VITALI'S THEOREM AND WWKL

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Abstract. Continuing the investigations of X. Yu and others, we study the role of set existence axioms in classical Lebesgue measure theory. We show that pairwise disjoint countable additivity for open sets of reals is provable in RCA_0. We show that several well-known measure-theoretic propositions including the Vitali Covering Theorem are equivalent to WWKL over RCA_0.

1. Introduction

The purpose of Reverse Mathematics is to study the role of set existence axioms, with an eye to determining which axioms are needed in order to prove specific mathematical theorems. In many cases, it is shown that a specific mathematical theorem is equivalent to the set existence axiom which is needed to prove it. Such equivalences are often proved in the weak base theory RCA_0. RCA_0 may be viewed as a kind of formalized constructive or recursive mathematics, with full classical logic but severely restricted comprehension and induction. The program of Reverse Mathematics has been developed in many publications; see for instance [5, 10, 11, 12, 20].

In this paper we carry out a Reverse Mathematics study of some aspects of classical Lebesgue measure theory. Historically, the subject of measure theory developed hand in hand with the nonconstructive, set-theoretic approach to mathematics. Errett Bishop has remarked that the foundations of measure theory present a special challenge to the constructive mathematician. Although our program of Reverse Mathematics is quite different from Bishop-style constructivism, we feel that Bishop's remark implicitly raises an interesting question: *Which nonconstructive set existence axioms are needed for measure theory?*

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This paper, together with earlier papers of Yu and others [21, 22, 23, 24, 25, 26], constitute an answer to that question.

The results of this paper build upon and clarify some early results of Yu and Simpson. The reader of this paper will find that familiarity with Yu–Simpson [26] is desirable but not essential. We begin in section 2 by exploring the extent to which measure theory can be developed in RCA_0. We show that pairwise disjoint countable additivity for open sets of reals is provable in RCA_0. This is in contrast to a result of Yu–Simpson [26]: countable additivity for open sets of reals is equivalent over RCA_0 to a nonconstructive set existence axiom known as Weak Weak König's Lemma (WWKL). We show in sections 3 and 4 that several other basic propositions of measure theory are also equivalent to WWKL over RCA_0. Finally in section 5 we show that the Vitali Covering Theorem is likewise equivalent to WWKL over RCA_0.

2. Measure Theory in RCA_0

Recall that RCA_0 is the subsystem of second order arithmetic with Δ^0_1 comprehension and Σ^0_1 induction. The purpose of this section is to show that some measure-theoretic results can be proved in RCA_0.

Within RCA_0, let X be a compact separable metric space. We define $C(X) = \widehat{A}$, the completion of A, where A is the vector space of rational "polynomials" over X under the sup-norm, $\|f\| = \sup_{x \in X} |f(x)|$. For the precise definitions within RCA_0, see [26] and section III.E of Brown's thesis [4]. The construction of $C(X)$ within RCA_0 is inspired by the constructive Stone–Weierstrass theorem in section 4.5 of Bishop and Bridges [2]. It is provable in RCA_0 that there is a natural one-to-one correspondence between points of $C(X)$ and continuous functions $f : X \to \mathbb{R}$ which are equipped with a *modulus of uniform continuity*, that is to say, a function $h : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$ and x, $y \in X$

$$d(x,y) \;<\; \frac{1}{2^{h(n)}} \qquad \text{implies} \qquad |f(x) - f(y)| \;<\; \frac{1}{2^n}\,.$$

Within RCA_0 we define a *measure* (more accurately, a nonnegative Borel probability measure) on X to be a nonnegative bounded linear functional $\mu : C(X) \to \mathbb{R}$ such that $\mu(1) = 1$. (Here $\mu(1)$ denotes $\mu(f)$, $f \in C(X)$, $f(x) = 1$ for all $x \in X$.) For example, if $X = [0,1]$, the unit interval, then there is an obvious measure $\mu_L : C([0,1]) \to \mathbb{R}$ given by $\mu_L(f) = \int_0^1 f(x)\,dx$, the Riemann integral of f from 0 to 1. We refer to μ_L as *Lebesgue measure* on $[0,1]$. There is also the obvious generalization to Lebesgue measure μ_L on $X = [0,1]^n$, the n-cube.

Definition 2.1 (measure of an open set)**.** This definition is made in RCA_0. Let X be any compact separable metric space, and let μ be any measure on X. If U is an open set in X, we define

$$\mu(U) \;=\; \sup\{\,\mu(f) \mid f \in C(X)\,,\, 0 \le f \le 1\,,\, f = 0 \text{ on } X \setminus U\,\}\,.$$

Within RCA_0 this supremum need not exist as a real number. (Indeed, the existence of $\mu(U)$ for all open sets U is equivalent to ACA_0 over RCA_0.) Therefore, when working within RCA_0, we interpret assertions about $\mu(U)$ in a "virtual" or comparative sense. For example, $\mu(U) \le \mu(V)$ is taken to mean that for all $\epsilon > 0$ and all $f \in C(X)$ with $0 \le f \le 1$ and $f = 0$ on $X \setminus U$, there exists $g \in C(X)$ with $0 \le g \le 1$ and $g = 0$ on $X \setminus V$ such that $\mu(f) \le \mu(g) + \epsilon$. See also [26].

Some basic properties of Lebesgue measure are easily proved in RCA_0. For instance, it is straightforward to show that the Lebesgue measure of the union of a finite set of pairwise disjoint open intervals is equal to the sum of the lengths of the intervals.

We define $L_1(X, \mu)$ to be the completion of $C(X)$ under the L_1-norm given by $\|f\|_1 = \mu(|f|)$. (For the precise definitions, see [5] and [26].) In RCA_0 we see that $L_1(X, \mu)$ is a separable Banach space, but to assert within RCA_0 that points of the Banach space $L_1(X, \mu)$ represent measurable functions $f : X \to \mathbb{R}$ is problematic. We shall comment further on this question in section 4 below.

