, although the proof of the Cantor/Bendixson theorem uses countable ordinal numbers, the statement of the theorem does not mention them. For details see §§VI.1 and V.4.

The Cantor/Bendixson theorem also applies more generally, to complete separable metric spaces other than $\mathbb{R}^n$. An important special case is the Baire space $\mathbb{N}^\mathbb{N}$. Note that points of $\mathbb{N}^\mathbb{N}$ may be identified with functions $f : \mathbb{N} \to \mathbb{N}$. The Cantor/Bendixson theorem for $\mathbb{N}^\mathbb{N}$ is closely related to the analysis of trees:

Definition I.6.5 (trees). Within ACA_0 we let

$$\mathrm{Seq} \quad = \quad \mathbb{N}^{<\mathbb{N}} \quad = \quad \bigcup_{k \in \mathbb{N}} \mathbb{N}^k$$

denote the set of (codes for) finite sequences of natural numbers. For $\sigma, \tau \in \mathbb{N}^{<\mathbb{N}}$ there is $\sigma^\frown\tau \in \mathbb{N}^{<\mathbb{N}}$ which is the *concatenation*, σ followed by τ. A *tree* is a set $T \subseteq \mathbb{N}^{<\mathbb{N}}$ such that any initial segment of a sequence in T belongs to T. A *path* or *infinite path* through T is a function $f : \mathbb{N} \to \mathbb{N}$ such that for all $k \in \mathbb{N}$, the initial sequence

$$f[k] \,=\, \langle f(0), f(1), \dots, f(k-1) \rangle$$

belongs to T. The set of paths through T is denoted $[T]$. Thus T may be viewed as a code for the closed set $[T] \subseteq \mathbb{N}^\mathbb{N}$. If T has no infinite path, we say that T is *well founded*. An *end node* of T is a sequence $\tau \in T$ which has no proper extension in T.

Definition I.6.6 (perfect trees). Two sequences in $\mathbb{N}^{<\mathbb{N}}$ are said to be *compatible* if they are equal or one is an initial segment of the other. Given a tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ and a sequence $\sigma \in T$, we denote by T_σ the set of $\tau \in T$ such that σ is compatible with τ. Given a tree T, there is a *derived tree* $T' \subseteq T$ consisting of all $\sigma \in T$ such that T_σ contains a pair of incompatible sequences. We say that T is *perfect* if $T' = T$, *i.e.*, every $\sigma \in T$ has a pair of incompatible extensions $\tau_1, \tau_2 \in T$.

Given a tree T, we may consider a transfinite sequence of trees defined by $T_0 = T$, $T_{\alpha+1}$ = the derived tree of T_α, and for limit ordinals δ, $T_\delta = \bigcap_{\alpha<\delta} T_\alpha$. We write $T_\infty = T_\beta$ where β is an ordinal such that $T_\beta = T_{\beta+1}$. Thus T_∞ is the largest perfect subtree of T. These notions concerning trees are analogous to example I.6.4 concerning closed sets. Indeed, the closed set $[T_\infty]$ is the perfect kernel of the closed set $[T]$ in the Baire space $\mathbb{N}^\mathbb{N}$. As in example I.6.4, it turns out that the existence of T_∞ is provable in Π^1_1-CA_0 but not in weaker systems such as ACA_0. This result will be proved in §VI.1.

Turning to another topic in mathematics, we point out that Π^1_1-CA_0 is strong enough to prove many of the basic results of classical descriptive set theory. By *classical descriptive set theory* we mean the study of Borel and analytic sets in complete separable metric spaces. The relevant definitions within ACA_0 are as follows.

DEFINITION I.6.7 (Borel sets). Let $\widehat{A}$ be a complete separable metric space. A (code for a) *Borel set* B in $\widehat{A}$ is defined to be a set $B \subseteq \mathbb{N}^{<\mathbb{N}}$ such that

(i) B is a well founded tree;
(ii) for any end node $\langle m_0, m_1, \ldots, m_k\rangle$ of B, we have $m_k = (a, r)$ for some $(a, r) \in A \times \mathbb{Q}^+$;
(iii) B contains exactly one sequence $\langle m_0\rangle$ of length 1.

In particular, for each $a \in A$ and $r \in \mathbb{Q}^+$ there is a Borel code $\langle (a, r)\rangle$. We take $\langle (a, r)\rangle$ to be a code for the basic open neighborhood $\mathrm{B}(a, r)$ as in definition I.4.7. Thus for all points $x \in \widehat{A}$ we have, by definition, $x \in \mathrm{B}(a, r)$ if and only if $d(a, x) < r$. If B is a Borel code which is not of the form $\langle (a, r)\rangle$, then for each $\langle m_0, n\rangle \in B$ we have another Borel code

$$B_n = \{\langle\rangle\} \cup \{\langle n\rangle^\frown\tau : \langle m_0, n\rangle^\frown\tau \in B\} \,.$$

We use transfinite recursion to define the notion of a point $x \in \widehat{A}$ belonging to (the Borel set coded by) B, in such a way that $x \in B$ if and only if either m_0 is odd and $x \in B_n$ for some n, or m_0 is even and $x \notin B_n$ for some n. This recursion can be carried out in Π^1_1-CA_0; see §V.3.

Thus the Borel sets form a σ-algebra containing the basic open sets and closed under countable union, countable intersection, and complementation.

DEFINITION I.6.8 (analytic sets). Let $\widehat{A}$ be a complete separable metric space. A (code for an) *analytic set* $S \subseteq \widehat{A}$ is defined to be a (code for a) continuous function $\phi : \mathbb{N}^\mathbb{N} \to \widehat{A}$. We put $x \in S$ if and only if

$$\exists f\, (f \in \mathbb{N}^\mathbb{N} \wedge \phi(f) = x)\,.$$

It can be proved in ACA_0 that a set is analytic if and only if it is defined by a Σ^1_1 formula with parameters.

Example I.6.9 (classical descriptive set theory). Within Π^1_1-CA_0 we can emulate the standard proofs of some well known classical results on Borel and analytic sets. This is possible because Π^1_1-CA_0 includes a good theory of countable well orderings and countable well founded trees. In particular Souslin's theorem ("a set S is Borel if and only if S and its complement are analytic"), Lusin's theorem ("any two disjoint analytic sets can be separated by a Borel set"), and Kondo's theorem (coanalytic uniformization) are provable in Π^1_1-CA_0 but not in ACA_0. For details, see §§V.3 and VI.2.

With the above examples, Π^1_1-CA_0 emerges as being of considerable interest with respect to the development of ordinary mathematics. Other examples of ordinary mathematical theorems which are provable in Π^1_1-CA_0 are: determinacy of open sets in $\mathbb{N}^\mathbb{N}$ (see §V.8), and the Ramsey property for open sets in $[\mathbb{N}]^\mathbb{N}$ (see §V.9). These theorems, like Ulm's theorem and the Cantor/Bendixson theorem, are exceptional in that they are not provable in ACA_0.

Remark I.6.10 (Friedman-style independence results). There are a small number of even more exceptional theorems which, for instance, are provable in ZFC (*i.e.*, Zermelo/Fraenkel set theory with the axiom of choice) but not in full Z_2. As an example, consider the following corollary, due to Friedman [71], of a theorem of Martin [177, 178]: Given a symmetric Borel set $B \subseteq I \times I$, $I = [0,1]$, there exists a Borel function $\phi : I \to I$ such that the graph of ϕ is either included in or disjoint from B. Friedman [71] has shown that this result is not provable in Z_2 or even in simple type theory. This is related to Friedman's earlier result [66, 71] that Borel determinacy is not provable in simple type theory. More results of this kind are in [72] and in the Friedman volume [102].

Notes for §I.6. Chapters V and VI of this book deal with the development of mathematics in Π^1_1-CA_0. The crucial role of comparablility of countable well orderings (remark I.6.2) was pointed out by Friedman [62, chapter II] and Steel [256, chapter I]; recent refinements are due to Friedman/Hirst [74] and Shore [223]. The impredicative nature of the Cantor/Bendixson theorem and Ulm's theorem was noted by Kreisel [149] and Feferman [58], respectively. An up-to-date textbook of classical descriptive set theory is Kechris [138]. Friedman has discovered a number of mathematically natural statements whose proofs require strong set existence axioms; see the Friedman volume [102] and recent papers such as [73].

I.7. The System RCA_0

In this section we introduce RCA_0, an important subsystem of Z_2 which is much weaker than ACA_0.

The acronym RCA stands for recursive comprehension axiom. This is because RCA_0 contains axioms asserting the existence of any set A which is recursive in given sets $B_1, \ldots, B_k$ (*i.e.*, such that the characteristic function of A is computable assuming oracles for the characteristic functions of $B_1, \ldots, B_k$). As in ACA_0 and Π^1_1-CA_0, the subscript 0 in RCA_0 denotes restricted induction. The axioms of RCA_0 include Σ^0_1 induction, a form of induction which is weaker than arithmetical induction (as defined in §I.3) but stronger than the induction axiom I.2.4(ii).

We now proceed to the definition of RCA_0.

Let n be a number variable, let t be a numerical term not containing n, and let φ be a formula of L_2. We use the following abbreviations:

$$\forall n < t\ \varphi \quad \equiv \quad \forall n\,(n < t \rightarrow \varphi)\ ,$$

$$\exists n < t\ \varphi \quad \equiv \quad \exists n\,(n < t \wedge \varphi)\ .$$

Thus $\forall n < t$ means "for all n less than t", and $\exists n < t$ means "there exists n less than t such that". We may also write $\forall n \leq t$ instead of $\forall n < t + 1$, and $\exists n \leq t$ instead of $\exists n < t + 1$.

The expressions $\forall n < t$, $\forall n \leq t$, $\exists n < t$, $\exists n \leq t$ are called *bounded number quantifiers*, or simply *bounded quantifiers*. A *bounded quantifier formula* is a formula φ such that all of the quantifiers occurring in φ are bounded number quantifiers. Thus the bounded quantifier formulas are a subclass of the arithmetical formulas. Examples of bounded quantifier formulas are

$$\exists m \leq n\ (n = m + m)\ ,$$

asserting that n is even, and

$$\forall m < 2n\ (m \in X \leftrightarrow \exists k < m\,(m = 2k + 1))\ ,$$

asserting that the first n elements of X are 1, 3, 5, ... , $2n - 1$.

DEFINITION I.7.1 (Σ^0_1 and Π^0_1 formulas). An L_2-formula φ is said to be Σ^0_1 if it is of the form $\exists m\,\theta$, where m is a number variable and θ is a bounded quantifier formula. An L_2-formula φ is said to be Π^0_1 if it is of the form $\forall m\,\theta$, where m is a number variable and θ is a bounded quantifier formula.

It can be shown that Σ^0_1 formulas are closely related to the notion of relative recursive enumerability in recursion theory. Namely, for $A, B \in P(\omega)$, A is recursively enumerable in B if and only if A is definable over some or any ω-model $(\omega, \mathcal{S}, +, \cdot, 0, 1, <)$, $B \in \mathcal{S} \subseteq P(\omega)$, by a Σ^0_1 formula with B as a parameter. (See also remarks I.3.4 and I.7.5.)

DEFINITION I.7.2 (Σ^0_1 induction). The Σ^0_1 *induction scheme*, Σ^0_1-IND, is the restriction of the second order induction scheme (as defined in §I.2) to L_2-formulas $\varphi(n)$ which are Σ^0_1. Thus we have the universal closure of

$$(\varphi(0) \wedge \forall n\,(\varphi(n) \rightarrow \varphi(n+1))) \rightarrow \forall n\,\varphi(n)$$

where $\varphi(n)$ is any Σ^0_1 formula of L_2.

The Π^0_1 *induction scheme*, Π^0_1-IND, is defined similarly. It can be shown that Σ^0_1-IND and Π^0_1-IND are equivalent (in the presence of the basic axioms I.2.4(i)). This easy but useful result is proved in §II.3.

DEFINITION I.7.3 (Δ^0_1 comprehension). The Δ^0_1 *comprehension scheme* consists of (the universal closures of) all formulas of the form

$$\forall n\,(\varphi(n) \leftrightarrow \psi(n)) \rightarrow \exists X\,\forall n\,(n \in X \leftrightarrow \varphi(n))\ ,$$

where $\varphi(n)$ is any Σ^0_1 formula, $\psi(n)$ is any Π^0_1 formula, n is any number variable, and X is a set variable which does not occur freely in $\varphi(n)$.

In the Δ^0_1 comprehension scheme, note that $\varphi(n)$ and $\psi(n)$ may contain parameters, *i.e.*, free set variables and free number variables in addition to n. Thus an L_2-structure M satisfies Δ^0_1 comprehension if and only if $\mathcal{S}_M$ contains all subsets of $|M|$ which are both Σ^0_1 and Π^0_1 definable over M allowing parameters from $|M| \cup \mathcal{S}_M$.

DEFINITION I.7.4 (definition of RCA_0). RCA_0 is the subsystem of Z_2 consisting of the basic axioms I.2.4(i), the Σ^0_1 induction scheme I.7.2, and the Δ^0_1 comprehension scheme I.7.3.

REMARK I.7.5 (ω-models of RCA_0). In remark I.3.4, we characterized the ω-models of ACA_0 in terms of recursion theory. We can characterize the ω-models of RCA_0 in similar terms, as follows. $\mathcal{S} \subseteq P(\omega)$ is an ω-model of RCA_0 if and only if

(i) $\mathcal{S} \neq \emptyset$;
(ii) $A \in \mathcal{S}$ and $B \in \mathcal{S}$ imply $A \oplus B \in \mathcal{S}$;
(iii) $A \in \mathcal{S}$ and $B \leq_{\mathrm{T}} A$ imply $B \in \mathcal{S}$.

(This result is proved in §VIII.1.) In particular, RCA_0 has a minimum (*i.e.*, unique smallest) ω-model, namely

$$\mathrm{REC}\ =\ \{A \in P(\omega) : A \text{ is recursive}\}\ .$$

More generally, given a set $B \in P(\omega)$, there is a unique smallest ω-model of RCA_0 containing B, consisting of all sets $A \in P(\omega)$ which are recursive in B.

The system RCA_0 plays two key roles in this book and in foundational studies generally. First, as we shall see in chapter II, the development of ordinary mathematics within RCA_0 corresponds roughly to the positive content of what is known as "computable mathematics" or "recursive analysis". Thus RCA_0 is a kind of formalized recursive mathematics. Second, RCA_0 frequently plays the role of a weak base theory in Reverse Mathematics. Most of the results of Reverse Mathematics in chapters III, IV, V, and VI will be stated formally as theorems of RCA_0.

