A Symmetric β-Model

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1 The Main Results

Our context is the study of ω-models of subsystems of second order arithmetic [5, Chapter VIII]. As in [5, Chapter VII], a *β-model* is an ω-model M such that, for all Σ^1_1 sentences φ with parameters from M, φ is true if and only if $M \models \varphi$. Theorems 1.1 and 1.3 below are an interesting supplement to the results on β-models which have been presented in Simpson [5, §§VII.2 and VIII.6].

Let HYP denote the set of hyperarithmetical reals. It is well known that, for any β-model M, HYP is properly included in M, and each $X \in$ HYP is definable in M.

Theorem 1.1. There exists a countable β-model satisfying

$$\forall X \,(\text{if } X \text{ is definable, then } X \in \text{HYP}).$$

Proof. Fix a recursive enumeration S_e, $e \in \omega$, of the Σ^1_1 sets of reals. If p is a finite subset of $\omega \times \omega^{<\omega}$, say that $\langle X_n \rangle_{n\in\omega}$ *meets* p if $X_{n_1} \oplus \cdots \oplus X_{n_k} \in S_e$ for all $(e, \langle n_1, \ldots, n_k \rangle) \in p$. Let $\mathcal{P}$ be the set of p such that there exists $\langle X_n \rangle_{n\in\omega}$ meeting p. Put $p \leq q$ if and only if $p \supseteq q$. Say that $\mathcal{D} \subseteq \mathcal{P}$ is *dense* if for all $p \in \mathcal{P}$ there exists $q \in \mathcal{D}$ such that $q \leq p$. Say that $\mathcal{D}$ is *definable* if it is definable over the ω-model HYP, i.e., arithmetical in the complete Π^1_1 subset of ω. Say that $\langle G_n \rangle_{n\in\omega}$ is *generic* if for every dense definable $\mathcal{D} \subseteq \mathcal{P}$ there exists $p \in \mathcal{D}$ such that $\langle G_n \rangle_{n\in\omega}$ meets p. We can show that for every $p \in \mathcal{P}$ there exists a generic $\langle G_n \rangle_{n\in\omega}$ meeting p. (This is a fusion argument, a la Gandy forcing.) Clearly $\{G_n : n \in \omega\}$ is a β-model. We can also show that, if C is countable and $C \cap \text{HYP} = \emptyset$, then there exists a generic $\langle G_n \rangle_{n\in\omega}$ such that $C \cap \{G_n : n \in \omega\} = \emptyset$.

Let $L_2(\langle X_n \rangle_{n\in\omega})$ be the language of second order arithmetic with additional set constants X_n, $n \in \omega$. Let φ be a sentence of $L_2(\langle X_n \rangle_{n\in\omega})$. We say that

p *forces* φ, written $p \Vdash \varphi$, if for all generic $\langle G_n \rangle_{n\in\omega}$ meeting p, the β-model $\{G_n : n \in \omega\}$ satisfies $\varphi[\langle X_n/G_n \rangle_{n\in\omega}]$. It can be shown that, for all generic $\langle G_n \rangle_{n\in\omega}$, the β-model $\{G_n : n \in \omega\}$ satisfies $\varphi[\langle X_n/G_n \rangle_{n\in\omega}]$ if and only if $\langle G_n \rangle_{n\in\omega}$ meets some p such that $p \Vdash \varphi$.

If π is a permutation of ω, define an action of π on $\mathcal{P}$ and $L_2(\langle X_n \rangle_{n\in\omega})$ by $\pi(p) = \{(e, \langle \pi(n_1), \ldots, \pi(n_k)\rangle) : (e, \langle n_1, \ldots, n_k\rangle) \in p\}$ and $\pi(X_n) = X_{\pi(n)}$. It is straightforward to show that $p \Vdash \varphi$ if and only if $\pi(p) \Vdash \pi(\varphi)$. The *support* of $p \in \mathcal{P}$ is defined by $\mathrm{supp}(p) = \bigcup\{\{n_1, \ldots, n_k\} : (e, \langle n_1, \ldots, n_k\rangle) \in p\}$. Clearly if $p, q \in \mathcal{P}$ and $\mathrm{supp}(p) \cap \mathrm{supp}(q) = \emptyset$, then $p \cup q \in \mathcal{P}$.

