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SETS WHICH DO NOT HAVE SUBSETS OF EVERY HIGHER DEGREE[1]

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Let A be a subset of ω, the set of natural numbers. The *degree* of A is its degree of recursive unsolvability. We say that A is *rich* if every degree above that of A is represented by a subset of A. We say that A is *poor* if no degree strictly above that of A is represented by a subset of A. The existence of infinite poor (and hence nonrich) sets was proved by Soare [9].

THEOREM 1. *Suppose that A is infinite and not rich. Then every hyperarithmetical subset H of ω is recursive in A.*

In the special case when H is arithmetical, Theorem 1 was proved by Jockusch [4] who employed a degree-theoretic analysis of Ramsey's theorem [3]. In our proof of Theorem 1 we employ a similar, degree-theoretic analysis of a certain generalization of Ramsey's theorem. The generalization of Ramsey's theorem is due to Nash-Williams [6]. If $A \subseteq \omega$ we write $[A]^\omega$ for the set of all infinite subsets of A. If $P \subseteq [\omega]^\omega$ we let $H(P)$ be the set of all infinite sets A such that either $[A]^\omega \subseteq P$ or $[A]^\omega \cap P = \emptyset$. Nash-Williams' theorem is essentially the statement that if $P \subseteq [\omega]^\omega$ is *clopen* (in the usual, Baire topology on $[\omega]^\omega$) then $H(P)$ is nonempty. Subsequent, further generalizations of Ramsey's theorem were proved by Galvin and Prikry [1], Silver [8], Mathias [5], and analyzed degree-theoretically by Solovay [10]; those results are not needed for this paper.

A subset of $[\omega]^\omega$ is said to be *recursively enumerable* if it can be written in the form $\{A \in [\omega]^\omega \mid \exists y R(\bar{A}(y))\}$ where $\bar{A}(y) = A \cap \{0, 1, \ldots, y-1\}$ and R is a recursive predicate of finite sets. A set $P \subseteq [\omega]^\omega$ is said to be *recursive* if both P and $[\omega]^\omega - P$ are recursively enumerable. A recursive subset of $[\omega]^\omega$ is clopen; indeed, a subset of $[\omega]^\omega$ is clopen if and only if it is recursive in some subset of ω (cf. [7, pp. 351–353]). Below we study degrees of members of $H(P)$ where P is recursive. It follows from a result of Solovay [10] that, for P recursive, $H(P)$ contains a hyperarithmetical set. A strong converse to this result is the following:

LEMMA 1. *For any hyperarithmetical set $B \subseteq \omega$ there exists a recursive set $P \subseteq [\omega]^\omega$ such that B is recursive in A for all $A \in H(P)$.*

PROOF. A set $P \subseteq [\omega]^\omega$ is said to be *unbalanced* if $[A]^\omega \subseteq P$ for all $A \in H(P)$. Our notation for the hyperarithmetical hierarchy is from Spector

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[11]. For each $b \in O$ we shall construct a recursive, unbalanced set $P_b \subseteq [\omega]^\omega$ such that H_b is recursive in A uniformly for all $A \in H(P_b)$. Here the word "uniformly" means that for each $b \in O$ there will exist a number e_b such that $\lambda i\{e_b\}(A, i)$ is the characteristic function of H_b for all $A \in H(P_b)$. Furthermore, P_b and e_b will be obtained recursively from b by means of the recursion theorem [7, §11.7].

The case $b = 1$ is handled trivially by putting $P_1 = [\omega]^\omega$. The case $b = 2$ is handled by the following sublemma which is essentially a special case of [3, Lemma 5.9].

SUBLEMMA. *Let $K = H_2$ be the complete, recursively enumerable subset of ω. There exists a recursive, unbalanced set $P \subseteq [\omega]^\omega$ such that K is recursive in A uniformly for all $A \in H(P)$.*

PROOF. Let $K = \{x \mid \exists y R(x, y)\}$ where R is recursive. For $A = \{a, b, c, \dots\}$ with $a < b < c < \cdots$ put A into P if and only if

$$\forall x < a(\exists y < b \,.\, R(x, y) \leftrightarrow \exists z < c \,.\, R(x, z)).$$

Then P is easily seen to satisfy the conclusions of the sublemma.