Lemma 2.2. *The following is provable in* RCA_0. *If* U_n, $n \in \mathbb{N}$, *is a sequence of open sets, then*

$$\mu\left(\bigcup_{n=0}^{\infty} U_n\right) \;\ge\; \lim_{k\to\infty} \mu\left(\bigcup_{n=0}^{k} U_n\right)\,.$$

Proof. Trivial. □

Lemma 2.3. *The following is provable in* RCA_0. *If* $U_0, U_1, \ldots, U_k$ *is a finite, pairwise disjoint sequence of open sets, then*

$$\mu\left(\bigcup_{n=0}^{k} U_n\right) \;\ge\; \sum_{n=0}^{k} \mu(U_n)\,.$$

Proof. Trivial. □

An open set is said to be *connected* if it is not the union of two disjoint nonempty open sets. Let us say that a compact separable metric space X is *nice* if for all sufficiently small $\delta > 0$ and all $x \in X$, the open ball

$$B(x, \delta) \;=\; \{\, y \in X \mid d(x, y) < \delta\,\}$$

is connected. Such a δ is called a *modulus of niceness* for X.

For example, the unit interval $[0, 1]$ and the n-cube $[0, 1]^n$ are nice, but the Cantor space $2^{\mathbb{N}}$ is not nice.

Theorem 2.4 (disjoint countable additivity)**.** *The following is provable in* RCA_0*. Assume that* X *is nice. If* U_n*,* $n \in \mathbb{N}$*, is a pairwise disjoint sequence of open sets in* X*, then*

$$\mu\left(\bigcup_{n=0}^{\infty} U_n\right) = \sum_{n=0}^{\infty} \mu\left(U_n\right).$$

Proof. Put $U = \bigcup_{n=0}^{\infty} U_n$. Note that U is an open set. By Lemmas 2.2 and 2.3, we have in RCA_0 that $\mu(U) \geq \sum_{n=0}^{\infty} \mu(U_n)$. It remains to prove in RCA_0 that $\mu(U) \leq \sum_{n=0}^{\infty} \mu(U_n)$. Let $f \in C(X)$ be such that $0 \leq f \leq 1$ and $f = 0$ on $X \setminus U$. It suffices to prove that $\mu(f) \leq \sum_{n=0}^{\infty} \mu(U_n)$.

Claim 1: There is a sequence of continuous functions $f_n : X \to \mathbb{R}$, $n \in \mathbb{N}$, defined by $f_n(x) = f(x)$ for all $x \in U_n$, $f_n(x) = 0$ for all $x \in X \setminus U_n$.

To prove this in RCA_0, recall from [6] or [20] that a *code* for a continuous function g from X to Y is a collection G of quadruples (a, r, b, s) with certain properties, the idea being that $d(a, x) < r$ implies $d(b, g(x)) \leq s$. Also, a code for an open set U is a collection of pairs (a, r) with certain properties, the idea being that $d(a, x) < r$ implies $x \in U$. In this case we write $(a, r) < U$ to mean that $d(a, b)+r < s$ for some (b, s) belonging to the code of U. Now let F be a code for $f : X \to \mathbb{R}$. Define a sequence of codes F_n, $n \in \mathbb{N}$, by putting (a, r, b, s) into F_n if and only if

1. (a, r, b, s) belongs to F and $(a, r) < U_n$, or
2. (a, r, b, s) belongs to F and $b - s \leq 0 \leq b + s$, or
3. $b - s \leq 0 \leq b + s$ and $(a, r) < U_m$ for some $m \neq n$.

It is straightforward to verify that F_n is a code for f_n as required by claim 1.

Claim 2: The sequence f_n, $n \in \mathbb{N}$, is a sequence of elements of $C(X)$.

To prove this in RCA_0, we must show that the sequence of f_n's has a sequence of moduli of uniform continuity. Let $h : \mathbb{N} \to \mathbb{N}$ be a modulus of uniform continuity for f, and let k be so large that $1/2^k$ is a modulus of niceness for X. We shall show that $h' : \mathbb{N} \to \mathbb{N}$ defined by $h'(m) = \max(h(m), k)$ is a modulus of uniform continuity for all of the f_n's. Let $x, y \in X$ and $m \in \mathbb{N}$ be such that $d(x, y) < 1/2^{h'(m)}$. To show that $|f_n(x) - f_n(y)| < 1/2^m$, we consider three cases. Case 1:

$x, y \in U_n$. In this case we have

$$|f_n(x) - f_n(y)| \;=\; |f(x) - f(y)| \;<\; \frac{1}{2^m}\;.$$

Case 2: $x, y \in X \setminus U_n$. In this case we have $f_n(x) = f_n(y) = 0$ so $|f_n(x) - f_n(x)| = 0$. Case 3: $x \in U_n$, $y \in X \setminus U_n$. Note that $f_n(x) = f(x)$, while $f_n(y) = 0$. Put $B = B(x, 1/2^{h'(m)})$. Then B is connected and $x, y \in B$. If $B \subseteq U$, then we would have

$$B \;=\; (B \cap U_n) \cup (B \cap (U \setminus U_n))$$

and this would be a decomposition of B into two disjoint nonempty open sets, a contradiction. Thus $B \setminus U$ is nonempty. Let z be a point of $B \setminus U$. Then $f(z) = 0$ and hence $|f_n(x) - f_n(y)| = |f_n(x)| = |f(x)| = |f(x) - f(z)| < 1/2^m$. This proves claim 2.