We claim that if $\langle G_n \rangle_{n\in\omega}$ and $\langle G'_n \rangle_{n\in\omega}$ are generic, then the β-models $\{G_n : n \in \omega\}$ and $\{G'_n : n \in \omega\}$ satisfy the same L_2-sentences. Suppose not. Then for some $p, q \in \mathcal{P}$ we have $p \Vdash \varphi$ and $q \Vdash \neg\varphi$, for some L_2-sentence φ. Let π be a permutation of ω such that $\mathrm{supp}(\pi(p)) \cap \mathrm{supp}(q) = \emptyset$. Since $\pi(\varphi) = \varphi$, we have $\pi(p) \Vdash \varphi$, hence $\pi(p) \cup q \Vdash \varphi$, a contradiction. This proves our claim.

Finally, let $M = \{G_n : n \in \omega\}$ where $\langle G_n \rangle_{n\in\omega}$ is generic. Suppose $A \in M$ is definable in M. Let $\langle G'_n \rangle_{n\in\omega}$ be generic such that $M' = \{G'_n : n \in \omega\}$ has $M \cap M' = \mathrm{HYP}$. Let $\varphi(X)$ be an L_2-formula with X as its only free variable, such that $M \models (\exists \text{ exactly one } X)\, \varphi(X)$, and $M \models \varphi(A)$. Then $M' \models (\exists \text{ exactly one } X)\, \varphi(X)$. Let $A' \in M'$ be such that $M' \models \varphi(A')$. Then for each $n \in \omega$, we have that $n \in A$ if and only if $M \models \exists X\, (\varphi(X) \text{ and } n \in X)$, if and only if $M' \models \exists X\, (\varphi(X) \text{ and } n \in X)$, if and only if $n \in A'$. Thus $A = A'$. Hence $A \in \mathrm{HYP}$. This completes the proof. □

Remark 1.2. Theorem 1.1 has been announced without proof by Friedman [3, Theorem 4.3]. Until now, a proof of Theorem 1.1 has not been available.

We now improve Theorem 1.1 as follows.

Let $\leq_{\mathrm{HYP}}$ denote hyperarithmetical reducibility, i.e., $X \leq_{\mathrm{HYP}} Y$ if and only if X is hyperarithmetical in Y.

Theorem 1.3. There exists a countable β-model satisfying

$$(*) \qquad \forall X\, \forall Y\, (\text{if } X \text{ is definable from } Y, \text{ then } X \leq_{\mathrm{HYP}} Y).$$

The β-model which we shall use to prove Theorem 1.3 is the same as for Theorem 1.1, namely $M = \{G_n : n \in \omega\}$ where $\langle G_n \rangle_{n\in\omega}$ is generic. In order to see that M has the desired property, we first relativize the proof of Theorem 1.1, as follows. Given Y, let $\mathcal{P}^Y$ be the set of $p \in \mathcal{P}$ such that there exists $\langle X_n \rangle_{n\in\omega}$ meeting p with $X_0 = Y$. (Obviously 0 plays no special role here.) Say that $\langle G_n \rangle_{n\in\omega}$ is *generic over* Y if, for every dense $\mathcal{D}^Y \subseteq \mathcal{P}^Y$ definable from Y over $\mathrm{HYP}(Y) = \{X : X \leq_{\mathrm{HYP}} Y\}$, there exists $p \in \mathcal{D}^Y$ such that $\langle G_n \rangle_{n\in\omega}$ meets p.

Lemma 1.4. If $\langle G_n \rangle_{n\in\omega}$ is generic over Y, then $G_0 = Y$, and $\{G_n : n \in \omega\}$ is a β-model satisfying $\forall X$ (if X is definable from Y, then $X \leq_{\mathrm{HYP}} Y$).

Proof. The proof of this lemma is a straightforward relativization to Y of the proof of Theorem 1.1. □

Consequently, in order to prove Theorem 1.3, it suffices to prove the following lemma.

Lemma 1.5. If $\langle G_n\rangle_{n\in\omega}$ is generic, then $\langle G_n\rangle_{n\in\omega}$ is generic over G_0.

Proof. It suffices to show that, for all p forcing ($\mathcal{D}^{X_0}$ is dense in $\mathcal{P}^{X_0}$), there exists $q \le p$ forcing $\exists r\,(r \in \mathcal{D}^{X_0}$ and $\langle X_n\rangle_{n\in\omega}$ meets r).