REMARK I.7.6 (first order part of RCA_0). By remark I.3.3, the first order part of ACA_0 is first order arithmetic, PA. In a similar vein, we can characterize the first order part of RCA_0. Namely, let Σ^0_1-PA be PA with induction restricted to Σ^0_1 formulas. (Thus Σ^0_1-PA is a formal system whose language is L_1 and whose axioms are the basic axioms I.2.4(i) plus the universal closure of

$$(\varphi(0) \wedge \forall n\,(\varphi(n) \rightarrow \varphi(n+1))) \rightarrow \forall n\,\varphi(n)$$

for any formula $\varphi(n)$ of L_1 which is Σ^0_1.) Clearly the axioms of Σ^0_1-PA are included in those of RCA_0. Conversely, given any model

$$(|M|, +_M, \cdot_M, 0_M, 1_M, <_M) \tag{2}$$

of Σ^0_1-PA, it can be shown that there exists $\mathcal{S}_M \subseteq P(|M|)$ such that

$$(|M|, \mathcal{S}_M, +_M, \cdot_M, 0_M, 1_M, <_M)$$

is a model of RCA_0. (Namely, we can take $\mathcal{S}_M = \Delta^0_1\text{-Def}(M) =$ the set of all $A \subseteq |M|$ such that A is both Σ^0_1 and Π^0_1 definable over (2) allowing parameters from $|M|$.) It follows that, for any sentence σ in the language of first order arithmetic, σ is a theorem of RCA_0 if and only if σ is a theorem of Σ^0_1-PA. In other words, Σ^0_1-PA is the first order part of RCA_0. (These results are proved in §IX.1.)

Models of RCA_0 are discussed further in §§VIII.1, IX.1, IX.2, and IX.3. The development of ordinary mathematics within RCA_0 is outlined in §I.8 and is discussed thoroughly in chapter II.

REMARK I.7.7 (Σ^0_1 comprehension). It would be possible to define a system Σ^0_1-CA_0 consisting of the basic axioms I.2.4(i), the induction axiom I.2.4(ii), and the Σ^0_1 *comprehension scheme*, *i.e.*, the universal closure of

$$\exists X\,\forall n\,(n \in X \leftrightarrow \varphi(n))$$

for all Σ^0_1 formulas $\varphi(n)$ of L_2 in which X does not occur freely. However, the introduction of Σ^0_1-CA_0 as a distinct subsystem of Z_2 is unnecessary, because it turns out that Σ^0_1-CA_0 is equivalent to ACA_0. This easy but important result will be proved in §III.1.

Generalizing the notion of Σ^0_1 and Π^0_1 formulas, we have:

DEFINITION I.7.8 (Σ^0_k and Π^0_k formulas). For $0 \le k \in \omega$, an L_2-formula φ is said to be Σ^0_k (respectively Π^0_k) if it is of the form

$$\exists n_1 \, \forall n_2 \, \exists n_3 \cdots n_k \, \theta$$

(respectively $\forall n_1 \exists n_2 \forall n_3 \cdots n_k \, \theta$), where $n_1, \ldots, n_k$ are number variables and θ is a bounded quantifier formula. In both cases, φ consists of k alternating unbounded number quantifiers followed by a formula containing only bounded number quantifiers. In the Σ^0_k case, the first unbounded number quantifier is existential, while in the Π^0_k case it is universal (assuming $k \ge 1$). Thus for instance a Π^0_2 formula is of the form $\forall m \, \exists n \, \theta$, where θ is a bounded quantifier formula. A Σ^0_0 or Π^0_0 formula is the same thing as a bounded quantifier formula.

Clearly any Σ^0_k formula is logically equivalent to the negation of a Π^0_k formula, and vice versa. Moreover, up to logical equivalence of formulas, we have $\Sigma^0_k \cup \Pi^0_k \subseteq \Sigma^0_{k+1} \cap \Pi^0_{k+1}$, for all $k \in \omega$.

REMARK I.7.9 (induction and comprehension schemes). Generalizing definition I.7.2, we can introduce induction schemes Σ^i_k-$\mathsf{IND}$and Π^i_k-IND, for all $k \in \omega$ and $i \in \{0, 1\}$. Clearly Σ^0_∞-$\mathsf{IND} = \bigcup_{k \in \omega} \Sigma^0_k$-$\mathsf{IND}$ is equivalent to arithmetical induction, and Σ^1_∞-$\mathsf{IND} = \bigcup_{k \in \omega} \Sigma^1_k$-$\mathsf{IND}$ is equivalent to the full second order induction scheme. It can be shown that, for all $k \in \omega$ and $i \in \{0, 1\}$, Σ^i_k-IND is equivalent to Π^i_k-IND and is properly weaker than Σ^i_{k+1}-IND. As for comprehension schemes, it follows from remark I.7.7 that the systems Σ^0_k-CA_0 and Π^0_k-CA_0, $1 \le k \in \omega$, are all equivalent to each other and to ACA_0, *i.e.*, Π^1_0-CA_0. On the other hand, we have remarked in §I.5 that, for each $k \in \omega$, Π^1_k-CA_0 is equivalent to Σ^1_k-CA_0 and is properly weaker than Π^1_{k+1}-CA_0. In chapter VII we shall introduce the systems Δ^1_k-CA_0, $1 \le k \in \omega$, and we shall show that Δ^1_k-CA_0 is properly stronger than Π^1_{k-1}-CA_0 and properly weaker than Π^1_k-CA_0.

Notes for §I.7. In connection with remark I.7.5, note that the literature of recursion theory sometimes uses the term *Turing ideals* referring to what we call ω-models of RCA_0. See for instance Lerman [161, page 29]. The system RCA_0 was first introduced by Friedman [69] (in an equivalent form, using a somewhat different language and axioms). The system Σ^0_1-PA was first studied by Parsons [201]. For a thorough discussion of Σ^0_1-PA and other subsystems of first order arithmetic, see Hájek/Pudlák [100] and Kaye [137].

I.8. Mathematics Within RCA_0

In this section we sketch how some concepts and results of ordinary mathematics can be developed in RCA_0. This portion of ordinary mathematics is roughly parallel to the positive content of recursive analysis and recursive algebra. We shall also give some recursive counterexamples showing that certain other theorems of ordinary mathematics are recursively false and hence, although provable in ACA_0, cannot be proved in RCA_0.

As already remarked in I.4.4 and I.4.8, the strictures of RCA_0 require us to modify our definitions of "real number" and "point of a complete separable metric space". The needed modifications are as follows:

DEFINITION I.8.1 (partially replacing I.4.2). Within RCA_0, a (code for a) *real number* $x \in \mathbb{R}$ is defined to be a sequence of rational numbers $x = \langle q_n : n \in \mathbb{N} \rangle$, $q_n \in \mathbb{Q}$, such that

$$\forall m \, \forall n \, (m < n \rightarrow |q_m - q_n| < 1/2^m) \ .$$

For real numbers x and y we have $x =_{\mathbb{R}} y$ if and only if

$$\forall m \, (|q_m - q'_m| \leq 1/2^{m-1}) \ ,$$

and $x <_{\mathbb{R}} y$ if and only if

$$\exists m \, (q_m + 1/2^m < q'_m) \ .$$

Note that with definition I.8.1 we now have that the predicate $x < y$ is Σ^0_1, and the predicates $x \leq y$ and $x = y$ are Π^0_1, for $x, y \in \mathbb{R}$. Thus real number comparisons have become easier, and therein lies the superiority of I.8.1 over I.4.2 within RCA_0.

DEFINITION I.8.2 (partially replacing I.4.5). Within RCA_0, a (code for a) complete separable metric space is defined as in I.4.5. However, a (code for a) *point of the complete separable metric space* $\widehat{A}$ is now defined in RCA_0 to be a sequence $x = \langle a_n : n \in \mathbb{N} \rangle$, $a_n \in A$, satisfying $\forall m \, \forall n \, (m < n \rightarrow d(a_m, a_n) < 1/2^m)$. The extension of d to $\widehat{A}$ is as in I.4.5.

Under definition I.8.2, the predicate $d(x, y) < r$ for $x, y \in \widehat{A}$ and $r \in \mathbb{R}$ becomes Σ^0_1. This makes I.8.2 far more appropriate than I.4.5 for use in RCA_0. We shall also need to modify slightly our earlier definitions of "continuous function" in I.4.6 and "open set" in I.4.7; the modified definitions will be presented in II.6.1 and II.5.6.

With these new definitions, the development of mathematics within RCA_0 is broadly similar to the development within ACA_0 as already outlined in §I.4 above. For the most part, Δ^0_1 comprehension is an adequate substitute for arithmetical comprehension. Thus RCA_0 is strong enough to prove basic results of real and complex linear and polynomial algebra, up to and including the fundamental theorem of algebra, and basic properties of countable algebraic structures and of continuous functions on complete separable metric spaces. Also within RCA_0 we can introduce sequences of real numbers, sequences of continuous functions, and separable Banach spaces including examples such as $\mathrm{C}[0, 1]$ and $\mathrm{L}_p[0, 1]$, $1 \leq p < \infty$, just as in ACA_0 (§I.4). This detailed development within RCA_0 will be presented in chapter II.

In addition to basic results (*e.g.*, the fact that the composition of two continuous functions is continuous), a number of nontrivial theorems are also provable in RCA_0. We have:

Theorem I.8.3 (mathematics in RCA_0). *The following ordinary mathematical theorems are provable in* RCA_0*:*

1. *the Baire category theorem* (§§II.4, II.5)*;*
2. *the intermediate value theorem* (§II.6)*;*
3. *Urysohn's lemma and the Tietze extension theorem for complete separable metric spaces* (§II.7)*;*
4. *the soundness theorem and a version of Gödel's completeness theorem in mathematical logic* (§II.8)*;*
5. *existence of an algebraic closure of a countable field* (§II.9)*;*
6. *existence of a unique real closure of a countable ordered field* (§II.9)*;*
7. *the Banach/Steinhaus uniform boundedness principle* (§II.10).

On the other hand, a phenomenon of great interest for us is that many well known and important mathematical theorems which are routinely provable in ACA_0 turn out not to be provable at all in RCA_0. We now present an example of this phenomenon.

Example I.8.4 (the Bolzano/Weierstraß theorem). Let us denote by BW the statement of the Bolzano/Weierstraß theorem: "Every bounded sequence of real numbers contains a convergent subsequence." It is straightforward to show that BW is provable in ACA_0.

We claim that BW is not provable in RCA_0.

To see this, consider the ω-model REC consisting of all recursive subsets of ω. We have seen in I.7.5 that REC is a model of RCA_0. We shall now show that BW is false in REC.

We use some basic results of recursive function theory. Let A be a recursively enumerable subset of ω which is not recursive. For instance, we may take $A = K = \{n : \{n\}(n) \text{ is defined}\}$. Let $f : \omega \to \omega$ be a one-to-one recursive function such that $A =$ the range of f. Define a bounded increasing sequence of rational numbers a_k, $k \in \omega$, by putting

$$a_k \;=\; \sum_{m=0}^{k} \frac{1}{2^{f(m)}} \;.$$

Clearly the sequence $\langle a_k \rangle_{k\in\omega}$, or more precisely its code, is recursive and hence is an element of REC. On the other hand, it can be shown that the real number

$$r \;=\; \sup_{k\in\omega} a_k \;=\; \sum_{m=0}^{\infty} \frac{1}{2^{f(m)}} \;=\; \sum_{n\in A} \frac{1}{2^n}$$

is not recursive, *i.e.* (any code of) r is not an element of REC. One way to see this would be to note that the characteristic function of the nonrecursive set A would be computable if we allowed (any code of) r as a Turing oracle.

Thus the ω-model REC satisfies "$\langle a_k\rangle_{k\in\mathbb{N}}$ is a bounded increasing sequence of rational numbers, and $\langle a_k\rangle_{k\in\mathbb{N}}$ has no least upper bound". In particular, REC satisfies "$\langle a_k\rangle_{k\in\mathbb{N}}$ is a bounded sequence of real numbers which has no convergent subsequence". Hence BW is false in the ω-model REC. Hence BW is not provable in RCA_0.

REMARK I.8.5 (recursive counterexamples). There is an extensive literature of what is known as "recursive analysis" or "computable mathematics", *i.e.*, the systematic development of portions of ordinary mathematics within the particular ω-model REC. (See the notes at the end of this section.) This literature contains many so-called "recursive counterexamples", where methods of recursive function theory are used to show that particular mathematical theorems are false in REC. Such results are of great interest with respect to our Main Question, §I.1, because they imply that the set existence axioms of RCA_0 are not strong enough to prove the mathematical theorems under consideration. We have already presented one such recursive counterexample, showing that the Bolzano/Weierstraß theorem is false in REC, hence not provable in RCA_0. Other recursive counterexamples will be presented below.

EXAMPLE I.8.6 (the Heine/Borel covering lemma). Let us denote by HB the statement of the Heine/Borel covering lemma: Every covering of the closed interval $[0,1]$ by a sequence of open intervals has a finite subcovering. Again HB is provable in ACA_0. We shall exhibit a recursive counterexample showing that HB is false in REC, hence not provable in RCA_0.

Consider the well known Cantor middle third set $C \subseteq [0,1]$ defined by

$$C \;=\; [0,1] \setminus ((1/3,2/3) \cup (1/9,2/9) \cup (7/9,8/9) \cup \dots)\,.$$

There is a well known and obvious recursive homeomorphism $H : C \cong \{0,1\}^\omega$, where $\{0,1\}^\omega$ is the product of ω copies of the two-point discrete space $\{0,1\}$. Points $h \in \{0,1\}^\omega$ may be identified with functions $h : \omega \to \{0,1\}$. For each $\varepsilon \in \{0,1\}$ and $n \in \omega$, let U^ε_n be the union of 2^n effectively chosen rational open intervals such that

$$H(U^\varepsilon_n \cap C) \;=\; \{h \in \{0,1\}^\omega : h(n) = \varepsilon\}\,.$$

For instance, corresponding to $\varepsilon = 0$ and $n = 2$ we could choose $U^0_2 = (-1,1/18) \cup (1/6,5/18) \cup (1/2,13/18) \cup (5/6,17/18)$.