Claim 3:

$$\mu(f) \;=\; \sum_{n=0}^{\infty} \mu(f_n)\;. \tag{1}$$

Note that by definition we already have

$$f(x) \;=\; \sum_{n=0}^{\infty} f_n(x)$$

for all $x \in X$. Since $\mu : C(X) \to \mathbb{R}$ is a bounded linear functional, (1) will follow if we show that the series $\sum_{n=0}^{\infty} f_n$ converges uniformly to f. If this were not the case, then there would be an $\epsilon > 0$ such that, for infinitely many n, $f(x) > \epsilon$ for some $x \in U_n$. By uniform continuity of f, let $\delta > 0$ be such that, for all x and $y \in X$, $d(x, y) < \delta$ implies $|f(x) - f(y)| < \epsilon$. We may safely assume that δ is a modulus of niceness for X. Now consider n and x such that $x \in U_n$ and $f(x) > \epsilon$. Clearly $f(y) > 0$ for all $y \in B(x, \delta)$; hence $B(x, \delta) \subseteq U$. Since both U_n and $U \setminus U_n$ are open, and since $B(x, \delta)$ is connected, it follows that $B(x, \delta) \subseteq U_n$. Thus, for infinitely many n, we have $\exists x\, B(x, \delta) \subseteq U_n$. On the other hand, by compactness of X, there exists a finite sequence of points $x_1, \ldots, x_k \in X$ such that $X = \bigcup_{i=1}^{k} B(x_i, \delta)$. It follows that for infinitely many n we have $x_i \in U_n$ for some i, $1 \le i \le k$. Since the predicate $x_i \in U_n$ is Σ^0_1, it follows by Σ^0_1 induction and bounded Σ^0_1 comprehension in RCA_0 that there exist m, n and i such that $m \neq n$ and $1 \le i \le k$ and $x_i \in U_m$ and $x_i \in U_n$. This contradicts our assumption that the sets U_n, $n \in \mathbb{N}$, are pairwise disjoint. Equation (1) and claim 3 are now proved.

From (1) we see that for each $\epsilon > 0$ there exists k such that $\mu(f)-\epsilon \leq \sum_{n=0}^{k} \mu(f_n)$. Thus we have

$$\mu(f) - \epsilon \;\leq\; \sum_{n=0}^{k} \mu(f_n) \;\leq\; \sum_{n=0}^{k} \mu(U_n) \;\leq\; \sum_{n=0}^{\infty} \mu(U_n) \;.$$

Since this holds for all $\epsilon > 0$, it follows that $\mu(f) \leq \sum_{n=0}^{\infty} \mu(U_n)$. Thus $\mu(U) \leq \sum_{n=0}^{\infty} \mu(U_n)$ and the proof of Theorem 2.4 is complete. □

Corollary 2.5. *The following is provable in* RCA_0. *If* (a_n, b_n), $n \in \mathbb{N}$ *is a sequence of pairwise disjoint open intervals, then*

$$\mu_L\left(\bigcup_{n=0}^{\infty}(a_n, b_n)\right) \;=\; \sum_{n=0}^{\infty} |a_n - b_n| \;.$$

Proof. This is a special case of Theorem 2.4. □

Remark 2.6. Theorem 2.4 fails if we drop the assumption that X is nice. Indeed, let μ_C be the familiar "fair coin" measure on the Cantor space $X = 2^{\mathbb{N}}$, given by $\mu_C(\{x|x(n) = i\}) = 1/2$ for all $n \in \mathbb{N}$ and $i \in \{0, 1\}$. It can be shown that disjoint finite additivity for μ_C is equivalent to WWKL over RCA_0. (WWKL is defined and discussed in the next section.) In particular, disjoint finite additivity for μ_C is not provable in RCA_0.

3. Measure Theory in WWKL_0

Yu and Simpson [26] introduced a subsystem of second order arithmetic known as WWKL_0, consisting of RCA_0 plus the following axiom: *if T is a subtree of $2^{<\mathbb{N}}$ with no infinite path, then*

$$\lim_{n\to\infty} \frac{|\{\,\sigma \in T \mid \text{length}(\sigma) = n\}|}{2^n} \;=\; 0\,. \tag{2}$$

This axiom is known as Weak Weak König's Lemma (WWKL). It is a weaker axiom than Weak König's Lemma (WKL), which reads as follows: *if T is a subtree of $2^{<\mathbb{N}}$ with no infinite path, then T is finite.*

Remark 3.1. Yu and Simpson [26] constructed an ω-model of WWKL_0 (namely a random real model) which is not an ω-model of WKL_0. In addition, Yu and Simpson [26] pointed out that the recursive sets form an ω-model of RCA_0 which is not an ω-model of WWKL_0. Thus WWKL_0 is properly weaker than WKL_0 and properly stronger than RCA_0. Furthermore, the mathematical content of WKL_0 and WWKL_0 is known to be nonconstructive. On the other hand, WKL_0 and therefore WWKL_0 are known to be conservative over Primitive Recursive Arithmetic for Π^0_2 sentences. This conservation result for WKL_0 is due to Friedman [9];

see also Sieg [18]. In this sense, every mathematical theorem provable in WKL_0 or WWKL_0 is finitistically reducible in the sense of Hilbert's Program; see [19, 6, 20].

Remark 3.2. The study of ω-models of WWKL_0 is closely related to the theory of 1-random sequences, as initiated by Martin-Löf [16] and continued by Kučera [7, 13, 14, 15]. At the time of writing of [26], Yu and Simpson were unaware of this work of Martin-Löf and Kučera.

The purpose of this section and the next is to review and extend the results of [26] and [21] concerning measure theory in WWKL_0.

A measure $\mu : C(X) \to \mathbb{R}$ on a compact separable metric space X is said to be *countably additive* if

$$\mu\left(\bigcup_{n=0}^{\infty} U_n\right) \;=\; \lim_{k\to\infty} \mu\left(\bigcup_{n=0}^{k} U_n\right)$$

for any sequence of open sets U_n, $n \in \mathbb{N}$, in X. The following theorem is implicit in [26] and [21].