Assume $p \Vdash$ ($\mathcal{D}^{X_0}$ is dense in $\mathcal{P}^{X_0}$). Since $p \Vdash p \in \mathcal{P}^{X_0}$, it follows that $p \Vdash \exists q\,(q \le p$ and $q \in \mathcal{D}^{X_0})$. Fix $p' \le p$ and $q' \le p$ such that $p' \Vdash q' \in \mathcal{D}^{X_0}$. Put $S' = \{X_0 : \langle X_n\rangle_{n\in\omega}$ meets $p'\}$. Then S' is a Σ^1_1 set, so let $e \in \omega$ be such that $S' = S_e$. Claim 1: $\{(e, \langle 0\rangle)\} \Vdash q' \in \mathcal{D}^{X_0}$. If not, let $p'' \le \{(e, \langle 0\rangle)\}$ be such that $p'' \Vdash q' \notin \mathcal{D}^{X_0}$. Let π be a permutation such that $\pi(0) = 0$ and $\mathrm{supp}(p') \cap \mathrm{supp}(\pi(p'')) = \{0\}$. Then $p' \cup \pi(p'') \in \mathcal{P}$ and $\pi(p'') \Vdash q' \notin \mathcal{D}^{X_0}$, a contradiction. This proves Claim 1.

Claim 2: $q' \cup \{(e, \langle 0\rangle)\} \in \mathcal{P}$. To see this, let $\langle G'_n\rangle_{n\in\omega}$ be generic meeting $\{(e, \langle 0\rangle)\}$. By Claim 1 we have $q' \in \mathcal{D}^{G'_0}$. Hence $q' \in \mathcal{P}^{G'_0}$, i.e., there exists $\langle X_n\rangle_{n\in\omega}$ meeting q' with $X_0 = G'_0$. Thus $\langle X_n\rangle_{n\in\omega}$ meets $q' \cup \{(e, \langle 0\rangle)\}$. This proves Claim 2. Finally, put $q'' = q' \cup \{(e, \langle 0\rangle)\}$. Then $q'' \le q' \le p$ and $q'' \Vdash (q' \in \mathcal{D}^{X_0}$ and $\langle X_n\rangle_{n\in\omega}$ meets $q')$. This proves our lemma. □

The proof of Theorem 1.3 is immediate from Lemmas 1.4 and 1.5.

2 Conservation Results

In this section we generalize the construction of §1 to a wider setting. We then use this idea to obtain some conservation results involving the scheme $(*)$ of Theorem 1.3.

Two important subsystems of second order arithmetic are ATR_0 (arithmetical transfinite recursion with restricted induction) and Π^1_∞-TI_0 (the transfinite induction scheme). For general background, see Simpson [5]. It is known [5, §VII.2] that $\mathsf{ATR}_0 \subseteq \Pi^1_\infty$-$\mathsf{TI}_0$, and that every β-model is a model of Π^1_∞-TI_0.

Let $(N, \mathcal{S})$ be a countable model of ATR_0, where $\mathcal{S} \subseteq P(|N|)$. Define $\mathcal{P}_{(N,\mathcal{S})} = \{p : (N, \mathcal{S}) \models (p$ is a finite subset of $\omega \times \omega^{<\omega}$ and there exists $\langle X_n\rangle_{n\in\omega}$ meeting $p)\}$. The notion of $\langle G_n\rangle_{n\in|N|}$ being *generic over* $(N, \mathcal{S})$ is defined in the obvious way. As in §1, the basic forcing lemmas can be proved. Let $\langle G_n\rangle_{n\in|N|}$ be generic over $(N, \mathcal{S})$. Put $\mathcal{S}' = \{G_n : n \in |N|\}$.

Lemma 2.1. $(N, \mathcal{S}')$ satisfies the scheme $(*)$ of Theorem 1.3.

Proof. The proof is a straightforward generalization of the arguments of §1. □

Lemma 2.2. Let ψ be a Π^1_2 sentence with parameters from $|N|$. If $(N, \mathcal{S}) \models \psi$, then $(N, \mathcal{S}') \models \psi$.