Now let A, B be a disjoint pair of recursively inseparable, recursively enumerable subsets of ω. For instance, we could take $A = \{n : \{n\}(n) \simeq 0\}$ and $B = \{n : \{n\}(n) \simeq 1\}$. Since A and B are recursively inseparable, it follows that for any recursive point $h \in \{0,1\}^\omega$ we have either $h(n) = 0$ for some $n \in A$, or $h(n) = 1$ for some $n \in B$. Let $f, g : \omega \to \omega$ be recursive functions such that $A = \mathrm{rng}(f)$ and $B = \mathrm{rng}(g)$. Then $U^0_{f(m)}$, $U^1_{g(m)}$, $m \in \omega$, give a recursive sequence of rational open intervals which cover the recursive reals in C but not all of C. Combining this with the middle third intervals $(1/3, 2/3)$, $(1/9, 2/9)$, $(7/9, 8/9)$, $\ldots$, we obtain a recursive sequence of rational open intervals which cover the recursive reals in $[0,1]$ but not all of $[0,1]$. Thus the ω-model REC satisfies "there exists a sequence of rational open intervals which is a covering of $[0,1]$ but has no finite subcovering". Hence HB is false in REC. Hence HB is not provable in RCA_0.

EXAMPLE I.8.7 (the maximum principle). Another ordinary mathematical theorem not provable in RCA_0 is the maximum principle: Every continuous real-valued function on $[0,1]$ attains a supremum. To see this, let C, f, g, U^ε_n, $\varepsilon \in \{0,1\}$, $n \in \omega$ be as in I.8.6, and let r, a_k, $k \in \omega$ be as in I.8.4. It is straightforward to construct a recursive code Φ for a function ϕ such that REC satisfies "$\phi : C \to \mathbb{R}$ is continuous and, for all $x \in C$, $\phi(x) = a_k$ where $k =$ the least m such that $x \in U^0_{f(m)} \cup U^1_{g(m)}$". Thus $\sup\{\phi(x) : x \in C \cap \mathrm{REC}\} = \sup_{k \in \omega} a_k = r$ is a nonrecursive real number, so REC satisfies "$\sup_{x \in C} \phi(x)$ does not exist". Since $0 < a_k < 2$ for all k, we actually have $\phi : C \to [0,2]$ in REC. Also, we can extend ϕ uniquely to a continuous function $\psi : [0,1] \to [0,2]$ which is linear on intervals disjoint from C. Thus REC satisfies "$\psi : [0,1] \to [0,2]$ is continuous and $\sup_{x \in C} \psi(x)$ does not exist". Hence the maximum principle is false in REC and therefore not provable in RCA_0.

EXAMPLE I.8.8 (König's lemma). Recall our notion of tree as defined in I.6.5. A tree T is said to be *finitely branching* if for each $\sigma \in T$ there are only finitely many n such that $\sigma^\frown\langle n\rangle \in T$. *König's lemma* is the following statement: every infinite, finitely branching tree has an infinite path.

We claim that König's lemma is provable in ACA_0. An outline of the argument within ACA_0 is as follows. Let $T \subseteq \mathbb{N}^{<\mathbb{N}}$ be an infinite, finitely branching tree. By arithmetical comprehension, there is a subtree $T^* \subseteq T$ consisting of all $\sigma \in T$ such that T_σ (see definition I.6.6) is infinite. Since T is infinite, the empty sequence $\langle\rangle$ belongs to T^*. Moreover, by the pigeonhole principle, T^* has no end nodes. Define $f : \mathbb{N} \to \mathbb{N}$ by primitive recursion by putting $f(m) =$ the least n such that $f[m]^\frown\langle n\rangle \in T^*$, for all $m \in \mathbb{N}$. Then f is a path through T^*, hence through T, Q.E.D.

We claim that König's lemma is not provable in RCA_0. To see this, let A, B, f, g be as in I.8.6. Let $\{0,1\}^{<\omega}$ be the full binary tree, *i.e.*, the tree of finite sequences of 0's and 1's. Let T be the set of all $\tau \in \{0,1\}^{<\omega}$ such that, if $k =$ the length of τ, then for all $m, n < k$, $f(m) = n$ implies $\tau(n) = 1$, and $g(m) = n$ implies $\tau(n) = 0$. Note that T is recursive. Moreover, $h \in \{0,1\}^{\omega}$ is a path through T if and only if h separates A and B, *i.e.*, $h(n) = 1$ for all $n \in A$ and $h(n) = 0$ for all $n \in B$. Thus T is an infinite, recursive, finitely branching tree with no recursive path. Hence we have a recursive counterexample to König's lemma, showing that König's lemma is false in REC, hence not provable in RCA_0.

The recursive counterexamples presented above show that, although RCA_0 is able to accommodate a large and significant portion of ordinary mathematical practice, it is also subject to some severe limitations. We shall eventually see that, in order to prove ordinary mathematical theorems such as the Bolzano/Weierstraß theorem, the Heine/Borel covering lemma, the maximum principle, and König's lemma, it is necessary to pass to subsystems of Z_2 that are considerably stronger than RCA_0. This investigation will lead us to another important theme: Reverse Mathematics (§§I.9, I.10, I.11, I.12).

REMARK I.8.9 (constructive mathematics). In some respects, our formal development of ordinary mathematics within RCA_0 resembles the practice of Bishop-style constructivism [20]. However, there are some substantial differences (see also the notes below):

1. The constructivists believe that mathematical objects are purely mental constructions, while we make no such assumption.
2. The meaning which the constructivists assign to the propositional connectives and quantifiers is incompatible with our classical interpretation.
3. The constructivists assume unrestricted induction on the natural numbers, while in RCA_0 we assume only Σ^0_1 induction.
4. We always assume the law of the excluded middle, while the constructivists deny it.
5. The typical constructivist response to a nonconstructive mathematical theorem is to modify the theorem by adding hypotheses or "extra data". In contrast, our approach in this book is to analyze the provability of mathematical theorems as they stand, passing to stronger subsystems of Z_2 if necessary. See also our discussion of Reverse Mathematics in §I.9.

Notes for §I.8. Some references on recursive and constructive mathematics are Aberth [2], Beeson [17], Bishop/Bridges [20], Demuth/Kučera

[46], Mines/Richman/Ruitenburg [189], Pour-El/Richards [203], and Troelstra/van Dalen [268]. The relationship between Bishop-style constructivism and RCA_0 is discussed in [78, §0]. Chapter II of this book is devoted to the development of mathematics within RCA_0. Some earlier literature presenting some of this development in a less systematic manner is Simpson [236], Friedman/Simpson/Smith [78], Brown/Simpson [27].

I.9. Reverse Mathematics

We begin this section with a quote from Aristotle.

> Reciprocation of premisses and conclusion is more frequent in mathematics, because mathematics takes definitions, but never an accident, for its premisses—a second characteristic distinguishing mathematical reasoning from dialectical disputations.
>
> Aristotle, *Posterior Analytics* [184, 78a10].

The purpose of this section is to introduce one of the major themes of this book: Reverse Mathematics.

In order to motivate Reverse Mathematics from a foundational standpoint, consider the Main Question as defined in §I.1, concerning the role of set existence axioms. In §§I.4 and I.6, we have sketched an approximate answer to the Main Question. Namely, we have suggested that most theorems of ordinary mathematics can be proved in ACA_0, and that of the exceptions, most can be proved in Π^1_1-CA_0.

Consider now the following sharpened form of the Main Question: *Given a theorem τ of ordinary mathematics, what is the weakest natural subsystem $S(\tau)$ of Z_2 in which τ is provable?*

Surprisingly, it turns out that for many specific theorems τ this question has a precise and definitive answer. Furthermore, $S(\tau)$ often turns out to be one of five specific subsystems of Z_2. For convenience we shall now list these systems as S_1, S_2, S_3, S_4 and S_5 in order of increasing ability to accommodate ordinary mathematical practice. The odd numbered systems S_1, S_3 and S_5 have already been introduced as RCA_0, ACA_0 and Π^1_1-CA_0 respectively. The even numbered systems S_2 and S_4 are intermediate systems which will be introduced in §§I.10 and I.11 below.

Our method for establishing results of the form $S(\tau) = S_j$, $2 \leq j \leq 5$ is based on the following empirical phenomenon: "When the theorem is proved from the right axioms, the axioms can be proved from the theorem." (Friedman [68].) Specifically, let τ be an ordinary mathematical theorem which is not provable in the weak base theory $S_1 = \mathsf{RCA}_0$. Then very often, τ turns out to be equivalent to S_j for some $j = 2, 3, 4$ or 5. The equivalence is provable in S_i for some $i < j$, usually $i = 1$.

For example, let τ = BW = the Bolzano/Weierstraß theorem: every bounded sequence of real numbers has a convergent subsequence. We have seen in I.8.4 that BW is false in the ω-model REC. An adaptation of that argument gives the following result:

THEOREM I.9.1. BW *is equivalent to* ACA_0*, the equivalence being provable in* RCA_0.

PROOF. Note first that $\mathsf{ACA}_0 = \mathsf{RCA}_0$ plus arithmetical comprehension. Thus the forward direction of our theorem is obtained by observing that the usual proof of BW goes through in ACA_0, as already remarked in §I.4.

For the reverse direction (*i.e.*, the converse), we reason within RCA_0 and assume BW. We are trying to prove arithmetical comprehension. Recall that, by relativization, arithmetical comprehension is equivalent to Σ^0_1 comprehension (see remark I.7.7). So let $\varphi(n)$ be a Σ^0_1 formula, say $\varphi(n) \equiv \exists m\, \theta(m,n)$ where θ is a bounded quantifier formula. For each $k \in \mathbb{N}$ define

$$c_k \;=\; \sum \{2^{-n} : n < k \wedge (\exists m < k)\, \theta(m,n)\}\,.$$

Then $\langle c_k : k \in \mathbb{N}\rangle$ is a bounded increasing sequence of rational numbers. This sequence exists by Δ^0_1 comprehension, which is available to us since we are working in RCA_0. Now by BW the limit $c = \lim_k c_k$ exists. Then we have

$$\forall n\, (\varphi(n) \leftrightarrow \forall k\, (|c - c_k| < 2^{-n} \rightarrow (\exists m < k)\, \theta(m,n)))\,.$$

This gives the equivalence of a Σ^0_1 formula with a Π^0_1 formula. Hence by Δ^0_1 comprehension we conclude $\exists X\, \forall n\, (n \in X \leftrightarrow \varphi(n))$. This proves Σ^0_1 comprehension and hence arithmetical comprehension, Q.E.D. □

REMARK I.9.2 (on Reverse Mathematics). Theorem I.9.1 implies that $S_3 = \mathsf{ACA}_0$ is the weakest natural subsystem of Z_2 in which τ = BW is provable. Thus, for this particular case involving the Bolzano/Weierstraß theorem, I.9.1 provides a definitive answer to our sharpened form of the Main Question.

Note that the proof of theorem I.9.1 involved the deduction of a set existence axiom (namely arithmetical comprehension) from an ordinary mathematical theorem (namely BW). This is the opposite of the usual pattern of ordinary mathematical practice, in which theorems are deduced from axioms. The deduction of axioms from theorems is known as *Reverse Mathematics*. Theorem I.9.1 illustrates how Reverse Mathematics is the key to obtaining precise answers for instances of the Main Question. This point will be discussed more fully in §I.12.

We shall now state a number of results, similar to I.9.1, showing that particular ordinary mathematical theorems are equivalent to the axioms needed to prove them. These Reverse Mathematics results with respect to ACA_0 and Π^1_1-CA_0 will be summarized in theorems I.9.3 and I.9.4 and proved in chapters III and VI, respectively.

Theorem I.9.3 (Reverse Mathematics for ACA_0). *Within* RCA_0 *one can prove that* ACA_0 *is equivalent to each of the following ordinary mathematical theorems:*

1. *Every bounded, or bounded increasing, sequence of real numbers has a least upper bound* (§III.2).
2. *The Bolzano/Weierstraß theorem: Every bounded sequence of real numbers, or of points in* $\mathbb{R}^n$, *has a convergent subsequence* (§III.2).
3. *Every sequence of points in a compact metric space has a convergent subsequence* (§III.2).
4. *The Ascoli lemma: Every bounded equicontinuous sequence of real-valued continuous functions on a bounded interval has a uniformly convergent subsequence* (§III.2).
5. *Every countable commutative ring has a maximal ideal* (§III.5).
6. *Every countable vector space over* $\mathbb{Q}$, *or over any countable field, has a basis* (§III.4).
7. *Every countable field* (*of characteristic* 0) *has a transcendence basis* (§III.4).
8. *Every countable Abelian group has a unique divisible closure* (§III.6).
9. *König's lemma: Every infinite, finitely branching tree has an infinite path* (§III.7).
10. *Ramsey's theorem for colorings of* $[\mathbb{N}]^3$, *or of* $[\mathbb{N}]^4$, $[\mathbb{N}]^5$, … (§III.7).

Theorem I.9.4 (Reverse Mathematics for Π^1_1-CA_0). *Within* RCA_0 *one can prove that* Π^1_1-CA_0 *is equivalent to each of the following ordinary mathematical statements:*

1. *Every tree has a largest perfect subtree* (§VI.1).
2. *The Cantor/Bendixson theorem: Every closed subset of* $\mathbb{R}$, *or of any complete separable metric space, is the union of a countable set and a perfect set* (§VI.1).
3. *Every countable Abelian group is the direct sum of a divisible group and a reduced group* (§VI.4).
4. *Every difference of two open sets in the Baire space* $\mathbb{N}^{\mathbb{N}}$ *is determined* (§VI.5).
5. *Every* G_δ *set in* $[\mathbb{N}]^{\mathbb{N}}$ *has the Ramsey property* (§VI.6).
6. *Silver's theorem: For every Borel* (*or coanalytic, or* F_σ) *equivalence relation with uncountably many equivalence classes, there exists a nonempty perfect set of inequivalent elements* (§VI.3).

More Reverse Mathematics results will be stated in §§I.10 and I.11 and proved in chapters IV and V, respectively. The significance of Reverse Mathematics for our Main Question will be discussed in §I.12.

Notes for §I.9. Historically, Reverse Mathematics may be viewed as a spin-off of Friedman's work [65, 66, 71, 72, 73] attempting to demonstrate the necessary use of higher set theory in mathematical practice. The theme of Reverse Mathematics in the context of subsystems of Z_2 first appeared in Steel's thesis [256, chapter I] (an outcome of Steel's reading of Friedman's thesis [62, chapter II] under Simpson's supervision [230]) and in Friedman [68, 69]; see also Simpson [238]. This theme was taken up by Simpson and his collaborators in numerous studies [236, 241, 76, 235, 234, 78, 79, 250, 243, 246, 245, 21, 27, 28, 280, 80, 113, 112, 247, 127, 128, 26, 93, 248] which established it as a subject. The slogan "Reverse Mathematics" was coined by Friedman during a special session of the American Mathematical Society organized by Simpson.