Theorem 3.3. *The following assertions are pairwise equivalent over* RCA_0.

1. *WWKL.*
2. (countable additivity) *For any compact separable metric space X and any measure μ on X, μ is countably additive.*
3. *For any covering of the closed unit interval* $[0, 1]$ *by a sequence of open intervals (a_n, b_n), $n \in \mathbb{N}$, we have $\sum_{n=0}^{\infty} |a_n - b_n| \geq 1$.*

Proof. That WWKL implies statement 2 is proved in Theorem 1 of [26]. The implication $2 \to 3$ is trivial. It remains to prove that statement 3 implies WWKL.

Reasoning in RCA_0, let T be a subtree of $2^{<\mathbb{N}}$ with no infinite path. Put

$$\widetilde{T} \;=\; \{\sigma^\frown\langle i\rangle \mid \sigma \in T,\, \sigma^\frown\langle i\rangle \notin T,\, i < 2\}\ .$$

For $\sigma \in 2^{<\mathbb{N}}$ put $\mathrm{lh}(\sigma) =$ length of σ and

$$a_\sigma \;=\; \sum_{n=0}^{\mathrm{lh}(\sigma)-1} \frac{\sigma(n)}{2^{n+1}}\ , \qquad b_\sigma \;=\; a_\sigma + \frac{1}{2^{\mathrm{lh}(\sigma)}}\ .$$

Note that $|a_\sigma - b_\sigma| = 1/2^{\mathrm{lh}(\sigma)}$. Note also that $\sigma, \tau \in 2^{<\mathbb{N}}$ are incomparable if and only if $(a_\sigma, b_\sigma) \cap (a_\tau, b_\tau) = \emptyset$. In particular, the intervals (a_τ, b_τ), $\tau \in \widetilde{T}$, are pairwise disjoint and cover $[0, 1)$ except for some of the points a_σ, $\sigma \in 2^{<\mathbb{N}}$. Fix $\epsilon > 0$ and put $c_\sigma = a_\sigma - \epsilon/4^{\mathrm{lh}(\sigma)}$, $d_\sigma = a_\sigma + \epsilon/4^{\mathrm{lh}(\sigma)}$. Then the open intervals (a_τ, b_τ), $\tau \in \widetilde{T}$, (c_σ, d_σ),

$\sigma \in 2^{<\mathbb{N}}$ and $(1-\epsilon, 1+\epsilon)$ form a covering of $[0,1]$. Applying statement 3, we see that the sum of the lengths of these intervals is ≥ 1, *i.e.*

$$\sum_{\tau \in \widetilde{T}} \frac{1}{2^{\mathrm{lh}(\tau)}} + 6\epsilon \;\geq\; 1 \; .$$

Since this holds for all $\epsilon > 0$, we see that

$$\sum_{\tau \in \widetilde{T}} \frac{1}{2^{\mathrm{lh}(\tau)}} \;=\; 1 \; .$$

From this, equation (2) follows easily. Thus we have proved that statement 3 implies WWKL. This completes the proof of the theorem. □

It is possible to take a somewhat different approach to measure theory in RCA_0. Note that the definition of $\mu(U)$ that we have given (Definition 2.1) is *extensional* in RCA_0. This means that if U and V contain the same points then $\mu(U) = \mu(V)$, provably in RCA_0. An alternative approach is the *intensional* one, embodied in Definition 3.4 below.

Recall that an open set U is given in RCA_0 as a sequence of basic open sets. In the case of the real line, basic open sets are just intervals with rational endpoints.

Definition 3.4 (intensional Lebesgue measure)**.** We make this definition in RCA_0. Let $U = \langle (a_n, b_n) \rangle_{n \in \mathbb{N}}$ be an open set in the real line. The *intensional Lebesgue measure* of U is defined by

$$\mu_I(U) \;=\; \lim_{k \to \infty} \mu_L\left(\bigcup_{n=0}^{k} (a_n, b_n) \right) \; .$$

Theorem 3.5. *It is provable in* RCA_0 *that intensional Lebesgue measure* μ_I *is countably additive on open sets. In other words, if* U_n, $n \in \mathbb{N}$, *is a sequence of open sets, then*

$$\mu_I\left(\bigcup_{n=0}^{\infty} U_n \right) \;=\; \lim_{k \to \infty} \mu_I\left(\bigcup_{n=0}^{k} U_n \right) \; .$$

Proof. This is immediate from the definitions, since $\bigcup_{n=0}^{\infty} U_n$ is defined as the union of the sequences of basic open intervals in U_n, $n \in \mathbb{N}$. □

Returning now to WWKL_0, we can prove that intensional Lebesgue measure concides with extensional Lebesgue measure. In fact, we have the following easy result.

Theorem 3.6. *The following assertions are pairwise equivalent over* RCA_0.

1. *WWKL.*
2. $\mu_I(U) = \mu_L(U)$ *for all open sets* $U \subseteq [0,1]$.
3. μ_I *is extensional on open sets. In other words, for all open sets* $U, V \subseteq [0,1]$, *if* $\forall x(x \in U \leftrightarrow x \in V)$ *then* $\mu_I(U) = \mu_I(V)$.
4. *For all open sets* $U \supseteq [0,1]$, *we have* $\mu_I(U) \geq 1$.

Proof. This is immediate from Theorems 3.3 and 3.5. □

4. More Measure Theory in WWKL_0

In this section we show that a good theory of measurable functions and measurable sets can be developed within WWKL_0.

We first consider pointwise values of measurable functions. Our approach is due to Yu [21, 24].

Let X be a compact separable metric space and let $\mu : C(X) \to \mathbb{R}$ be a positive Borel probability measure on X. Recall that $L_1(X,\mu)$ is defined within RCA_0 as the completion of $C(X)$ under the L_1-norm. In what sense or to what extent can we prove that a point of the Banach space $L_1(X,\mu)$ gives rise to a function $f : X \to \mathbb{R}$?