Proof. Write ψ as $\forall X\,(X \in S_e)$ for some fixed $e \in |N|$. If $(N, \mathcal{S}) \models \psi$, then for each $n \in |N|$ we have that $\{p \in \mathcal{P}_{(N,\mathcal{S})} : (e, \langle n\rangle) \in p\}$ is dense in $\mathcal{P}_{(N,\mathcal{S})}$, hence $\emptyset \Vdash X_n \in S_e$. Thus $\emptyset \Vdash \forall X\,(X \in S_e)$, i.e., $\emptyset \Vdash \psi$, so $(N, \mathcal{S}') \models \psi$. □

Remark 2.3. Since $(N, \mathcal{S}) \models \mathsf{ATR}_0$ and ATR_0 is axiomatized by a Π^1_2 sentence, it follows by Lemma 2.2 that $(N, \mathcal{S}') \models \mathsf{ATR}_0$. Lemma 2.2 also implies that $(N, \mathcal{S})$ and $(N, \mathcal{S}')$ satisfy the same Π^1_1 sentences with parameters from $|N|$. From this it follows that the recursive well orderings of $(N, \mathcal{S})$ and $(N, \mathcal{S}')$ are the same, and that $\mathrm{HYP}_{(N,\mathcal{S})} = \mathrm{HYP}_{(N,\mathcal{S}')}$. It can also be shown that $\mathrm{HYP}_{(N,\mathcal{S})} = \mathcal{S} \cap \mathcal{S}'$.

We now have:

Theorem 2.4. $\mathsf{ATR}_0 + (*)$ is conservative over ATR_0 for Σ^1_2 sentences.

Proof. Let φ be a Σ^1_2 sentence. Suppose $\mathsf{ATR}_0 \nvdash \varphi$. Let $(N, \mathcal{S})$ be a countable model of $\mathsf{ATR}_0 + \neg\varphi$. Let $\mathcal{S}' = \{G_n : n \in |N|\}$ as above. Since $\neg\varphi$ is a Π^1_2 sentence, we have by Lemmas 2.1 and 2.2 that $(N, \mathcal{S}') \models \mathsf{ATR}_0 + (*) + \neg\varphi$. Thus $\mathsf{ATR}_0 + (*) \nvdash \varphi$. $\square$

In order to obtain a similar result for Π^1_∞-TI_0, we first prove:

Lemma 2.5. $(N, \mathcal{S}') \models$ "all ordinals are recursive", i.e., "every well ordering is isomorphic to a recursive well ordering".

Proof. Recall from [5, §§VIII.3 and VIII.6] that all of the basic results of hyperarithmetical theory are provable in ATR_0. In particular, by [5, Theorem VIII.6.7], the Gandy/Kreisel/Tait Theorem holds in ATR_0. Thus for all $p \in \mathcal{P}_{(N,\mathcal{S})}$ we have

$$(N, \mathcal{S}) \models \text{there exists } \langle X_n \rangle_{n\in\omega} \text{ meeting } p \text{ such that } \forall n\, (\omega_1^{X_n} = \omega_1^{\mathrm{CK}}).$$

Now, it is provable in ATR_0 that the predicate $\omega_1^X = \omega_1^{\mathrm{CK}}$ is Σ^1_1. Thus we have $\emptyset \Vdash \forall X\, (\omega_1^X = \omega_1^{\mathrm{CK}})$, i.e., $\emptyset \Vdash$ "all ordinals are recursive". This proves our lemma. $\square$

Remark 2.6. An alternative proof of Lemma 2.5 is to note that $\mathsf{ATR}_0 + (*) \vdash$ "O does not exist", because O would be definable but not hyperarithmetical. And "O does not exist" is equivalent over ATR_0 to "all ordinals are recursive". (Here O denotes the complete Π^1_1 set of integers. See [5, §VIII.3].)

Theorem 2.7. Π^1_∞-$\mathsf{TI}_0 + (*)$ is conservative over Π^1_∞-TI_0 for Σ^1_2 sentences.

Proof. By Lemma 2.5 it suffices to prove: if $(N, \mathcal{S}) \models$ transfinite induction for recursive well orderings, then $(N, \mathcal{S}') \models$ transfinite induction for recursive well orderings. Let $e \in |N|$ be such that $(N, \mathcal{S}) \models$ "$<_e$ is a recursive well ordering". Suppose $p \Vdash \exists n\, (n \in \mathrm{field}(<_e) \text{ and } \varphi(n))$. Put $A = \{n \in \mathrm{field}(<_e) : p \nVdash \neg\varphi(n)\}$. Clearly $A \neq \emptyset$. By definability of forcing, A is definable over $(N, \mathcal{S})$. Hence there exists $a \in A$ such that $(N, \mathcal{S}) \models$ "a is the $<_e$-least element of A". For each $n <_e a$ we have $p \Vdash \neg\varphi(n)$, but $p \nVdash \neg\varphi(a)$, so let $q \le p$ be such that $q \Vdash \varphi(a)$. Then $q \Vdash$ "a is the $<_e$-least n such that $\varphi(n)$". Thus $(N, \mathcal{S}') \models$ transfinite induction for recursive well orderings. $\square$