I.10. The System WKL_0

In this section we introduce WKL_0, a subsystem of Z_2 consisting of RCA_0 plus a set existence axiom known as *weak König's lemma*. We shall see that, in the notation of §I.9, $\mathsf{WKL}_0 = S_2$ is intermediate between $\mathsf{RCA}_0 = S_1$ and $\mathsf{ACA}_0 = S_3$. We shall also state several results of Reverse Mathematics with respect to WKL_0 (theorem I.10.3 below).

In order to motivate WKL_0 in terms of foundations of mathematics, consider our Main Question (§I.1) as it applies to three specific theorems of ordinary mathematics: the Bolzano/Weierstraß theorem, the Heine/Borel covering lemma, the maximum principle. We have seen in I.8.4, I.8.6, I.8.7 that these three theorems are not provable in RCA_0. However, we have definitively answered the Main Question only for the Bolzano/Weierstraß theorem, not for the other two. We have seen in I.9.1 that Bolzano/Weierstraß is equivalent to ACA_0 over RCA_0.

It will turn out (theorem I.10.3) that the Heine/Borel covering lemma, the maximum principle, and many other ordinary mathematical theorems are equivalent to each other and to weak König's lemma, over RCA_0. Thus WKL_0 is the weakest natural subsystem of Z_2 in which these ordinary mathematical theorems are provable. Thus WKL_0 provides the answer to these instances of the Main Question.

It will also turn out that WKL_0 is sufficiently strong to accommodate a large portion of mathematical practice, far beyond what is available in RCA_0, including many of the best-known non-constructive theorems. This will become clear in chapter IV.

We now present the definition of WKL_0.

Definition I.10.1 (weak König's lemma). The following definitions are made within RCA_0. We use $\{0,1\}^{<\mathbb{N}}$ or $2^{<\mathbb{N}}$ to denote the full binary tree, *i.e.*, the set of (codes for) finite sequences of 0's and 1's. *Weak König's lemma* is the following statement: Every infinite subtree of $2^{<\mathbb{N}}$ has an infinite path. (Compare definition I.6.5 and example I.8.8.)

WKL_0 is defined to be the subsystem of Z_2 consisting of RCA_0 plus weak König's lemma.

Remark I.10.2 (ω-models of WKL_0). By example I.8.8, the ω-model REC consisting of all recursive subsets of ω does not satisfy weak König's lemma. Hence REC is not a model of WKL_0. Since REC is the minimum ω-model of RCA_0 (remark I.7.5), it follows that RCA_0 is a proper subsystem of WKL_0. In addition, I.8.8 implies that WKL_0 is a subsystem of ACA_0. That it is a proper subsystem is not so obvious, but we shall see this in §VIII.2, where it is shown for instance that REC is the intersection of all ω-models of WKL_0. Thus we have

$$\mathsf{RCA}_0 \quad \subsetneqq \quad \mathsf{WKL}_0 \quad \subsetneqq \quad \mathsf{ACA}_0$$

and there are ω-models for the independence.

We now list several results of Reverse Mathematics with respect to WKL_0. These results will be proved in chapter IV.

Theorem I.10.3 (Reverse Mathematics for WKL_0). *Within* RCA_0 *one can prove that* WKL_0 *is equivalent to each of the following ordinary mathematical statements:*

1. *The Heine/Borel covering lemma: Every covering of the closed interval* $[0,1]$ *by a sequence of open intervals has a finite subcovering* (§IV.1).
2. *Every covering of a compact metric space by a sequence of open sets has a finite subcovering* (§IV.1).
3. *Every continuous real-valued function on* $[0,1]$*, or on any compact metric space, is bounded* (§IV.2).
4. *Every continuous real-valued function on* $[0,1]$*, or on any compact metric space, is uniformly continuous* (§IV.2).
5. *Every continuous real-valued function on* $[0,1]$ *is Riemann integrable* (§IV.2).
6. *The maximum principle: Every continuous real-valued function on* $[0,1]$*, or on any compact metric space, has, or attains, a supremum* (§IV.2).
7. *The local existence theorem for solutions of* (*finite systems of*) *ordinary differential equations* (§IV.8).

8. *Gödel's completeness theorem: every finite, or countable, consistent set of sentences in the predicate calculus has a countable model* (§IV.3)*.*
9. *Every countable commutative ring has a prime ideal* (§IV.6)*.*
10. *Every countable field* (*of characteristic* 0) *has a unique algebraic closure* (§IV.5)*.*
11. *Every countable formally real field is orderable* (§IV.4)*.*
12. *Every countable formally real field has a* (*unique*) *real closure* (§IV.4)*.*
13. *Brouwer's fixed point theorem: Every uniformly continuous function* $\phi : [0,1]^n \to [0,1]^n$ *has a fixed point* (§IV.7)*.*
14. *The separable Hahn/Banach theorem: If* f *is a bounded linear functional on a subspace of a separable Banach space, and if* $\|f\| \leq 1$*, then* f *has an extension* $\widetilde{f}$ *to the whole space such that* $\|\widetilde{f}\| \leq 1$ (§IV.9)*.*

REMARK I.10.4 (mathematics within WKL_0). Theorem I.10.3 illustrates how WKL_0 is much stronger than RCA_0 from the viewpoint of mathematical practice. In fact, WKL_0 is strong enough to prove many well known nonconstructive theorems that are extremely important for mathematical practice but not true in the ω-model REC, hence not provable in RCA_0 (see §I.8).

REMARK I.10.5 (first order part of WKL_0). We have seen that WKL_0 is much stronger than RCA_0 with respect to both ω-models (remark I.10.2) and mathematical practice (theorem I.10.3, remark I.10.4). Nevertheless, it can be shown that WKL_0 is of the same strength as RCA_0 in a proof-theoretic sense. Namely, the first order part of WKL_0 is the same as that of RCA_0, *viz.* Σ^0_1-PA. (See also remark I.7.6.) In fact, given any model M of RCA_0, there exists a model $M' \supseteq M$ of WKL_0 having the same first order part as M. This model-theoretic conservation result will be proved in §IX.2.

Another key conservation result is that WKL_0 is conservative over the formal system known as PRA or *primitive recursive arithmetic*, with respect to Π^0_2 sentences. In particular, given a Σ^0_1 formula $\varphi(m,n)$ and a proof of $\forall m\, \exists n\, \varphi(m,n)$ in WKL_0, we can find a primitive recursive function $f : \omega \to \omega$ such that $\varphi(m, f(m))$ holds for all $m \in \omega$. This interesting and important result will be proved in §IX.3.

REMARK I.10.6 (Hilbert's program). The results of chapters IV and IX are of great importance with respect to the foundations of mathematics, specifically Hilbert's program. Hilbert's intention [114] was to justify all of mathematics (including infinitistic, set-theoretic mathematics) by reducing it to a restricted form of reasoning known as finitism. Gödel's [94, 115, 55, 222] limitative results show that there is no hope of realizing Hilbert's program completely. However, results along the lines of theorem I.10.3 and remark I.10.5 show that a large portion of infinitistic mathematical practice is in fact finitistically reducible, because it can be carried out in WKL_0. Thus we have a significant partial realization of Hilbert's program of finitistic reductionism. See also remark IX.3.18.

Notes for §I.10. The formal system WKL_0 was first introduced by Friedman [69]. In the model-theoretic literature, ω-models of WKL_0 are sometimes known as *Scott systems*, referring to Scott [217]. Chapter IV of this book is devoted to the development of mathematics within WKL_0 and Reverse Mathematics for WKL_0. Models of WKL_0 are discussed in §§VIII.2, IX.2, and IX.3 of this book. The original paper on Hilbert's program is Hilbert [114]. The significance of WKL_0 and Reverse Mathematics for partial realizations of Hilbert's program is expounded in Simpson [246].

I.11. The System ATR_0

In this section we introduce and discuss ATR_0, a subsystem of Z_2 consisting of ACA_0 plus a set existence axiom known as *arithmetical transfinite recursion*. Informally, arithmetical transfinite recursion can be described as the assertion that the Turing jump operator can be iterated along any countable well ordering starting at any set. The precise statement is given in definition I.11.1 below.

From the standpoint of foundations of mathematics, the motivation for ATR_0 is similar to the motivation for WKL_0, as explained in §I.10. (See also the analogy in I.11.7 below.) Using the notation of §I.9, $\mathsf{ATR}_0 = S_4$ is intermediate between $\mathsf{ACA}_0 = S_3$ and $\Pi^1_1\text{-}\mathsf{CA}_0 = S_5$. It turns out that ATR_0 is equivalent to several theorems of ordinary mathematics which are provable in $\Pi^1_1\text{-}\mathsf{CA}_0$ but not in ACA_0.

As an example, consider the *perfect set theorem*: Every uncountable closed set (or analytic set) has a perfect subset. We shall see that ATR_0 is equivalent over RCA_0 to (either form of) the perfect set theorem. Thus ATR_0 is the weakest natural subsystem of Z_2 in which the perfect set theorem is provable. Actually, ATR_0 provides the answer not only to this instance of the Main Question (§I.9) but also to many other instances of it; see theorem I.11.5 below. Moreover, ATR_0 is sufficiently strong to accommodate a large portion of mathematical practice beyond ACA_0, including many basic theorems of infinitary combinatorics and classical descriptive set theory.

We now proceed to the definition of ATR_0.

Definition I.11.1 (arithmetical transfinite recursion). Consider an arithmetical formula $\theta(n, X)$ with a free number variable n and a free set variable X. Note that $\theta(n, X)$ may also contain parameters, *i.e.*, additional free number and set variables. Fixing these parameters, we may view θ as an "arithmetical operator" $\Theta : P(\mathbb{N}) \to P(\mathbb{N})$, defined by

$$\Theta(X) \;=\; \{n \in \mathbb{N} : \theta(n, X)\}\ .$$

Now let $A, <_A$ be any countable well ordering (definition I.6.1), and consider the set $Y \subseteq \mathbb{N}$ obtained by transfinitely iterating the operator Θ along $A, <_A$. This set Y is defined by the following conditions: $Y \subseteq \mathbb{N} \times A$ and, for each $a \in A$, $Y_a = \Theta(Y^a)$, where $Y_a = \{m : (m, a) \in Y\}$ and $Y^a = \{(n, b) : n \in Y_b \wedge b <_A a\}$. Thus, for each $a \in A$, Y^a is the result of iterating Θ along the initial segment of $A, <_A$ up to but not including a, and Y_a is the result of applying Θ one more time.

Finally, *arithmetical transfinite recursion* is the axiom scheme asserting that such a set Y exists, for every arithmetical operator Θ and every countable well ordering $A, <_A$. We define ATR_0 to consist of ACA_0 plus the scheme of arithmetical transfinite recursion. It is easy to see that ATR_0 is a subsystem of Π^1_1-CA_0, and we shall see below that it is a proper subsystem.

EXAMPLE I.11.2 (the ω-model ARITH). Recall the ω-model

$$\begin{aligned}\mathrm{ARITH} &= \mathrm{Def}((\omega, +, \cdot, 0, 1, <)) \\ &= \{X \subseteq \omega : \exists n \in \omega\, X \leq_{\mathrm{T}} \mathrm{TJ}(n, \emptyset)\}\end{aligned}$$

consisting of all arithmetically definable subsets of ω (remarks I.3.3 and I.3.4). We have seen that ARITH is the minimum ω-model of ACA_0. Trivially for each $n \in \omega$ we have $\mathrm{TJ}(n, \emptyset) \in \mathrm{ARITH}$; here $\mathrm{TJ}(n, \emptyset)$ is the result of iterating the Turing jump operator n times, *i.e.*, along a finite well ordering of order type n. On the other hand, ARITH does not contain $\mathrm{TJ}(\omega, \emptyset)$, the result of iterating the Turing jump operator ω times, *i.e.*, along the well ordering $(\omega, <)$. Thus ARITH fails to satisfy this instance of arithmetical transfinite recursion. Hence ARITH is not an ω-model of ATR_0.

EXAMPLE I.11.3 (the ω-model HYP). Another important ω-model is

$$\begin{aligned}\mathrm{HYP} &= \{X \subseteq \omega : X \leq_{\mathrm{H}} \emptyset\} \\ &= \{X \subseteq \omega : X \text{ is hyperarithmetical}\} \\ &= \{X \subseteq \omega : \exists \alpha < \omega_1^{\mathrm{CK}}\, X \leq_{\mathrm{T}} \mathrm{TJ}(\alpha, \emptyset)\}\ .\end{aligned}$$

Here α ranges over the recursive ordinals, *i.e.*, the countable ordinals which are order types of recursive well orderings of ω. We use ω_1^{CK} to denote Church/Kleene ω_1, *i.e.*, the least nonrecursive ordinal. Clearly HYP is much larger than ARITH, and HYP contains many sets which are defined by arithmetical transfinite recursion. However, as we shall see in §VIII.3, HYP does not contain enough sets to be an ω-model of ATR_0.

REMARK I.11.4 (ω-models of ATR_0). In §§VII.2 and VIII.6 we shall prove two facts: (1) every β-model is an ω-model of ATR_0; (2) the intersection of all β-models is HYP, the ω-model consisting of the hyperarithmetical sets. From this it follows that HYP, although not itself an ω-model of ATR_0, is the intersection of all such ω-models. Hence ATR_0 does not have a minimum ω-model or a minimum β-model. Combining these observations with what we already know about ω-models of ACA_0 and Π^1_1-CA_0 (remarks I.3.4 and I.5.4), we see that

$$\mathsf{ACA}_0 \quad \subsetneqq \quad \mathsf{ATR}_0 \quad \subsetneqq \quad \Pi^1_1\text{-}\mathsf{CA}_0$$

and there are ω-models for the independence.

We now list several results of Reverse Mathematics with respect to ATR_0. These results will be proved in chapter V.

THEOREM I.11.5 (Reverse Mathematics for ATR_0). *Within* RCA_0 *one can prove that* ATR_0 *is equivalent to each of the following ordinary mathematical statements:*

1. *Any two countable well orderings are comparable* (§V.6).
2. *Ulm's theorem: Any two countable reduced Abelian p-groups which have the same Ulm invariants are isomorphic* (§V.7).
3. *The perfect set theorem: Every uncountable closed, or analytic, set has a perfect subset* (§V.4, V.5).
4. *Lusin's separation theorem: Any two disjoint analytic sets can be separated by a Borel set* (§§V.3, V.5).
5. *The domain of any single-valued Borel set in the plane is a Borel set* (§V.3, V.5).
6. *Every open, or clopen, subset of* $\mathbb{N}^{\mathbb{N}}$ *is determined* (§V.8).
7. *Every open, or clopen, subset of* $[\mathbb{N}]^{\mathbb{N}}$ *has the Ramsey property* (§V.9).