In order to answer this question, recall that $f \in L_1(X,\mu)$ is given by a sequence $f_n \in C(X)$, $n \in \mathbb{N}$, which converges to f in the L_1-norm; more precisely

$$\|f_n - f_{n+1}\|_1 \ \leq \ \frac{1}{2^n}$$

for all $n \in \mathbb{N}$. We now observe that it is provable in WWKL_0 that the sequence of continuous functions f_n, $n \in \mathbb{N}$, converges pointwise almost everywhere. This is established by the following proposition:

Proposition 4.1 (Yu [21])**.** *Provably in* WWKL_0 *we have a sequence of closed sets*

$$C_0^f \subseteq C_1^f \subseteq \cdots \subseteq C_n^f \subseteq \dots\,, \ n \in \mathbb{N}$$

such that

$$\mu(X \setminus C_n^f) \ \leq \ \frac{1}{2^n}$$

for all n, *and*

$$|f_m(x) - f_{m'}(x)| \ \leq \ \frac{1}{2^k}$$

for all $x \in C_n^f$ *and all* m, m', k *such that* $m, m' \geq n + 2k + 2$.

Proof. Put

$$C_n^f \ = \ \left\{ x \ \middle| \ \forall k \sum_{i=n+2k+2}^{\infty} |f_i(x) - f_{i+1}(x)| \ \leq \ \frac{1}{2^k} \right\} .$$

Then for $x \in C_n^f$ and $m' \geq m \geq n + 2k + 2$ we have

$$\begin{aligned} |f_m(x) - f_{m'}(x)| &\leq \sum_{i=m}^{m'-1} |f_i(x) - f_{i+1}(x)| \\ &\leq \sum_{i=n+2k+2}^{\infty} |f_i(x) - f_{i+1}(x)| \\ &\leq \frac{1}{2^k} \,. \end{aligned}$$

Moreover C_n^f is a closed set. It remains to show that $\mu(X \setminus C_n^f) \leq 1/2^n$. To see this, note that $C_n^f = \bigcap_{k=0}^{\infty} C_{nk}^f$ where

$$C_{nk}^f = \left\{ x \;\middle|\; \sum_{i=n+2k+2}^{\infty} |f_i(x) - f_{i+1}(x)| \leq \frac{1}{2^k} \right\} .$$

We need a lemma:

Lemma 4.2. *The following is provable in* RCA_0. *For* $f \in C(X)$ *and* $\epsilon > 0$, *we have* $\mu(\{x \mid f(x) > \epsilon\}) \leq \|f\|_1/\epsilon$.

Proof. Put $U = \{x \mid f(x) > \epsilon\}$. Note that U is an open set. If $g \in C(X)$, $0 \leq g \leq 1$, $g = 0$ on $X \setminus U$, then we have $\epsilon g \leq |f|$, hence $\epsilon\mu(g) = \mu(\epsilon g) \leq \mu(|f|) = \|f\|_1$, hence $\mu(g) \leq \|f\|_1/\epsilon$. Thus $\mu(U) \leq \|f\|_1/\epsilon$ and the lemma is proved. □

Using this lemma we have

$$\begin{aligned} \mu(X \setminus C_{nk}^f) &= \mu\left(\left\{ x \;\middle|\; \sum_{i=n+2k+2}^{\infty} |f_i(x) - f_{i+1}(x)| > \frac{1}{2^k} \right\}\right) \\ &\leq 2^k \left\| \sum_{i=n+2k+2}^{\infty} |f_i - f_{i+1}| \right\|_1 \\ &\leq 2^k \sum_{i=n+2k+2}^{\infty} \|f_i - f_{i+1}\|_1 \\ &\leq 2^k \sum_{i=n+2k+2}^{\infty} \frac{1}{2^i} \\ &= \frac{1}{2^{n+k+1}} \,, \end{aligned}$$

hence by countable additivity

$$\mu(X \setminus C_n^f) \leq \sum_{k=0}^{\infty} \mu(X \setminus C_{nk}^f) \leq \sum_{k=0}^{\infty} \frac{1}{2^{n+k+1}} = \frac{1}{2^n} .$$

This completes the proof of Proposition 4.1. □

Remark 4.3 (Yu [21])**.** In accordance with Proposition 4.1, for

$$f = \langle f_n \rangle_{n\in\mathbb{N}} \in L_1(X,\mu)$$

and $x \in \bigcup_{n=0}^{\infty} C_n^f$, we define $f(x) = \lim_{n\to\infty} f_n(x)$. Thus we see that $f(x)$ is defined on an F_σ set of measure 1. Moreover, if $f = g$ in $L_1(X,\mu)$, *i.e.* if $\|f-g\|_1 = 0$, then $f(x) = g(x)$ for all x in an F_σ set of measure 1. These facts are provable in WWKL_0.

We now turn to a discussion of measurable sets within WWKL_0. We sketch two approaches to this topic. Our first approach is to identify measurable sets with their characteristic functions in $L_1(X,\mu)$, according to the following definition.

Definition 4.4. This definition is made within WWKL_0. We say that $f \in L_1(X,\mu)$ is a *measurable characteristic function* if there exists a sequence of closed sets

$$C_0 \subseteq C_1 \subseteq \cdots \subseteq C_n \subseteq \ldots , \quad n \in \mathbb{N} ,$$

such that $\mu(X \setminus C_n) \leq 1/2^n$ for all n, and $f(x) \in \{0,1\}$ for all $x \in \bigcup_{n=0}^{\infty} C_n$. Here $f(x)$ is as defined in Remark 4.3.