Remark 2.8. Theorems 2.4 and 2.7 are best possible, in the sense that we cannot replace Σ^1_2 by Π^1_2. An example is the Π^1_2 sentence "all ordinals are recursive", which is provable in $\mathsf{ATR}_0 + (*)$ but not in Π^1_∞-TI_0.

3 Recursion-Theoretic Analogs

The results of §§1 and 2 are in the realm of hyperarithmetical theory. We now present the analogous results in the realm of recursion theory, concerning models of WKL_0. For background on this topic, see Simpson [5, §§VIII.2 and XI.2].

Let REC denote the set of recursive reals. It is well known that, for any ω-model M of WKL_0, REC is properly included in M, and each $X \in \mathrm{REC}$ is definable in M. The recursion-theoretic analog of Theorem 1.1 is:

Theorem 3.1. There exists a countable ω-model of WKL_0 satisfying

$$\forall X\,(\text{if } X \text{ is definable, then } X \in \mathrm{REC}).$$

Proof. Use exactly the same construction and argument as for Theorem 1.1, replacing Σ^1_1 sets by Π^0_1 subsets of 2^ω. □

Remark 3.2. Theorem 3.1 is originally due to Friedman [2, Theorem 1.10, unpublished] (see also [3, Theorem 1.6]). It was later proved again by Simpson [6] (see also Simpson/Tanaka/Yamazaki [7]). All three proofs of Theorem 3.1 are different from one another.

Let $\leq_T$ denote Turing reducibility, i.e., $X \leq_T Y$ if and only if X is computable using Y as an oracle. The recursion-theoretic analog of Theorem 1.3 is:

Theorem 3.3. There exists a countable ω-model of WKL_0 satisfying

$$(**)\qquad \forall X\,\forall Y\,(\text{if } X \text{ is definable from } Y, \text{ then } X \leq_T Y).$$

Proof. Use exactly the same construction and argument as for Theorem 1.3, replacing Σ^1_1 sets by Π^0_1 subsets of 2^ω. □

The recursion-theoretic analog of Theorems 2.4 and 2.7 is:

Theorem 3.4. $\mathsf{WKL}_0 + (**)$ is conservative over WKL_0 for arithmetical sentences.

Proof. The proof is analogous to the arguments of §2. □

Remark 3.5. Theorems 3.3 and 3.4 are originally due to Simpson [6]. The proofs given here are different from the proofs that were given in [6].

4 Some Additional Results

In this section we present some additional results and open questions.

Lemma 4.1. Let $\varphi(X)$ be a Σ^1_1 formula with no free set variables other than X. The following is provable in ATR_0. If $\exists X\,(X \notin \mathrm{HYP}$ and $\varphi(X))$, then $\exists P\,(P$ is a perfect tree and $\forall X\,(\text{if } X \text{ is a path through } P \text{ then } \varphi(X)))$.

Proof. This is a well known consequence of formalizing the Perfect Set Theorem within ATR_0. See Simpson [5, §§V.4 and VIII.3]. See also Sacks [4, §III.6]. □

Lemma 4.2. Let $\varphi(X)$ be a Σ^1_2 formula with no free set variables other than X. The following is provable in $\mathsf{ATR}_0 +$ "all ordinals are recursive". If $\exists X\,(X \notin \mathrm{HYP}$ and $\varphi(X))$, then $\exists P\,(P$ is a perfect tree and $\forall X$ (if X is a path through P then $\varphi(X)))$.