Returning to the proof of Lemma 1, given $b = 2^a$ we may suppose inductively that we are in possession of a recursive, unbalanced set P_a such that H_a is recursive in A uniformly for all $A \in H(P_a)$. We have

$$H_b = \text{jump of } H_a = K^{H_a}.$$

Relativizing the previous sublemma to H_a we obtain an unbalanced set $Q \subseteq [\omega]^\omega$ such that Q is recursive in H_a, and H_b is recursive in the pair H_a, A uniformly for all $A \in H(Q)$. Since P_a is recursive, $[\omega]^\omega - H(P_a)$ is recursively enumerable. But Q is recursive in A uniformly for all $A \in H(P_a)$. Therefore, we can define a recursive set $R \subseteq [\omega]^\omega$ such that $H(P_a) \cap R = H(P_a) \cap Q$. We then put $P_b = P_a \cap R$. Clearly P_b is recursive. By the Nash-Williams theorem, it is easy to see that P_b is unbalanced and, in fact, $H(P_b) = H(P_a) \cap H(Q)$. Hence H_b is recursive in A uniformly for all $A \in H(P_b)$.

Finally, given $b = 3 \cdot 5^e$, we may suppose inductively that we are in possession of recursive, unbalanced sets $P_{\{e\}(n)}$ such that $H_{\{e\}(n)}$ is recursive in A uniformly for all $n \in \omega$, $A \in H(P_{\{e\}(n)})$. Let P_b be the set of all A such that $A - \{n\}$ belongs to $P_{\{e\}(n)}$ where n is the least element of A. Then clearly P_b is recursive and unbalanced, and H_b is recursive in A uniformly for all $A \in H(P_b)$. This completes the proof of Lemma 1.

REMARK. Let ATR be the formal system of "arithmetical transfinite recursion", discussed by H. Friedman in [12]. Let $\boldsymbol{\Delta}^0_1 - \text{CR}$ (respectively $\boldsymbol{\Sigma}^0_1 - \text{CR}$) be the assertion, in the language of second order arithmetic, that $H(P)$ is nonempty whenever P is a clopen (open) subset of $[\omega]^\omega$. Here CR stands for "completely Ramsey", cf. Silver [8]. By adapting Lemma 1 above and a lemma of Solovay [10], I can prove (in a very weak formal system) that $\boldsymbol{\Delta}^0_1 - \text{CR}$ and $\boldsymbol{\Sigma}^0_1 - \text{CR}$ are equivalent to each other and to (the principal axiom of) ATR. Earlier, in 1973, J. Steel [13] had proved that $\boldsymbol{\Delta}^0_1 - \text{AD}$ and $\boldsymbol{\Sigma}^0_1 - \text{AD}$ are equivalent to each other and to ATR, $\boldsymbol{\Delta}^0_1 - \text{AD}$ (respectively $\boldsymbol{\Sigma}^0_1 - \text{AD}$) being the assertion that every clopen (open) subset of ω^ω is determined.

If $P \subseteq [\omega]^\omega$ let $H^+(P)$ be the set of all $A \subseteq \omega$ such that $A - F \in H(P)$ for some finite set F. The next lemma is a generalization of [4, Lemma 2].