Our second approach is more direct, but in its present form it applies only to certain specific situations. For concreteness we consider only Lebesgue measure μ_L on the unit interval $[0,1]$. Our discussion can easily be extended to Lebesgue measure on the n-cube $[0,1]^n$, the "fair coin" measure on the Cantor space $2^{\mathbb{N}}$, *etc.*

Definition 4.5. The following definition is made within RCA_0. Let S be the Boolean algebra of finite unions of intervals in $[0,1]$ with rational endpoints. For E_1, $E_2 \in S$ we define the distance

$$d(E_1,E_2) = \mu_L((E_1 \setminus E_2) \cup (E_2 \setminus E_1)) ,$$

the Lebesgue measure of the symmetric difference of E_1 and E_2. Thus d is a pseudometric on S, and we define $\widehat{S}$ to be the compact separable metric space which is the completion of S under d. A point $E \in \widehat{S}$ is called a *Lebesgue measurable set* in $[0,1]$.

We shall show that these two approaches to measurable sets (Definitions 4.4 and 4.5) are equivalent in WWKL_0.

Begin by defining an isometry $\chi : S \to L_1([0,1], \mu_L)$ as follows. For $0 \le a < b \le 1$ define

$$\chi([a,b]) \;=\; \langle f_n \rangle_{n\in\mathbb{N}} \;\in\; L_1([0,1],\mu_L)$$

where $f_n(0) = f_n(a) = f_n(b) = f_n(1) = 0$ and

$$f_n\left(a + \frac{b-a}{2^{n+1}}\right) \;=\; f_n\left(b - \frac{b-a}{2^{n+1}}\right) \;=\; 1$$

and $f_n \in C([0,1])$ is piecewise linear otherwise. Thus $\chi([a,b])$ is a measurable characteristic function corresponding to the interval $[a,b]$. For $0 \le a_1 < b_1 < \cdots < a_k < b_k \le 1$ define

$$\chi([a_1,b_1] \cup \cdots \cup [a_k,b_k]) \;=\; \chi([a_1,b_1]) + \cdots + \chi([a_k,b_k])\,.$$

It is straightforward to prove in RCA_0 that χ extends to an isometry

$$\widehat{\chi} : \widehat{S} \to L_1([0,1],\mu_L)\ .$$

Proposition 4.6. *The following is provable in* WWKL_0*. If* $E \in \widehat{S}$ *is a Lebesgue measurable set, then* $\widehat{\chi}(E)$ *is a measurable characteristic function in* $L_1([0,1],\mu_L)$*. Conversely, given a measurable characteristic function* $f \in L_1([0,1],\mu_L)$*, we can find* $E \in \widehat{S}$ *such that* $\widehat{\chi}(E) = f$ *in* $L_1([0,1],\mu_L)$*.*

Proof. It is straightforward to prove in RCA_0 that for all $E \in \widehat{S}$, $\widehat{\chi}(E)$ is a measurable characteristic function.

For the converse, let f be a measurable characteristic function. By Definition 4.4 we have that $f(x) \in \{0,1\}$ for all $x \in \bigcup_{n=0}^{\infty} C_n$. By Proposition 4.1 we have $|f(x) - f_{3n+3}(x)| < 1/2^n$ for all $x \in C_n^f$. Put $U_n = \{x \mid |f_{3n+3}(x) - 1| < 1/2^n\}$ and $V_n = \{x \mid |f_{3n+3}(x)| < 1/2^n\}$. Then for $n \ge 1$, U_n and V_n are disjoint open sets. Moreover $C_n \cap C_n^f \subseteq U_n \cup V_n$, hence $\mu_L(U_n \cup V_n) \ge 1 - 1/2^{n-1}$. By countable additivity (Theorem 3.3) we can effectively find $E_n, F_n \in S$ such that $E_n \subseteq U_n$ and $F_n \subseteq V_n$ and $\mu_L(E_n \cup F_n) \ge 1 - 1/2^{n-2}$. Put $E = \langle E_{n+5} \rangle_{n\in\mathbb{N}}$. It is straightforward to show that E belongs to $\widehat{S}$ and that $\widehat{\chi}(E) = f$ in $L_1([0,1],\mu_L)$. This completes the proof. □

Remark 4.7. We have presented two notions of Lebesgue measurable set and shown that they are equivalent in WWKL_0. Our first notion (Definition 4.4) has the advantage of generality in that it applies to any measure on a compact separable metric space. Our second notion (Definition 4.5) is advantageous in other ways, namely it is more straightforward and works well in RCA_0. It would be desirable to find a single definition which combines all of these advantages.

5. Vitali's Theorem

Let $\mathcal{S}$ be a collection of sets. A point x is said to be *Vitali covered* by $\mathcal{S}$ if for all $\epsilon > 0$ there exists $S \in \mathcal{S}$ such that $x \in S$ and the diameter of S is less than ϵ. The *Vitali Covering Theorem* in its simplest form says the following: if $\mathcal{I}$ is a sequence of intervals which Vitali covers an interval E in the real line, then $\mathcal{I}$ contains a countable, pairwise disjoint set of intervals I_n, $n \in \mathbb{N}$, such that $\bigcup_{n=0}^{\infty} I_n$ covers E except for a set of Lebesgue measure 0.

The purpose of this section is to show that various forms of the Vitali Covering Theorem are provable in WWKL_0 and in fact equivalent to WWKL over RCA_0.

Throughout this section, we use μ to denote Lebesgue measure.