REMARK I.11.6 (mathematics within ATR_0). Theorem I.11.5 illustrates how ATR_0 is much stronger than ACA_0 from the viewpoint of mathematical practice. Namely, ATR_0 proves many well known ordinary mathematical theorems which fail in the ω-models ARITH and HYP and hence are not provable in ACA_0 (see §I.4) or even in somewhat stronger systems such as Σ^1_1-AC_0 (§VIII.4). A common feature of such theorems is that they require, implicitly or explicitly, a good theory of countable ordinal numbers.

REMARK I.11.7 (Σ^0_1 and Σ^1_1 separation). From the viewpoint of mathematical practice, we have already noted an interesting analogy between WKL_0 and ATR_0, suggested by the following equation:

$$\frac{\mathsf{WKL}_0}{\mathsf{ACA}_0} \quad \approx \quad \frac{\mathsf{ATR}_0}{\Pi^1_1\text{-}\mathsf{CA}_0}\ .$$

We shall now extend this analogy by reformulating WKL_0 and ATR_0 in terms of separation principles.

Define Σ^0_1 *separation* to be the axiom scheme consisting of (the universal closures of) all formulas of the form

$$(\forall n \neg (\varphi_1(n) \wedge \varphi_2(n))) \rightarrow \\ \exists X (\forall n (\varphi_1(n) \rightarrow n \in X) \wedge \forall n (\varphi_2(n) \rightarrow n \notin X)) ,$$

where $\varphi_1(n)$ and $\varphi_2(n)$ are any Σ^0_1 formulas, n is any number variable, and X is a set variable which does not occur freely in $\varphi_1(n) \wedge \varphi_2(n)$. Define Σ^1_1 *separation* similarly, with Σ^1_1 formulas instead of Σ^0_1 formulas. It turns out that

$$\mathsf{WKL}_0 \equiv \Sigma^0_1 \text{ separation} ,$$

and

$$\mathsf{ATR}_0 \equiv \Sigma^1_1 \text{ separation} ,$$

over RCA_0. These equivalences, which will be proved in §§IV.4 and V.5 respectively, serve to strengthen the above-mentioned analogy between WKL_0 and ATR_0. They will also be used as technical tools for proving several of the reversals given by theorems I.10.3 and I.11.5.

Remark I.11.8. Another analogy in the same vein as that of I.11.7 is

$$\frac{\mathsf{WKL}_0}{\mathsf{RCA}_0} \approx \frac{\mathsf{ATR}_0}{\Delta^1_1\text{-}\mathsf{CA}_0} .$$

The system Δ^1_1-CA_0 will be studied in §§VIII.3 and VIII.4, where we shall see that HYP is its minimum ω-model. Recall also (remark I.7.5) that REC is the minimum ω-model of

$$\mathsf{RCA}_0 \equiv \Delta^0_1\text{-}\mathsf{CA}_0 .$$

Remark I.11.9 (first order part of ATR_0). It is known that the first order part of ATR_0 is the same as that of Feferman's system IR of predicative analysis; indeed, these two systems prove the same Π^1_1 sentences. Thus our development of mathematics within ATR_0 (theorem I.11.5, remark I.11.6, chapter V) may be viewed as contributions to a program of "predicative reductionism," analogous to Hilbert's program of finitistic reductionism (remark I.10.6, section IX.3). See also the proof of theorem IX.5.7 below.

Notes for §I.11. The formal system ATR_0 was first investigated by Friedman [68, 69] (see also Friedman [62, chapter II]) and Steel [256, chapter I]. A key reference for ATR_0 is Friedman/McAloon/Simpson [76]. Chapter V of this book is devoted to the development of mathematics within ATR_0 and Reverse Mathematics for ATR_0. Models of ATR_0 are discussed in §§VII.2, VII.3 and VIII.6. The basic reference for formal systems of predicative analysis is Feferman [56, 57]. The significance of ATR_0 for predicative reductionism has been discussed by Simpson [238, 246].

I.12. The Main Question, Revisited

The Main Question was introduced in §I.1. We now reexamine it in light of the results outlined in §§I.2 through I.11.

The Main Question asks which set existence axioms are needed to support ordinary mathematical reasoning. We take "needed" to mean that the set existence axioms are to be as weak as possible. When developing precise formal versions of the Main Question, it is natural also to consider formal languages which are as weak as possible. The language L_2 comes to mind because it is just adequate to define the majority of ordinary mathematical concepts and to express the bulk of ordinary mathematical reasoning. This leads in §I.2 to the consideration of subsystems of Z_2.

Two of the most obvious subsystems of Z_2 are ACA_0 and Π^1_1-CA_0, and in §§I.3–I.6 we outline the development of ordinary mathematics in these systems. The upshot of this is that a great many ordinary mathematical theorems are provable in ACA_0, and that of the exceptions, most are provable in Π^1_1-CA_0. The exceptions tend to involve countable ordinal numbers, either explicitly or implicitly. Another important subsystem of Z_2 is RCA_0, which is seen in §§I.7 and I.8 to embody a kind of formalized computable or constructive mathematics. Thus we have an approximate answer to the Main Question.

We then turn to a sharpened form of the Main Question, where we insist that the ordinary mathematical theorems should be logically equivalent to the set existence axioms needed to prove them. Surprisingly, this demand can be met in some cases; several ordinary mathematical theorems turn out to be equivalent over RCA_0 to either ACA_0 or Π^1_1-CA_0. This is our theme of Reverse Mathematics in §I.9. But the situation is not entirely satisfactory, because many ordinary mathematical theorems seem to fall into the gaps.

In order to improve the situation, we introduce two additional systems: WKL_0 lying strictly between RCA_0 and ACA_0, and analogously ATR_0 lying strictly between ACA_0 and Π^1_1-CA_0. These systems are introduced in §§I.10 and I.11 respectively. With this expanded complement of subsystems of Z_2, a certain stability is achieved; it now seems possible to "calibrate" a great many ordinary mathematical theorems, by showing that they are either provable in RCA_0 or equivalent over RCA_0 to WKL_0, ACA_0, ATR_0, or Π^1_1-CA_0.

Historically, the intermediate systems WKL_0 and ATR_0 were discovered in exactly in this way, as a response to the needs of Reverse Mathematics. See for example the discussion in Simpson [246, §§4,5].

From the above it is clear that the five basic systems RCA_0, WKL_0, ACA_0, ATR_0, Π^1_1-CA_0 arise naturally from investigations of the Main Question.

The proof that these systems are mathematically natural is provided by Reverse Mathematics.

As a perhaps not unexpected byproduct, we note that these same five systems turn out to correspond to various well known, philosophically motivated programs in foundations of mathematics, as indicated in table 1. The foundational programs that we have in mind are: Bishop's program of constructivism [20] (see however remarks I.8.9 and IV.2.8); Hilbert's program of finitistic reductionism [114, 246] (see remarks I.10.6 and IX.3.18); Weyl's program of predicativity [274] as developed by Feferman [56, 57, 59]; predicative reductionism as developed by Friedman and Simpson [69, 76, 238, 247]; impredicativity as developed in Buchholz/Feferman/Pohlers/Sieg [29]. Thus, by studying the formalization of mathematics and Reverse Mathematics for the five basic systems, we can develop insight into the mathematical consequences of these philosophical proposals. Thus we can expect this book and other Reverse Mathematics studies to have a substantial impact on the philosophy of mathematics.

TABLE 1. Foundational programs and the five basic systems.

RCA_0	constructivism	Bishop
WKL_0	finitistic reductionism	Hilbert
ACA_0	predicativism	Weyl, Feferman
ATR_0	predicative reductionism	Friedman, Simpson
$\Pi^1_1\text{-}\mathsf{CA}_0$	impredicativity	Feferman *et al.*

I.13. Outline of Chapters II Through X

This section of our introductory chapter I consists of an outline of the remaining chapters.

The bulk of the material is organized in two parts. Part A consists of chapters II through VI and focuses on the development of mathematics within the five basic systems: RCA_0, WKL_0, ACA_0, ATR_0, $\Pi^1_1\text{-}\mathsf{CA}_0$. A principal theme of Part A is Reverse Mathematics (see also §I.9). Part B, consisting of chapters VII through IX, is concerned with metamathematical properties of various subsystems of Z_2, including but not limited to the five basic systems. Chapters VII, VIII, and IX deal with β-models, ω-models, and non-ω-models, respectively. At the end of the book there is an appendix, chapter X, in which additional results are presented without proof but with references to the published literature. See also table 2.

TABLE 2. An overview of the entire book.

Introduction	Chapter I	introductory survey
Part A (mathematics within the 5 basic systems)	Chapter II	RCA_0
	Chapter III	ACA_0
	Chapter IV	WKL_0
	Chapter V	ATR_0
	Chapter VI	Π^1_1-CA_0
Part B (models of various systems)	Chapter VII	β-models
	Chapter VIII	ω-models
	Chapter IX	non-ω-models
Appendix	Chapter X	additional results

Part A: Mathematics Within Subsystems of Z_2. Part A consists of a key chapter II on the development of ordinary mathematics within RCA_0, followed by chapters III, IV, V, and VI on ordinary mathematics within the other four basic systems: ACA_0, WKL_0, ATR_0, and Π^1_1-CA_0, respectively. These chapters present many results of Reverse Mathematics showing that particular set existence axioms are necessary and sufficient to prove particular ordinary mathematical theorems. Table 3 indicates in more detail exactly where some of these results may be found. Table 3 may serve as a guide or road map concerning the role of set existence axioms in ordinary mathematical reasoning.

Chapter II: RCA_0. In §II.1 we define the formal system RCA_0 consisting of Δ^0_1 comprehension and Σ^0_1 induction. After that, the rest of chapter II is concerned with the development of ordinary mathematics within RCA_0. Although chapter II does not itself contain any Reverse Mathematics, it is necessarily a prerequisite for all of the Reverse Mathematics results to be presented in later chapters. This is because RCA_0 serves as our weak base theory (see §I.9 above).

In §II.2 we employ a device reminiscent of *Gödel's beta function* to prove within RCA_0 that finite sequences of natural numbers can be encoded as single numbers. This encoding is essential for §II.3, where we prove within RCA_0 that the class of functions from $f : \mathbb{N}^k \to \mathbb{N}$, $k \in \mathbb{N}$, is closed under *primitive recursion*. Another key technical result of §II.3 is that RCA_0 proves *bounded Σ^0_1 comprehension*, *i.e.*, the existence of bounded subsets of $\mathbb{N}$ defined by Σ^0_1 formulas.

Armed with these preliminary results from §§II.2 and II.3, we begin the development of mathematics proper in §II.4 by discussing the *number*

TABLE 3. Ordinary mathematics within the five basic systems.

	RCA_0	WKL_0	ACA_0	ATR_0	Π^1_1-CA_0
analysis (separable):					
differential equations	IV.8	IV.8			
continuous functions	II.6, II.7	IV.2, IV.7	III.2		
completeness, *etc.*	II.4	IV.1	III.2		
Banach spaces	II.10	IV.9, X.2			X.2
open and closed sets	II.5	IV.1		V.4, V.5	VI.1
Borel and analytic sets	V.1			V.1, V.3	VI.2, VI.3
algebra (countable):					
countable fields	II.9	IV.4, IV.5	III.3		
commutative rings	III.5	IV.6	III.5		
vector spaces	III.4		III.4		
Abelian groups	III.6		III.6	V.7	VI.4
miscellaneous:					
mathematical logic	II.8	IV.3			
countable ordinals	V.1		V.6.10	V.1, V.6	
infinite matchings		X.3	X.3	X.3	
the Ramsey property			III.7	V.9	VI.6
infinite games			V.8	V.8	VI.5

systems $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$. Also in §II.4 we present an important completeness property of the real number system, known as *nested interval completeness*. An RCA_0 version of the *Baire category theorem* for k-dimensional Euclidean spaces $\mathbb{R}^k$, $k \in \mathbb{N}$, is stated; the proof is postponed to §II.5.

Sections II.5, II.6, and II.7 discuss *complete separable metric spaces* in RCA_0. Among the notions introduced (in a form appropriate for RCA_0) are *open sets*, *closed sets*, and *continuous functions*. We prove the following important technical result: An open set in a complete separable metric space $\widehat{A}$ is the same thing as a set in $\widehat{A}$ defined by a Σ^0_1 formula with an extensionality property (II.5.7). Nested interval completeness is used to prove the *intermediate value property* for continuous functions $\phi : \mathbb{R} \to \mathbb{R}$ in RCA_0 (II.6.6). A number of basic topological results for complete separable metric spaces are shown to be provable in RCA_0. Among these are *Urysohn's lemma* (II.7.3), the *Tietze extension theorem* (II.7.5), the *Baire category theorem* (II.5.8), and *paracompactness* (II.7.2).

Sections II.8 and II.9 deal with *mathematical logic* and *countable algebra*, respectively. We show in §II.8 that some surprisingly strong versions of basic results of mathematical logic can be proved in RCA_0. Among these are *Lindenbaum's lemma*, the *Gödel completeness theorem*, and the *strong soundness theorem*, via *cut elimination*. To illustrate the power of these results, we show that RCA_0 proves the consistency of *elementary function arithmetic*, EFA. In §II.9 we apply the results of §§II.3 and II.8 in a discussion of countable algebraically closed and real closed fields in RCA_0. We use *quantifier elimination* to prove within RCA_0 that every countable field has an *algebraic closure*, and that every countable ordered field has a *unique real closure*. (Uniqueness of algebraic closure is discussed later, in §IV.5.)

Section II.10 presents some basic concepts and results of the theory of *separable Banach spaces* and *bounded linear operators*, within RCA_0. It is shown that the standard proof of the *Banach/Steinhaus uniform boundedness principle*, via the Baire category theorem, goes through in this setting.

Chapter III: ACA_0**.** Chapter III is concerned with ACA_0, the formal system consisting of RCA_0 plus arithmetical comprehension. The focus of chapter III is Reverse Mathematics with respect to ACA_0. (See also §§I.4, I.3, and I.9.)

In §III.1 we define ACA_0 and show that it is equivalent over RCA_0 to Σ^0_1 comprehension and to the principle that for any function $f : \mathbb{N} \to \mathbb{N}$, the range of f exists. This equivalence is used to establish all of the Reverse Mathematics results which occupy the rest of the chapter. For example, it is shown in §III.2 that ACA_0 is equivalent to the *Bolzano/Weierstraß theorem*, *i.e.*, sequential compactness of the closed unit interval. Also in §III.2 we introduce the notion of *compact metric space*, and we show that ACA_0 is equivalent to the principle that any sequence of points in a compact metric space has a convergent subsequence. We end §III.2 by showing that ACA_0 is equivalent to the *Ascoli lemma* concerning bounded equicontinuous families of continuous functions.