Proof. Since $\varphi(X)$ is Σ^1_2, we can write $\varphi(X) \equiv \exists Y\,\forall f\,\exists n\,R(X[n], Y[n], f[n])$ where R is a primitive recursive predicate. Let $T_{X,Y}$ be the tree consisting of all τ such that $\neg\,(\exists n \leq \mathrm{lh}(\tau))\,R(X[n], Y[n], \tau[n])$. For $\alpha < \omega_1^{\mathrm{CK}}$ put

$$\varphi_\alpha(X) \;\equiv\; \exists Y\,(T_{X,Y} \text{ is well founded of height } \leq \alpha).$$

Note that, for each $\alpha < \omega_1^{\mathrm{CK}}$, $\varphi_\alpha(X)$ is Σ^1_1. Reasoning in $\mathsf{ATR}_0 +$ "all ordinals are recursive", we have $\forall X\,(\varphi(X)$ if and only if $(\exists \alpha < \omega_1^{\mathrm{CK}})\,\varphi_\alpha(X))$. Thus Lemma 4.2 follows easily from Lemma 4.1. □

Theorem 4.3. Let T be ATR_0 or Π^1_∞-TI_0. Let $\varphi(X)$ be a Σ^1_2 formula with no free set variables other than X. If $T \vdash \exists X\,(X \notin \mathrm{HYP}$ and $\varphi(X))$, then $T \vdash \exists P\,(P$ is a perfect tree and $\forall X$ (if X is a path through P then $\varphi(X)))$.

Proof. From Friedman [1] or Simpson [5, §VII.2], we have that $T \vdash$ the disjunction (1) all ordinals are recursive, or (2) there exists a countable coded β-model M satisfying $T +$ "all ordinals are recursive". In case (1), the desired conclusion follows from Lemma 4.2. In case (2), we have $M \models \exists X\,(X \notin \mathrm{HYP}$ and $\varphi(X))$, so the proof of Lemma 4.2 within M gives $\alpha < \omega_1^{\mathrm{CK}}$ such that $M \models \exists X\,(X \notin \mathrm{HYP}$ and $\varphi_\alpha(X))$. It follows that $\exists X\,(X \notin \mathrm{HYP}$ and $\varphi_\alpha(X))$. We can then apply Lemma 4.1 to $\varphi_\alpha(X)$ to obtain the desired conclusion. □

Corollary 4.4. Let T and $\varphi(X)$ be as in Theorem 4.3. If $T \vdash \exists X\,(X \notin \mathrm{HYP}$ and $\varphi(X))$, then $T \vdash \forall Y\,\exists X\,(\varphi(X)$ and $\forall n\,(X \neq (Y)_n))$.

Proof. This follows easily from Theorem 4.3. □

Theorem 4.5. Let T and $\varphi(X)$ be as in Theorem 4.3. If $T \vdash (\exists$ exactly one $X)\,\varphi(X)$, then $T \vdash \exists X\,(X \in \mathrm{HYP}$ and $\varphi(X))$.

Proof. Consider cases (1) and (2) as in the proof of Theorem 4.3. In both cases it suffices to show that, for all $\alpha < \omega_1^{\mathrm{CK}}$, if $(\exists$ exactly one $X)\,\varphi_\alpha(X)$ then $\exists X\,(X \in \mathrm{HYP}$ and $\varphi_\alpha(X))$. This follows from Lemma 4.1 applied to $\varphi_\alpha(X)$. □

Remark 4.6. Theorems 4.3 and 4.5 appear to be new. Corollary 4.4 has been stated without proof by Friedman [3, Theorems 3.4 and 4.4]. A recursion-theoretic analog of Corollary 4.4 has been stated without proof by Friedman [3, Theorem 1.7], but this statement of Friedman is known to be false, in view of Simpson [6]. A recursion-theoretic analog of Theorem 4.5 has been proved by Simpson/Tanaka/Yamazaki [7].

Question 4.7. Suppose $\mathsf{WKL}_0 \vdash \exists X\,(X \notin \mathrm{REC} \text{ and } \varphi(X))$ where $\varphi(X)$ is Σ^1_1, or even arithmetical, with no free set variables other than X. Does it follow that $\mathsf{WKL}_0 \vdash \exists X\,\exists Y\,(X \neq Y \wedge \varphi(X) \wedge \varphi(Y))$? A similar question has been asked by Friedman [2, unpublished].

Question 4.8. Suppose $\mathsf{WKL}_0 \vdash (\exists \text{ exactly one } X)\,\varphi(X)$ where $\varphi(X)$ is Σ^1_1 with no free set variables other than X. Does it follow that $\mathsf{WKL}_0 \vdash \exists X\,(X \in \mathrm{REC} \text{ and } \varphi(X))$? If $\varphi(X)$ is arithmetical or Π^1_1 then the answer is yes, by Simpson/Tanaka/Yamazaki [7].

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