Lemma 5.1 (Baby Vitali Lemma)**.** *The following is provable in* RCA_0*. Let* $I_0, \ldots, I_n$ *be a finite sequence of intervals. Then we can find a pairwise disjoint subsequence* $I_{k_0}, \ldots, I_{k_m}$ *such that*

$$\mu(I_{k_0} \cup \cdots \cup I_{k_m}) \ \geq \ \frac{1}{3}\, \mu(I_0 \cup \cdots \cup I_n)\ .$$

Proof. Put $N = \{0, \ldots, n\}$. By bounded Σ^0_1 comprehension in RCA_0, the finite sets

$$\{\, (i, j) \in N^2 \mid I_i \cap I_j = \emptyset \,\}$$

and

$$\{\, (i, j) \in N^2 \mid \mu(I_i) \leq \mu(I_j) \,\}$$

exist. Using these finite sets as parameters, we can carry out the following primitive recursion within RCA_0. Begin by letting $k_0 \leq n$ be such that $\mu(I_{k_0})$ is as large as possible. Then let k_1 be such that $I_{k_1} \cap I_{k_0} = \emptyset$ and $\mu(I_{k_1})$ is as large as possible. At stage j, let k_j be such that $I_{k_j} \cap I_{k_0} = \emptyset$, ... , $I_{k_j} \cap I_{k_{j-1}} = \emptyset$ and $\mu(I_{k_j})$ is as large as possible. The recursion ends with a finite, pairwise disjoint sequence of intervals I_{k_0}, ... , I_{k_m} such that

$$\forall i \leq n\ \exists j \leq m\ [\, I_i \cap I_{k_j} \neq \emptyset \,]\ .$$

By construction it follows easily that

$$\forall i \leq n\ \exists j \leq m\ [\, I_i \cap I_{k_j} \neq \emptyset \text{ and } \mu(I_i) \ \leq\ \mu(I_{k_j}) \,]\ .$$

For any such i and j, we have $I_i \subseteq I'_{k_j}$, where I'_{k_j} is an interval with the same midpoint as I_{k_j} and 3 times as long. (If $I = [a, b]$, then

$I' = [2a - b, 2b - a]$.) Thus

$$\begin{aligned}\mu(I_0 \cup \cdots \cup I_n) &\leq \mu(I'_{k_0} \cup \cdots \cup I'_{k_m}) \\ &\leq \mu(I'_{k_0}) + \cdots + \mu(I'_{k_m}) \\ &= 3\,\mu(I_{k_0}) + \cdots + 3\,\mu(I_{k_m}) \\ &= 3\,\mu(I_{k_0} \cup \cdots \cup I_{k_m})\end{aligned}$$

and the lemma is proved. □

Lemma 5.2. *The following is provable in* WWKL_0. *Let* E *be an interval, and let* I_n, $n \in \mathbb{N}$, *be a sequence of intervals. If* $E \subseteq \bigcup_{n=0}^{\infty} I_n$, *then*

$$\mu(E) \;\leq\; \lim_{k\to\infty} \mu\left(\bigcup_{n=0}^{k} I_n\right) \;.$$

Proof. If the intervals I_n are open, then the desired conclusion follows immediately from countable additivity (Theorem 3.3). Otherwise, fix $\epsilon > 0$ and let I'_n be an open interval with the same midpoint as I_n and

$$\mu(I'_n) \;=\; \mu(I_n) + \frac{\epsilon}{2^n} \;.$$

Then by countable additivity we have

$$\mu(E) \;\leq\; \lim_{k\to\infty} \mu\left(\bigcup_{n=0}^{k} I'_n\right) \;\leq\; \lim_{k\to\infty} \mu\left(\bigcup_{n=0}^{k} I_n\right) + 2\epsilon \;.$$

Since this holds for all $\epsilon > 0$, the desired conclusion follows. □

Lemma 5.3 (Vitali theorem for intervals)**.** *The following is provable in* WWKL_0. *Let* E *be an interval, and let* $\mathcal{I}$ *be a sequence of intervals which is a Vitali covering of* E. *Then* $\mathcal{I}$ *contains a pairwise disjoint sequence of intervals* I_n, $n \in \mathbb{N}$, *such that*

$$\mu\left(E \setminus \bigcup_{n=0}^{\infty} I_n\right) = 0 \;.$$

Proof. We reason in WWKL_0. Without loss of generality, let us assume that $\mathcal{I}$ consists of closed intervals. Let $\mathcal{I}^*$ be the set of finite unions $I_1 \cup \cdots \cup I_k$ where $I_1, \ldots, I_k$ are pairwise disjoint intervals from $\mathcal{I}$.

We claim: Given $A \in \mathcal{I}^*$, if $\mu(E \setminus A) > 0$ then we can find $B \in \mathcal{I}^*$ such that $A \cap B = \emptyset$ and

$$\mu(E \setminus (A \cup B)) \;<\; \frac{3}{4}\,\mu(E \setminus A) \;. \tag{3}$$

To prove the claim, use Lemma 5.2 and the Vitali property to find a finite set of intervals $J_1, \ldots, J_l \in \mathcal{I}$ such that $J_1, \ldots, J_l \subseteq E \setminus A$ and

$$\mu\left(E \setminus (A \cup J_1 \cup \cdots \cup J_l)\right) \ < \ \frac{1}{12}\,\mu(E \setminus A)\ .$$

By the Baby Vitali Lemma 5.1, we can find a pairwise disjoint subset $\{I_1, \ldots, I_k\} \subseteq \{J_1, \ldots, J_l\}$ such that

$$\mu(I_1 \cup \cdots \cup I_k) \ \geq \ \frac{1}{3}\,\mu(J_1 \cup \cdots \cup J_l)\ .$$

We then have

$$\begin{aligned}
&\mu(E \setminus (A \cup I_1 \cup \cdots \cup I_k)) \\
&\quad < \ \frac{2}{3}\,\mu(J_1 \cup \cdots \cup J_l) + \frac{1}{12}\,\mu(E \setminus A) \\
&\quad \leq \ \frac{2}{3}\,\mu(E \setminus A) + \frac{1}{12}\,\mu(E \setminus A) \\
&\quad = \ \frac{3}{4}\,\mu(E \setminus A)
\end{aligned}$$

Thus we may take $B = I_1 \cup \cdots \cup I_k$ and our claim is proved.