Sections III.3, III.4, III.5 and III.6 are concerned with countable algebra in ACA_0. It is perhaps interesting to note that chapter III has much more to say about algebra than about analysis.

We begin in §III.3 by reexamining the notion of an algebraic closure $h : K \to \widetilde{K}$ of a countable field K. We define a notion of *strong algebraic closure*, *i.e.*, an algebraic closure with the additional property that the range of the embedding h exists as a set. Although the existence of algebraic closures is provable in RCA_0, we show in §III.3 that the existence of strong algebraic closures is equivalent to ACA_0. Similarly, although it is provable in RCA_0 that any countable ordered field has a real closure, we show in §III.3 that ACA_0 is required to prove the existence of a *strong real closure*.

In §III.4 we show that ACA_0 is equivalent to the theorem that every countable *vector space* over a countable field (or over the rational field $\mathbb{Q}$) has a basis. We then refine this result (following Metakides/Nerode [187]) by showing that ACA_0 is also equivalent to the assertion that every countable, infinite dimensional vector space over $\mathbb{Q}$ has an infinite linearly independent set. We also obtain similar results for *transcendence bases* of countable fields.

In §III.5 we turn to countable commutative rings. We use localization to show that ACA_0 is equivalent to the assertion that every countable commutative ring has a *maximal ideal*. In §III.6 we discuss *countable Abelian groups*. We show that ACA_0 is equivalent to the assertion that, for every countable Abelian group G, the *torsion subgroup* of G exists. We also show that, although the existence of *divisible closures* is provable in RCA_0, the uniqueness requires ACA_0

In §III.7 we consider *Ramsey's theorem*. We define $\mathrm{RT}(k)$ to be Ramsey's theorem for exponent k, *i.e.*, the assertion that for every coloring of the k-element subsets of $\mathbb{N}$ with finitely many colors, there exists an infinite subset of $\mathbb{N}$ all of whose k-element subsets have the same color. We show that ACA_0 is equivalent to $\mathrm{RT}(k)$ for each "standard integer" $k \in \omega$, $k \geq 3$. From the viewpoint of Reverse Mathematics, the case $k = 2$ turns out to be anomalous: $\mathrm{RT}(2)$ is provable in ACA_0 but neither equivalent to ACA_0 nor provable in WKL_0. See also the notes at the end of §III.7. Another somewhat annoying anomaly is that the general assertion of Ramsey's theorem, $\forall k\, \mathrm{RT}(k)$, is slightly stronger than ACA_0, due to the fact that ACA_0 lacks full induction.

An interesting technical result of §III.7 is that ACA_0 is equivalent to *König's lemma*: every infinite, finitely branching tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ has an infinite path. It turns out that ACA_0 is also equivalent to a much weaker sounding statement, namely König's lemma restricted to *binary trees*. (A tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ is defined to be binary if each node of T has at most two immediate successors.) The binary tree version of König's lemma is to be contrasted with its special case, *weak König's lemma*: every infinite tree $T \subseteq 2^{<\mathbb{N}}$ has an infinite path. It is important to understand that, in terms of set existence axioms and Reverse Mathematics, weak König's lemma is much weaker than König's lemma for binary trees. These observations provide a transition to the next chapter, which is concerned only with weak König's lemma and not at all with König's lemma for binary trees.

Chapter IV: WKL_0. Chapter IV focuses on Reverse Mathematics with respect to the formal system WKL_0 consisting of RCA_0 plus weak König's lemma. (See also the previous paragraph and §I.10.)

We begin in §IV.1 by showing that weak König's lemma is equivalent over RCA_0 to the *Heine/Borel covering lemma*: every covering of the closed unit interval $[0,1]$ by a sequence of open intervals has a finite subcovering. We then generalize this result by showing that WKL_0 proves a Heine/Borel covering property for arbitrary *compact metric spaces*. In order to obtain this generalization, we first prove a technical result: WKL_0 proves *bounded König's lemma*, *i.e.*, König's lemma for subtrees of $\mathbb{N}^{<\mathbb{N}}$ which are bounded. (A tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ is said to be *bounded* if there exists a function $g : \mathbb{N} \to \mathbb{N}$ such that $\tau(m) < g(m)$ for all $\tau \in T$, $m < \mathrm{lh}(\tau)$.) We also develop some additional technical results which are needed in later sections.

Section IV.2 shows that various properties of continuous functions on compact metric spaces are provable in WKL_0 and in fact equivalent to weak König's lemma over RCA_0. Among the properties considered are *uniform continuity*, *Riemann integrability*, the *Weierstraß polynomial approximation theorem*, and the *maximum principle*. A key technical notion here is that of *modulus of uniform continuity* (definition IV.2.1).

In §IV.3 we return to mathematical logic. We show that several well known theorems of mathematical logic, such as the *completeness theorem* and the *compactness theorem* for both propositional logic and predicate calculus, are each equivalent to weak König's lemma over RCA_0. Our results here in §IV.3 are to be contrasted with those of §II.8.

Sections IV.4, IV.5 and IV.6 deal with countable algebra in WKL_0. We show in §IV.5 that weak König's lemma is equivalent to the assertion that every countable field has a *unique algebraic closure*. (We have already seen in §II.9 that the *existence* of algebraic closures is provable in RCA_0.) In §IV.4 we discuss *formally real fields*, *i.e.*, fields in which -1 cannot be written as a sum of squares. We show that weak König's lemma is equivalent over RCA_0 to the assertion that every countable formally real field is *orderable*, and to the assertion that every countable formally real field has a *real closure*. In order to prove these results of Reverse Mathematics, we first prove a technical result characterizing WKL_0 in terms of Σ^0_1 *separation*; see also §I.11.

In §IV.6 we show that WKL_0 proves the existence of *prime ideals* in countable commutative rings. The argument for this result is somewhat interesting in that it involves not only two applications of weak König's lemma but also bounded Σ^0_1 comprehension. In addition, we obtain reversals showing that weak König's lemma is equivalent over RCA_0 to the existence of prime ideals, or even of radical ideals, in countable commutative rings. These results stand in contrast to §III.5, where we saw that ACA_0 is needed to prove the existence of *maximal ideals* in countable commutative rings. Thus it emerges that the usual textbook proof of the existence of prime ideals, via maximal ideals, is far from optimal with respect to its use of set existence axioms.

Sections IV.7, IV.8 and IV.9 are concerned with certain advanced topics in analysis. We begin in §IV.7 by showing that the well known *fixed point theorems* of Brouwer and Schauder are provable in WKL_0. In §IV.8 we use a fixed point technique to prove *Peano's existence theorem for solutions of ordinary differential equations*, in WKL_0. We also obtain reversals showing weak König's lemma is needed to prove the Brouwer and Schauder fixed point theorems and Peano's existence theorem. On the other hand, we note that the more familiar *Picard existence and uniqueness theorem*, assuming a Lipschitz condition, is already provable in RCA_0 alone.

Section IV.9 is concerned with Banach space theory in WKL_0. We build on the concepts and results of §§II.10 and IV.7. We begin by showing that yet another fixed point theorem, the *Markov/Kakutani theorem* for commutative families of affine maps, is provable in WKL_0. We then use this result to show that WKL_0 proves a version of the *Hahn/Banach extension theorem* for bounded linear functionals on separable Banach spaces. A reversal is also obtained.

Chapter V: ATR_0. Chapter V deals with mathematics in ATR_0, the formal system consisting of ACA_0 plus arithmetical transfinite recursion. (See also §I.11.) Many of the ordinary mathematical theorems considered in chapters V and VI are in the areas of countable combinatorics and classical descriptive set theory. The first few sections of chapter V focus on proving ordinary mathematical theorems in ATR_0. Reverse Mathematics with respect to ATR_0 is postponed to §V.5.

Chapter V begins with a preliminary §V.1 whose purpose is to elucidate the relationships among Σ^1_1 formulas, analytic sets, countable well orderings, and trees. An important tool is the *Kleene/Brouwer ordering* $\mathrm{KB}(T)$ of an arbitrary tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$. Key properties of the Kleene/Brouwer construction are: (1) $\mathrm{KB}(T)$ is always a linear ordering; (2) $\mathrm{KB}(T)$ is a well ordering if and only if T is well founded. The Kleene normal form theorem is proved in ACA_0 and is then used to show that any Π^1_1 assertion ψ can be expressed in ACA_0 by saying that an appropriately chosen tree T_ψ is well founded, or equivalently, $\mathrm{KB}(T_\psi)$ is a well ordering.

In §V.2 we define the formal system ATR_0 and observe that it is strong enough to accommodate a good theory of *countable ordinal numbers*, encoded by countable well orderings. In §V.3 we show that ATR_0 is also strong enough to accommodate a good theory of *Borel and analytic sets* in the Cantor space $2^{\mathbb{N}}$. In this setting, the well known theorems of Souslin ("B is Borel if and only if B and its complement are analytic") and Lusin ("any two disjoint analytic sets can be separated by a Borel set") are proved, along with a lesser known closure property of Borel sets ("the domain of a single-valued Borel relation is Borel"). In §V.4 we advance our examination of classical descriptive set theory by showing that the *perfect set* *theorem* ("every uncountable analytic set has a nonempty perfect subset") is provable in ATR_0. This last result uses an interesting technique known as the *method of pseudohierarchies*, or "nonstandard H-sets", *i.e.*, arithmetical transfinite recursion along countable linear orderings which are not well orderings.

In §V.5, most of the descriptive set-theoretic theorems mentioned in §§V.3 and V.4 are reversed, *i.e.*, shown to be equivalent over RCA_0 to ATR_0. The reversals are based on our characterization of ATR_0 in terms of Σ^1_1 *separation*. See also §I.11. We also present the following alternative characterization: ATR_0 is equivalent to the assertion that, for any sequence of trees $\langle T_i : i \in \mathbb{N} \rangle$, if each T_i has at most one path, then the set $\{i : T_i$ has a path$\}$ exists. This equivalence is based on a sharpening of the Kleene normal form theorem.

We have already observed that the development of mathematics within ATR_0 seems to go hand in hand with a good theory of countable ordinal numbers. In §V.6 we sharpen this observation by showing that ATR_0 is actually equivalent over RCA_0 to a certain statement which is obviously indispensable for any such theory. The statement in question is, "any two countable well orderings are comparable", abbreviated CWO. The proof that CWO implies ATR_0 is rather technical and uses what are called *double descent trees*.

In §V.7 we return to the study of countable Abelian groups (see also §§III.6 and VI.4). We show that ATR_0 is needed to prove *Ulm's theorem* for reduced Abelian p-groups, as well as some consequences of Ulm's theorem. The reversals use the fact that ATR_0 is equivalent to CWO. Ulm's theorem is of interest with respect to our Main Question, because it seems to be one of the few places in analysis or algebra where transfinite recursion plays an apparently indispensable role.

In §§V.8 and V.9 we consider two other topics in ordinary mathematics where strong set existence axioms arise naturally. These are (1) infinite game theory, and (2) the Ramsey property.

The games considered in §V.8 are Gale/Stewart games, *i.e.*, infinite games with perfect information. A payoff set $S \subseteq \mathbb{N}^{\mathbb{N}}$ is specified. Two players take turns choosing nonnegative integers m_1, n_1, m_2, n_2, $\ldots$, with full disclosure. The first player is declared the winner if the infinite sequence $\langle m_1, n_1, m_2, n_2, \ldots \rangle$ belongs to S. Otherwise the second player is declared the winner. Such a game is said to be *determined* if one player or the other has a winning strategy. Letting $\mathcal{S}$ be any class of payoff sets, $\mathcal{S}$-*determinacy* is the assertion that all games of this class are determined. It is well known that strong set existence axioms are correlated to determinacy for large classes of games. A striking result of this kind is due to Friedman [66, 71], who showed that Borel determinacy requires $\aleph_1$ applications of the power set axiom.

We show in §V.8 that ATR_0 proves *open determinacy*, *i.e.*, determinacy for all games in which the payoff set $S \subseteq \mathbb{N}^{\mathbb{N}}$ is open. This result uses pseudohierarchies, just as for the perfect set theorem. We also obtain a reversal, showing that open determinacy or even *clopen determinacy* is equivalent to ATR_0 over RCA_0. Our argument for the reversal proceeds via CWO. Along the way we obtain the following preliminary result: *determinacy for games of length* 3 is equivalent to ACA_0 over RCA_0.

As a consequence of open determinacy in ATR_0, we obtain the following interesting theorem: ATR_0 proves the Σ^1_1 *axiom of choice*. (More information on Σ^1_1 choice is in §VIII.4.)

In §V.9 we deal with a well known topological generalization of Ramsey's theorem. Let $[\mathbb{N}]^{\mathbb{N}}$ be the *Ramsey space*, *i.e.*, the space of all infinite subsets of $\mathbb{N}$. Note that $[\mathbb{N}]^{\mathbb{N}}$ is canonically homeomorphic to the Baire space $\mathbb{N}^{\mathbb{N}}$ via $\Phi : [\mathbb{N}]^{\mathbb{N}} \cong \mathbb{N}^{\mathbb{N}}$ defined by

$$\Phi^{-1}(f) \;=\; \{f(0) + 1 + \cdots + 1 + f(n) : n \in \mathbb{N}\}\ .$$

A set $S \subseteq [\mathbb{N}]^{\mathbb{N}}$ is said to have the *Ramsey property* if there exists $X \in [\mathbb{N}]^{\mathbb{N}}$ such that either $[X]^{\mathbb{N}} \subseteq S$ or $[X]^{\mathbb{N}} \cap S = \emptyset$. (Here $[X]^{\mathbb{N}}$ denotes the set of infinite subsets of X.) The main result of §V.9 is that ATR_0 is equivalent over RCA_0 to the *open Ramsey theorem*, *i.e.*, the assertion that every open subset of $[\mathbb{N}]^{\mathbb{N}}$ has the Ramsey property. The *clopen Ramsey theorem* is also seen to be equivalent over RCA_0 to ATR_0.

Chapter VI: Π^1_1-CA_0. Chapter VI is concerned with mathematics and Reverse Mathematics with respect to the formal system Π^1_1-CA_0, consisting of ACA_0 plus Π^1_1 comprehension. We show that Π^1_1-CA_0 is just strong enough to prove several theorems of ordinary mathematics. It is interesting to note that several of these ordinary mathematical theorems, which are equivalent to Π^1_1 comprehension, have "ATR_0 counterparts" which are equivalent to arithmetical transfinite recursion. Thus chapter VI on Π^1_1-CA_0 goes hand in hand with chapter V on ATR_0.