LEMMA 2. *Suppose A is infinite and not rich. Then $A \in H^+(P)$ for every recursive $P \subseteq [\omega]^\omega$.*

PROOF. We shall assume that A is an infinite set not in $H^+(P)$ and prove that A is rich. Let B be a set in which A is recursive. We must show that A has a subset C of the same degree as B. Since P is recursive, there exist recursive predicates S and T such that

$$P = \{D \in [\omega]^\omega \mid \exists y\, S(\bar{D}(y))\} \quad \text{and}$$
$$[\omega]^\omega - P = \{D \in [\omega]^\omega \mid \exists y\, T(\bar{D}(y))\}.$$

Moreover we may choose S and T so that for each $D \in [\omega]^\omega$ there is exactly one initial segment $\bar{D}(y)$ of D such that $S(\bar{D}(y))$ or $T(\bar{D}(y))$ holds. We shall obtain C as $\bigcup \{C_i \mid i \geq 1\}$ where for each i

(i) C_{i+1} is a finite subset of A,
(ii) $\max(C_i) < \min(C_{i+1})$,
(iii) either $S(C_{i+1})$ or $T(C_{i+1})$,
(iv) $i \in B$ if and only if $S(C_{i+1})$.

The C_i are defined by recursion on i as follows. Put $C_0 = \{0\}$. Given C_i put $F_i = \{j \mid j \leq \max(C_i)\}$ and let C_{i+1} be the finite set of least index (in some effective indexing) which satisfies (i)–(iv). Such a finite set exists since $A - F_i$ is not in $H(P)$. Clearly C is recursive in B. On the other hand, C_{i+1} is the unique initial segment of $C - F_i$ such that (iii) holds. In particular, the sequence of C_i's is recursive in C. Hence by (iv) B is recursive in C. This proves Lemma 2.

Theorem 1 is an immediate consequence of Lemmas 1 and 2.

Theorem 1 is sharp in that no nonhyperarithmetical set is recursive (or even hyperarithmetical) in every infinite poor set. For, by a remark in [9], there exists a nonempty, arithmetical collection of infinite poor sets, so the Gandy–Kreisel–Tait theorem (cf. Grilliot [2]) is applicable.

Say that $\mathscr{C} \subseteq [\omega]^\omega$ is *downward closed* if $B \in \mathscr{C}$ whenever $B \in [A]^\omega$, $A \in \mathscr{C}$. A *$\mathscr{C}$-degree* is the degree of an element of $\mathscr{C}$. A set X of degrees is *upward closed if* $\boldsymbol{b} \in X$ whenever $\boldsymbol{b} \geq \boldsymbol{a} \in X$.

COROLLARY 1. *If $\mathscr{C} \subseteq [\omega]^\omega$ is downward closed and contains a hyperarithmetical element, then the set of all $\mathscr{C}$-degrees is upward closed.*

PROOF. Let $\boldsymbol{b} \geq \boldsymbol{a}$ where $\boldsymbol{a}$ is a $\mathscr{C}$-degree. Let A be a set of degree $\boldsymbol{a}$. If A is rich, there is $B \in [A]^\omega$ of degree $\boldsymbol{b}$. If A is not rich, consider a hyperarithmetical set $H \in \mathscr{C}$. By Theorem 1, H is recursive in A and H is rich. Hence there is $B \in [H]^\omega$ of degree $\boldsymbol{b}$. In either case $B \in \mathscr{C}$ so $\boldsymbol{b}$ is a $\mathscr{C}$-degree.

COROLLARY 2. *If $P \subseteq [\omega]^\omega$ is recursive, then the set of all $H(P)$-degrees is upward closed.*

PROOF. Clearly $H(P)$ is downward closed. Also, as remarked before Lemma 1, $H(P)$ contains a hyperarithmetical element. Now the desired conclusion follows from Corollary 1.

NOTE ADDED IN PROOF (JANUARY 12, 1978). In the proof of Corollary 2, use was made of a lemma of Solovay to the effect that if P is recursive the $H(P)$ contains a hyperarithmetical set. A simplified proof of Solovay's lemma has been discovered by R. B. Mansfield. Mansfield's paper is entitled *A footnote to a theorem of Solovay on recursive encodability* and will appear in the ***Proceedings of the Association for Symbolic Logic*** summer meeting in Wroclaw, Poland, August 1977, to be published by North-Holland.

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