Note that the predicates $A \cap B = \emptyset$ and (3) are Σ^0_1. Thus within RCA_0 we can apply our claim recursively to choose a pairwise disjoint sequence $A_0 = \emptyset$, A_1, A_2, $\ldots$ of sets in $\mathcal{I}^*$ such that for all $n \geq 1$,

$$\mu(E \setminus (A_1 \cup \cdots \cup A_n)) \ < \ \left(\frac{3}{4}\right)^n \mu(E)\ .$$

Then by countable additivity we have

$$\mu\left(E \setminus \bigcup_{n=1}^{\infty} A_n\right) \ = \ 0$$

and the lemma is proved. □

Remark 5.4. It is straightforward to generalize the previous lemma to the case of a Vitali covering of the n-cube $[0,1]^n$ by closed balls or n-dimensional cubes. In the case of closed balls, the constant 3 in the Baby Vitali Lemma 5.1 is replaced by 3^n.

Theorem 5.5. *The Vitali theorem for the interval* $[0,1]$ (as stated in Lemma 5.3) *is equivalent to WWKL over* RCA_0*.*

Proof. Lemma 5.3 shows that, in RCA_0, WWKL implies the Vitali theorem for intervals. It remains to prove within RCA_0 that the Vitali theorem for $[0,1]$ implies WWKL. Instead of proving WWKL, we shall prove the equivalent statement 3.3.3. Reasoning in RCA_0, suppose that

(a_n, b_n), $n \in \mathbb{N}$, is a sequence of open intervals which covers $[0,1]$. Let $\mathcal{I}$ be the countable set of intervals

$$(a_{nki}, b_{nki}) \ = \ \left(a_n + \frac{i}{k}(b_n - a_n)\,,\ a_n + \frac{i+1}{k}(b_n - a_n)\right)$$

where $i, k, n \in \mathbb{N}$ and $0 \le i < k$. Then $\mathcal{I}$ is a Vitali covering of $[0,1]$. By the Vitali theorem for intervals, $\mathcal{I}$ contains a sequence of pairwise disjoint intervals I_m, $m \in \mathbb{N}$, such that

$$\mu\left(\bigcup_{m=0}^{\infty} I_m\right) \ \geq \ 1 \ .$$

By disjoint countable additivity (Corollary 2.5), we have

$$\sum_{m=0}^{\infty} \mu(I_m) \ \geq \ 1 \ .$$

From this it follows easily that $\sum_{n=0}^{\infty} |a_n - b_n| \geq 1$. Thus we have 3.3.3 and our theorem is proved. □

We now turn to Vitali's theorem for measurable sets. Recall our discussion of measurable sets in section 4. A sequence of intervals $\mathcal{I}$ is said to *almost Vitali cover* a Lebesgue measurable set $E \subseteq [0,1]$ if for all $\epsilon > 0$ we have $\mu_L(E \setminus O_\epsilon) = 0$, where

$$O_\epsilon \ = \ \bigcup \{I \mid I \in \mathcal{I}\,,\ \mathrm{diam}(I) < \epsilon\} \ .$$

Theorem 5.6. *The following is provable in* WWKL_0. *Let* $E \subseteq [0,1]$ *be a Lebesgue measurable set with* $\mu(E) > 0$. *Let* $\mathcal{I}$ *be a sequence of intervals which almost Vitali covers* E. *Then* $\mathcal{I}$ *contains a pairwise disjoint sequence of intervals* I_n, $n \in \mathbb{N}$, *such that*

$$\mu\left(E \setminus \bigcup_{n=0}^{\infty} I_n\right) \ = \ 0 \ .$$

Proof. The proof of this theorem is similar to that of Lemma 5.3. The only modification needed is in the proof of the claim. Recall from Definition 4.5 that $E = \lim_{n\to\infty} E_n$ where each E_n is a finite union of intervals in $[0,1]$. Fix m so large that

$$\mu((E \setminus E_m) \cup (E_m \setminus E)) \ < \ \frac{1}{36}\,\mu(E \setminus A) \ .$$

As before, find a finite set of intervals $J_1, \ldots, J_l \in \mathcal{I}$ such that

$$J_1 \cup \cdots \cup J_l \ \subseteq \ E_m \setminus A$$

and

$$\mu\left(E_m \setminus (A \cup J_1 \cup \cdots \cup J_l)\right) \ < \ \frac{1}{36}\,\mu(E \setminus A)\ .$$

Find $\{I_1, \ldots, I_k\} \subseteq \{J_1, \ldots, J_l\}$ as before. We then have

$$\begin{aligned}
&\mu(E \setminus (A \cup I_1 \cup \cdots \cup I_k)) \\
&\quad < \ \mu(E_m \setminus (A \cup I_1 \cup \cdots \cup I_k)) + \frac{1}{36}\,\mu(E \setminus A) \\
&\quad < \ \frac{2}{3}\,\mu(J_1 \cup \cdots \cup J_l) + \frac{2}{36}\,\mu(E \setminus A) \\
&\quad \leq \ \frac{2}{3}\,\mu(E_m \setminus A) + \frac{2}{36}\,\mu(E \setminus A) \\
&\quad < \ \frac{2}{3}\,\mu(E \setminus A) + \frac{3}{36}\,\mu(E \setminus A) \\
&\quad = \ \frac{3}{4}\,\mu(E \setminus A)
\end{aligned}$$

Thus we may take $B = I_1 \cup \cdots \cup I_k$ and the claim is proved. The rest of the proof is as for Lemma 5.3. □

Remark 5.7. Once again, the previous theorem can be generalized to the case of a Lebesgue measurable set $E \subseteq [0,1]^n$ and a Vitali covering consisting of closed balls or n-dimensional cubes. Such versions of Vitali's theorem are also provable in WWKL_0.

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