In §§VI.1 through VI.3 we consider several well known theorems of *classical descriptive set theory* in Π^1_1-CA_0. We begin in §VI.1 by showing that the *Cantor/Bendixson theorem* ("every closed set consists of a perfect set plus a countable set") is equivalent to Π^1_1 comprehension. This result for the Baire space $\mathbb{N}^{\mathbb{N}}$ and the Cantor space $2^{\mathbb{N}}$ is closely related to an analysis of trees in $\mathbb{N}^{<\mathbb{N}}$ and $2^{<\mathbb{N}}$, respectively. The ATR_0 counterpart of the Cantor/Bendixson theorem is, of course, the perfect set theorem (§V.4).

In §VI.2 we show that *Kondo's theorem* (coanalytic uniformization) is provable in Π^1_1-CA_0 and in fact equivalent to Π^1_1 comprehension over ATR_0.

The reversal uses an ATR_0 formalization of *Suzuki's theorem* on Π^1_1 singletons.

In §VI.3 we consider *Silver's theorem*: For any coanalytic equivalence relation with uncountably many equivalence classes, there exists a nonempty perfect set of inequivalent elements. We show that a certain carefully stated reformulation of Silver's theorem is provable in ATR_0. (See lemma VI.3.1. The proof of this lemma is somewhat technical and uses *formalized hyperarithmetical theory* (§VIII.3) as well as *Gandy forcing* over countable coded ω-models.) We then use this ATR_0 result to show that Silver's theorem itself is provable in Π^1_1-CA_0. We also present a reversal showing that Silver's theorem specialized to Δ^0_2 equivalence relations is equivalent to Π^1_1 comprehension over RCA_0 (theorem VI.3.6).

In §VI.4 we resume our study of countable algebra. We show that Π^1_1 comprehension is equivalent over RCA_0 to the assertion that every countable Abelian group can be written as the direct sum of a divisible group and a reduced group. The ATR_0 counterpart of this assertion is Ulm's theorem (§V.7). Combining these results, we see that Π^1_1-CA_0 is just strong enough to develop the classical *structure theory of countable Abelian groups* as presented in, for instance, Kaplansky [136].

In §§VI.5 and VI.6 we resume our study of determinacy and the Ramsey property. We show that Π^1_1 comprehension is just strong enough to prove $\Sigma^0_1 \wedge \Pi^0_1$ *determinacy* and the Δ^0_2 *Ramsey theorem*. The ATR_0 counterparts of these results are, of course, Σ^0_1 determinacy (*i.e.*, open determinacy) and the Σ^0_1 Ramsey theorem (*i.e.*, the open Ramsey theorem). Our proof technique in §VI.6 uses countable coded β-models (§VII.2).

Section VI.7 serves as an appendix to §§VI.5 and VI.6. In it we remark that stronger forms of Ramsey's theorem and determinacy require *stronger set existence axioms*. For instance, the Δ^1_1 *Ramsey theorem* (*i.e.*, the Galvin/Prikry theorem) and Δ^0_2 *determinacy* each require Π^1_1 *transfinite recursion* (theorem VI.7.3). Moreover, there are yet stronger forms of Ramsey's theorem and determinacy which go beyond Z_2 (remarks VI.7.6 and VI.7.7).

Note: The results in §VI.7 are stated without proof but with appropriate references to the published literature.

This completes our summary of part A.

Part B: Models of Subsystems of Z_2. Part B is a fairly thorough study of metamathematical properties of subsystems of Z_2. We consider not only the five basic systems RCA_0, WKL_0, ACA_0, ATR_0, and Π^1_1-CA_0 but also many other systems, including Δ^1_k-CA_0 (Δ^1_k comprehension), Π^1_k-CA_0 (Π^1_k comprehension), Σ^1_k-AC_0 (Σ^1_k choice), Σ^1_k-DC_0 (Σ^1_k dependent choice), Π^1_k-TR_0 (Π^1_k transfinite recursion), and Π^1_k-TI_0 (Π^1_k transfinite induction), for arbitrary k in the range $1 \leq k \leq \infty$. Table 4 lists these systems in order of increasing *logical strength*, also known as *consistency strength*.

We have found it convenient to divide the metamathematical material of part B into three chapters dealing with *β-models*, *ω-models*, and *non-ω-models* respectively. This threefold partition is perhaps somewhat misleading, and there are many cross-connections among the three chapters. This is mostly because the chapters which are ostensibly about β- and ω-models actually present their results in greater generality, so as to apply also to β- and *ω-submodels* of a given model, which need not itself be a β- or ω-model. Table 4 indicates where the main results concerning β-, ω- and non-ω-models of the various systems may be found.

Chapter VII: β-models. Recall from definition I.5.3 that a *β-model* is an ω-model M such that for any arithmetical formula $\theta(X)$ with parameters from M, if $\exists X\, \theta(X)$ then $(\exists X \in M)\, \theta(X)$. Such models are of importance because the concept of well ordering is absolute to them.

Throughout chapter VII, we find it convenient to consider a more general notion: M is a *β-submodel of* M' if M is a submodel of M' and, for all arithmetical formulas $\theta(X)$ with parameters from M, $M \models \exists X\, \theta(X)$ if and only if $M' \models \exists X\, \theta(X)$. Thus a β-model is the same thing as a β-submodel of the intended model $P(\omega)$.

Section VII.1 is introductory in nature. In it we characterize β-models of Π^1_1-CA_0 in terms of familiar recursion-theoretic notions. Namely, M is a β-model of Π^1_1-CA_0 if and only if M is closed under *relative recursiveness* and the *hyperjump*. We also obtain the obvious generalization to β-submodels. This is based on a formalized ACA_0 version of the *Kleene basis theorem*, according to which the sets recursive in $\mathrm{HJ}(X)$ form a basis for predicates which are arithmetical in X, provided $\mathrm{HJ}(X)$ exists.

In §VII.2 we consider *countable coded β-models*, *i.e.*, β-models of the form $M = \{(W)_n : n \in \mathbb{N}\}$ where $W \subseteq \mathbb{N}$ and $(W)_n = \{m : (m,n) \in W\}$. Within ACA_0 we define the notion of *satisfaction* for such models, and we prove within ACA_0 that every such model satisfies ATR_0 and all instances of the *transfinite induction scheme*, Π^1_∞-TI_0, given by

$$\forall X(\mathrm{WO}(X) \rightarrow \mathrm{TI}(X, \varphi))$$

where φ is an arbitrary L_2-formula. Here $\mathrm{WO}(X)$ says that X is a countable well ordering, and $\mathrm{TI}(X, \varphi)$ expresses transfinite induction along X with respect to φ. We also prove within ACA_0 that if $\mathrm{HJ}(X)$ exists then there is a countable coded β-model $M \leq_{\mathrm{T}} \mathrm{HJ}(X)$ such that $X \in M$. These considerations have a number of interesting consequences: (1) Π^1_∞-TI_0 includes ATR_0; (2) Π^1_∞-TI_0 is not finitely axiomatizable; (3) there exists a β-model of Π^1_∞-TI_0 which is not a model of Π^1_1-CA_0; (4) Π^1_1-CA_0 proves the

TABLE 4. Models of subsystems of Z_2.

	β-models	ω-models	non-ω-models
RCA_0		VIII.1	IX.1
WKL_0		VIII.2; see note 1	IX.2–IX.3
Π^0_1-AC_0		”	”
Π^0_1-DC_0		”	”
strong Π^0_1-DC_0		”	”
ACA_0		VIII.1; see note 2	IX.1, IX.4.3–IX.4.6
Δ^1_1-CA_0		VIII.4; see note 2	IX.4.3–IX.4.6
Σ^1_1-AC_0		”	”
Σ^1_1-DC_0		VIII.4–VIII.5; notes 2, 3	
Π^1_1-TI_0		”	
ATR_0	VII.2–VII.3, VIII.6	VIII.5–VIII.6; note 2	IX.4.7
Π^1_2-TI_0	VII.2.26-VII.2.32	see note 2	
Π^1_∞-TI_0	VII.2.14-VII.2.25	VIII.5.1-VIII.5.10; note 2	
strong Σ^1_1-DC_0	VII.6–VII.7	see notes 2 and 4	IX.4.8–IX.4.10
Π^1_1-CA_0	VII.1–VII.5, VII.7	”	”
Δ^1_2-CA_0	VII.5–VII.7	”	”
Σ^1_2-AC_0	VII.6	”	”
Σ^1_2-DC_0	”	”	
Π^1_1-TR_0	VII.1.18, VII.5.20, VII.7.12	VIII.4.24; see note 2	
strong Σ^1_2-DC_0	VII.6–VII.7	see notes 2 and 4	IX.4.8–IX.4.14
Π^1_{k+2}-CA_0	VII.5–VII.7	see note 2	”
Δ^1_{k+3}-CA_0	”	”	”
Σ^1_{k+3}-AC_0	VII.6	”	”
Σ^1_{k+3}-DC_0	”	”	
Π^1_{k+2}-TR_0	VII.5.20, VII.7.12	VIII.4.24; see note 2	
strong Σ^1_{k+3}-DC_0	VII.6–VII.7	see note 2	IX.4.8–IX.4.14
Π^1_∞-CA_0	VII.5–VII.7	”	
Σ^1_∞-AC_0	VII.6–VII.7	”	
Σ^1_∞-DC_0	”	”	

Notes:

1. Each of Π^0_1-AC_0 and Π^0_1-DC_0 and strong Π^0_1-DC_0 is equivalent to WKL_0. See lemma VIII.2.5.
2. The ω-model incompleteness theorem VIII.5.6 applies to any system $S \supseteq \mathsf{ACA}_0$. The ω-model hard core theorem VIII.6.6 applies to any system $S \supseteq$ weak Σ^1_1-AC_0. Quinsey's theorem VIII.6.12 applies to any system $S \supseteq \mathsf{ATR}_0$.
3. Π^1_1-TI_0 is equivalent to Σ^1_1-DC_0. See theorem VIII.5.12.
4. Σ^1_2-AC_0 is equivalent to Δ^1_2-CA_0. Σ^1_2-DC_0 is equivalent to Δ^1_2-CA_0 plus Σ^1_2 induction. Strong Σ^1_1-DC_0 and strong Σ^1_2-DC_0 are equivalent to Π^1_1-CA_0 and Π^1_2-CA_0, respectively. See remarks VII.6.3–VII.6.5 and theorem VII.6.9.

consistency of Π^1_∞-TI_0. We also obtain some technical results characterizing Π^1_2 sentences that are provable in Π^1_∞-TI_0 and in Π^1_2-TI_0.

In §VII.3 we introduce set-theoretic methods. We employ the language $\mathrm{L}_{\mathrm{set}} = \{\in, =\}$ of Zermelo/Fraenkel set theory. Of key importance is an $\mathrm{L}_{\mathrm{set}}$-theory $\mathsf{ATR}^{\mathrm{set}}_0$, among whose axioms are the *Axiom of Countability*, asserting that all sets are hereditarily countable, and *Axiom Beta*, asserting that for any regular (*i.e.*, well founded) binary relation r there exists a *collapsing function*, *i.e.*, a function f such that $f(u) = \{f(v) : \langle v, u\rangle \in r\}$ for all $u \in \mathrm{field}(r)$. By using well founded trees to encode hereditarily countable sets, we define a close relationship of mutual interpretability between ATR_0 and $\mathsf{ATR}^{\mathrm{set}}_0$. Under this interpretation, Σ^1_{k+1} formulas of L_2 correspond to Σ^{set}_k formulas of $\mathrm{L}_{\mathrm{set}}$ (theorem VII.3.24). Thus any formal system $T_0 \supseteq \mathsf{ATR}_0$ in L_2 is seen to have a *set-theoretic counterpart* T^{set}_0 in $\mathrm{L}_{\mathrm{set}}$ (definition VII.3.33). We point out that several familiar subsystems of Z_2 have elegant characterizations in terms of their set-theoretic counterparts. For instance, the principal axiom of Π^0_∞-$\mathsf{TI}^{\mathrm{set}}_0$ is the $\in$-induction scheme, and the principal axiom of Σ^1_2-$\mathsf{AC}^{\mathrm{set}}_0$ is Σ^{set}_1 collection.

In §VII.4 we explore Gödel's theory of *constructible sets* in a form appropriate for the study of subsystems of Z_2. We begin by defining within $\mathsf{ATR}^{\mathrm{set}}_0$ the inner model L^u of sets constructible from u, where u is any given nonempty transitive set. After that, we turn to *absoluteness results*. We prove within Π^1_1-$\mathsf{CA}^{\mathrm{set}}_0$ that the formula "r is a regular relation" is absolute to L^u. This fact is used to prove Π^1_1-$\mathsf{CA}^{\mathrm{set}}_0$ versions of the well known absoluteness theorems of Shoenfield and Lévy. We consider the inner models $\mathrm{L}(X)$ and $\mathrm{HCL}(X)$ of sets that are constructible from X and *hereditarily constructibly countable* from X, respectively, where $X \subseteq \omega$. We prove within Π^1_1-$\mathsf{CA}^{\mathrm{set}}_0$ that $\mathrm{HCL}(X)$ satisfies Π^1_1-$\mathsf{CA}^{\mathrm{set}}_0$ plus $\mathrm{V} = \mathrm{HCL}(X)$, and that Σ^1_2 and Σ^{set}_1 formulas are absolute to $\mathrm{HCL}(X)$. We prove within $\mathsf{ATR}^{\mathrm{set}}_0$ that if $\mathrm{HCL}(X) \neq \mathrm{L}(X)$ then $\mathrm{HCL}(X)$ satisfies Π^1_∞-$\mathsf{CA}^{\mathrm{set}}_0$.

In §§VII.5, VII.6 and VII.7 we apply our results on constructible sets to the study of β-models of subsystems of second order arithmetic which are stronger than Π^1_1-CA_0.

Section VII.5 is concerned with *strong comprehension schemes*. The main result is that if T_0 is any one of the systems Π^1_1-CA_0, Δ^1_2-CA_0, Π^1_2-CA_0, Δ^1_3-CA_0, ... , then T_0 implies its own relativization to the inner models $\mathrm{L}(X) \cap P(\mathbb{N})$, $X \subseteq \mathbb{N}$. This has several interesting consequences: (1) $T_0 + \exists X\, \forall Y\, (Y \in \mathrm{L}(X))$ is conservative over T_0 for Π^1_4 sentences; (2) T_0 has a *minimum β-model*, and this minimum β-model is of the form $\mathrm{L}_\alpha \cap P(\omega)$ where α is an appropriately chosen countable ordinal. (These minimum β-models and their corresponding ordinals turn out to be distinct from one another; see §VII.7.) We also present generalizations involving minimum β-submodels of a given model.

Section VII.6 is concerned with several *strong choice schemes*, *i.e.*, instances of the axiom of choice expressible in the language of second order arithmetic. Among the schemes considered are Σ^1_k *choice*

$$\forall n\,\exists Y\,\eta(n,Y) \rightarrow \exists Z\,\forall n\,\eta(n,(Z)_n)\,,$$

Σ^1_k *dependent choice*

$$\forall n\,\forall X\,\exists Y\,\eta(n,Y) \rightarrow \exists Z\,\forall n\,\eta(n,(Z)^n,(Z)_n)\,,$$

and *strong* Σ^1_k *dependent choice*

$$\exists Z\,\forall n\,\forall Y(\eta(n,(Z)^n,Y) \rightarrow \eta(n,(Z)^n,(Z)_n))\,.$$

The corresponding formal systems are known as Σ^1_k-AC_0, Σ^1_k-DC_0, and strong Σ^1_k-DC_0, respectively. The case $k = 2$ is somewhat special. We show that Δ^1_2 comprehension implies Σ^1_2 choice, and even Σ^1_2 dependent choice provided Σ^1_2 induction is assumed. We also show that strong Σ^1_2 dependent choice is equivalent to Π^1_2 comprehension. These equivalences for $k = 2$ are based on the fact that Σ^1_2 *uniformization* is provable in Π^1_1-CA_0. Two proofs of this fact are given, one via Kondo's theorem and the other via Shoenfield absoluteness.

For $k \geq 3$ we obtain similar equivalences under the additional assumption $\exists X\,\forall Y\,(Y \in \mathrm{L}(X))$, via Σ^1_k *uniformization.* We then apply our conservation theorems of the previous section to see that, for each $k \geq 3$, Σ^1_k choice and strong Σ^1_k dependent choice are conservative for Π^1_4 sentences over Δ^1_k comprehension and Π^1_k comprehension, respectively. Other results of a similar character are obtained. The case $k = 1$ is of a completely different character, and its treatment is postponed to §VIII.4.

Section VII.7 begins by generalizing the concept of β-model to β_k*-model*, *i.e.*, an ω-model M such that all Σ^1_k formulas with parameters from M are absolute to M. (Thus a β_1-model is the same thing as a β-model.) It is shown that, for each $k \geq 1$,

$$\forall X\,\exists M\,(X \in M \wedge M \text{ is a countable coded } \beta_k\text{-model})$$

is equivalent to strong Σ^1_k dependent choice. This implies a kind of β_k*-model reflection principle* (theorem VII.7.6). Combining this with the results of §§VII.5 and VII.6, we obtain several noteworthy corollaries, *e.g.*, the fact that Δ^1_{k+1}-CA_0 proves the existence of a countable coded β-model of Π^1_k-CA_0 which in turn proves the existence of a countable coded β-model of Δ^1_k-CA_0. From this it follows that the minimum β-models of Π^1_1-CA_0, Δ^1_2-CA_0, Π^1_2-CA_0, Δ^1_3-CA_0, ... are all distinct.

Chapter VIII: ω-models. The purpose of chapter VIII is to study ω-models of various subsystems of Z_2. We focus primarily on the five basic systems: RCA_0, WKL_0, ACA_0, ATR_0, Π^1_1-CA_0. We note that each of these systems is finitely axiomatizable. We also obtain some general results about fairly arbitrarily L_2-theories, which may be stronger than Π^1_1-CA_0 and need not be finitely axiomatizable. Many of our results on ω-models are formulated more generally, so as to apply also to ω-submodels of a given non-ω-model.

Section VIII.1 is introductory in nature. We characterize models of RCA_0 and ACA_0 in terms of Turing reducibility and the Turing jump operator. We show that the *minimum* ω*-models of* RCA_0 *and* ACA_0 are $\mathrm{REC} = \{X : X$ is recursive$\}$ and $\mathrm{ARITH} = \{X : X$ is arithmetical$\}$ respectively. We apply the strong soundness theorem and countable coded ω-models to show that ATR_0 proves the *consistency of* ACA_0, which in turn proves the *consistency of* RCA_0.

In §VIII.2 we consider models of WKL_0. We begin by showing that WKL_0 proves *strong* Π^0_1 *dependent choice*, which in turn implies the existence of a countable coded *strict* β*-model*. Such a model necessarily satisfies WKL_0, so we are surprisingly close to asserting that WKL_0 proves its own consistency (see however remark VIII.2.14). In particular, ACA_0 actually does prove the *consistency of* WKL_0, via countable coded ω-models (corollary VIII.2.12). Moreover, WKL_0 *has no minimal* ω*-model* (corollary VIII.2.8).

The rest of §VIII.2 is concerned with the *basis problem*: Given an infinite recursive tree $T \subseteq 2^{<\omega}$, to find a path through T which is in some sense "close to being recursive." We obtain three results, the *low basis theorem*, the *almost recursive basis theorem*, and the *GKT basis theorem*, which provide various solutions of the basis problem. They also imply the existence of *countable* ω*-models of* WKL_0 with various properties (theorems VIII.2.17, VIII.2.21, VIII.2.24). In particular, REC *is the intersection of all* ω*-models of* WKL_0 (corollary VIII.2.27).

In §VIII.3 we develop the technical machinery of *formalized hyperarithmetical theory.* We define the H*-sets* H^X_a for $X \subseteq \mathbb{N}$ and $a \in \mathcal{O}^X$. We note that ATR_0 is equivalent to $\forall X\, \forall a\, (\mathcal{O}(a, X) \rightarrow \mathrm{H}^X_a$ exists). We prove ATR_0 versions of the major classical results: *invariance of Turing degree* (VIII.3.13); $\Delta^1_1 = \mathrm{HYP}$ (VIII.3.19); the theorem on *hyperarithmetical quantifiers* (VIII.3.20, VIII.3.27). The latter result involves *pseudohierarchies.* An unorthodox feature of our exposition is that we do not use the recursion theorem.

In §VIII.4 we use the machinery of §VIII.3 to study ω-models of the systems Δ^1_1-CA_0, Σ^1_1-AC_0, and Σ^1_1-DC_0. We also consider a closely related system known as *weak* Σ^1_1-AC_0. We show that $\mathrm{HYP} = \{X : X$ is hyperarithmetical$\}$ is the *minimum* ω*-model* of each of these four systems. The proof of this result uses Π^1_1 *uniformization.* Although the main results of classical hyperarithmetical theory are provable in ATR_0 (§VIII.3), the existence of the ω-model HYP is not (remark VIII.4.4). Nevertheless, we show that ATR_0 proves the existence of countable coded ω-models of

Σ^1_1-AC_0 *etc.* (theorem VIII.4.20). Indeed, ATR_0 proves that HYP is the intersection of all such ω-models (theorem VIII.4.23). In particular, ATR_0 proves the *consistency of* Σ^1_1-AC_0 *etc.*

In §VIII.5 we present two surprising theorems of Friedman which apply to fairly arbitrary L_2-theories $S \supseteq \mathsf{ACA}_0$. They are: (1) If S is recursively axiomatizable and has an ω-model, then so does $S \wedge \neg\,\exists$ countable coded ω-model of S. (2) If S is finitely axiomatizable, then Π^1_∞-TI_0 proves $S \to \exists$ countable coded ω-model of S. Note that (1) is an *ω-model incompleteness theorem*, while (2) is an *ω-model reflection principle*. Combining (1) and (2), we see that if S is finitely axiomatizable and has an ω-model, then there exists an ω-model of S which is does not satisfy Π^1_∞-TI_0 (corollary VIII.5.8).

At the end of §VIII.5 we prove that Π^1_1 *transfinite induction* is equivalent to *ω-model reflection for Σ^1_3 formulas*, which is equivalent to Σ^1_1 *dependent choice* (theorem VIII.5.12). From this it follows that there exists an ω-model of ATR_0 in which Σ^1_1-DC_0 fails (theorem VIII.5.13). This is in contrast to the fact that ATR_0 implies Σ^1_1-AC_0 (theorem V.8.3).

Section VIII.6 presents several *hard core theorems*. We show that any model M of ATR_0 has a proper β-submodel; indeed, by corollary VIII.6.10, HYP^M is the intersection of all such submodels. We also prove the following theorem of Quinsey: if M is any ω-model of a recursively axiomatizable L_2-theory $S \supseteq \mathsf{ATR}_0$, then M has a proper submodel which is again a model of S (theorem VIII.6.12). Indeed, HYP^M is the intersection of all such submodels (exercise VIII.6.23). In particular, no such S has a minimal ω-model.

Chapter IX: non-ω-models. In chapter IX we study non-ω-models of various subsystems of Z_2. Section IX.1 deals with RCA_0 and ACA_0. Sections IX.2 and IX.3 are concerned with WKL_0. Section IX.4 is concerned with various systems including Π^1_k-CA_0 and Σ^1_k-AC_0, $k \geq 0$. For most of the results of chapter IX, it is essential that our systems contain only restricted induction and not full induction. Many of the results can be phrased as conservation theorems. The methods of §§IX.3 and IX.4 depend crucially on the existence of nonstandard integers.

We begin in §IX.1 by showing that every model M of PA can be expanded to a model of ACA_0. The expansion is accomplished by letting $\mathcal{S}_M = \mathrm{Def}(M) = \{X \subseteq |M| : X$ is first order definable over M allowing parameters from $M\}$. From this it follows that PA is the first order part of ACA_0, and that ACA_0 has the same consistency strength as PA. We then prove analogous results for RCA_0. Namely, every model M of Σ^0_1-PA can be expanded to a model of RCA_0; the expansion is accomplished by letting $\mathcal{S}_M = \Delta^0_1$-$\mathrm{Def}(M) = \{X \subseteq |M| : X$ is Δ^0_1 definable over M allowing parameters from $M\}$. The delicate point of this argument is to show that the expansion preserves Σ^0_1 induction. It follows that Σ^0_1-PA is the first order part of RCA_0, and that RCA_0 has the same consistency strength as Σ^0_1-PA.

In §IX.2 we show that WKL_0 has the same first order part and consistency strength as RCA_0. This is based on the following model-theoretic result due to Harrington: Given a countable model M of RCA_0, we can construct a countable model M' of WKL_0 such that M is an ω-submodel of M'. The model M' is obtained from M by iterated forcing, where at each stage we force with trees to add a generic path through a tree. Again, the delicate point is to verify that Σ^0_1 induction is preserved. This model-theoretic result implies that WKL_0 is conservative over RCA_0 for Π^1_1 sentences.

In §IX.3 we introduce the well known formal system PRA of *primitive recursive arithmetic*. This theory of primitive recursive functions contains a function symbol and defining axioms for each such function. We prove the following result of Friedman: WKL_0 has the same consistency strength as PRA and is conservative over PRA for Π^0_2 sentences. Our proof uses a model-theoretic method due to Kirby and Paris, involving *semiregular cuts*. The foundational significance of PRA is that it embodies *Hilbert's concept of finitism*. Therefore, Friedman's theorem combined with the mathematical work of chapters II and IV shows that a significant portion of mathematical practice is finitistically reducible. Thus we have a *partial realization of Hilbert's program*; see also remark IX.3.18.

In §IX.4 we use *recursively saturated models* to prove some surprising conservation theorems for various subsystems of Z_2. The main results may be summarized as follows: For each $k \geq 0$, Σ^1_{k+1}-AC_0 has the same consistency strength as Π^1_k-CA_0 and is conservative over Π^1_k-CA_0 for Π^1_l sentences, $l = \min(k+2, 4)$. These results are due to Barwise/Schlipf, Feferman, Friedman, and Sieg. We also obtain a number of related results.

Section IX.5 is a very brief discussion of Gentzen-style proof theory, with emphasis on provable ordinals of subsystems of Z_2.

This completes our summary of part B.

Appendix: Chapter X: Additional Results. Chapter X is an appendix in which some additional Reverse Mathematics results and problems are presented without proof but with references to the published literature.

In §X.1 we consider *measure theory* in subsystems of Z_2. We introduce the formal system WWKL_0 consisting of RCA_0 plus *weak weak König's lemma* and show that it is just strong enough to prove several measure theoretic results, *e.g.*, the *Vitali covering theorem.* We also consider measure theory in stronger systems such as ACA_0.

In §X.2 we mention some additional results on *separable Banach spaces* in subsystems of Z_2. We note that WKL_0 is just strong enough to prove *Banach separation*. We develop various notions related to the *weak-$*$ topology* on X^*, the dual of a separable Banach space. We show that Π^1_1-CA_0 is just strong enough to prove the existence of the *weak-$*$-closed linear span* of a countable set Y in X^*.

In §X.3 we consider *countable combinatorics* in subsystems of Z_2. We note that *Hindman's theorem* lies between ACA_0 and a slightly stronger system, ACA_0^+. We mention a similar result for the closely related *Auslander/Ellis theorem* of topological dynamics. In the area of *matching theory*, we show that the *Podewski/Steffens theorem* ("every countable bipartite graph has a König covering") is equivalent to ATR_0. At the end of the section we consider *well quasiordering theory*, noting for instance that the *Nash-Williams transfinite sequence theorem* lies between ATR_0 and Π^1_1-CA_0.

In §X.4 we initiate a project of weakening the base theory for Reverse Mathematics. We introduce a system RCA_0^* which is essentially RCA_0 with Σ^0_1 induction weakened to Σ^0_0 induction. We also introduce a system WKL_0^* consisting of RCA_0^* plus weak König's lemma. We present some conservation results showing in particular that RCA_0^* and WKL_0^* have the same consistency strength as EFA, elementary function arithmetic. We note that several theorems of countable algebra are equivalent over RCA_0^* to Σ^0_1 induction. Among these are: (1) every polynomial over a countable field has an irreducible factor; (2) every finitely generated vector space over $\mathbb{Q}$ has a basis.

I.14. Conclusions

In this chapter we have presented and motivated the main themes of the book, including the Main Question (§§I.1, I.12) and Reverse Mathematics (§I.9). A detailed outline of the book is in section I.13. The five most important subsystems of second order arithmetic are RCA_0, WKL_0, ACA_0, ATR_0, Π^1_1-CA_0. Part A of the book consists of chapters II through VI and focuses on the development of mathematics in these five systems. Part B consists of chapters VII through IX and focuses on models of these and other subsystems of Z_2. Additional results are presented in an appendix, chapter X.

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