Some conservation results on weak König's lemma

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Abstract

By RCA_0, we denote the system of second order arithmetic based on recursive comprehension axioms and Σ^0_1 induction. WKL_0 is defined to be RCA_0 plus weak König's lemma: every infinite tree of sequences of 0's and 1's has an infinite path. In this paper, we first show that for any countable model M of RCA_0, there exists a countable model M' of WKL_0 whose first order part is the same as that of M, and whose second order part consists of the M-recursive sets and sets not in the second order part of M. By combining this fact with a certain forcing argument over universal trees, we obtain the following result (which has been called Tanaka's conjecture): if WKL_0 proves $\forall X \exists ! Y \varphi(X,Y)$ with φ arithmetical, so does RCA_0. We also discuss several improvements of this results.

Key words: RCA_0, WKL_0, weak König's lemma, conservation theorems, hard core theorem, productive functions, forcing, universal trees, genericity, pointed perfect trees

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1 Introduction

A celebrated metamathematical theorem due to L. Harrington asserts that WKL_0 is conservative over RCA_0 for the arithmetical (in fact, Π^1_1) sentences. In other words, if an arithmetical theorem can be obtained by some analytical methods involving the compactness argument over computable mathematics, it is already provable without it. This result can be viewed as a computable analogue of the Gödel-Kreisel theorem on set theory, which asserts that if an arithmetical sentence can be proved in ZF with the axiom of choice, it is already provable without it.

It is natural to think of extending Harrington's conservation result to analytical sentences, since the Gödel-Kreisel theorem has been extended to the Σ^1_2 (in fact, Π^1_3) sentences by J. Shoenfield. However, we can easily see that WKL_0 is not conservative over RCA_0 for all Σ^1_1 sentences, since an instance of weak König's lemma is Σ^1_1.

In this context, it has been conjectured by K. Tanaka [14] that if WKL_0 proves $\forall X \exists! Y \varphi(X, Y)$ with φ arithmetical, so does RCA_0. By $\exists! X \varphi(X)$, we mean that there exists a unique X satisfying $\varphi(X)$. The difficulty in solving the conjecture arises from the restricted induction of those systems. It was soon realized that Tanaka's conjecture holds under the assumption of arithmetical induction.

Some important results concerned with this conjecture were obtained by several people. Most notably, A. M. Fernandes [3] already proved the conjecture for the sentences of the form $\forall X \exists! Y \varphi(X, Y)$ with $\varphi \in \Sigma^0_3$. He also showed that $\mathsf{WKL}_0 + \Sigma^0_2$ induction is conservative over $\mathsf{RCA}_0 + \Sigma^0_2$ induction with respect to the sentences of the same form. In a different context, U. Kohlenbach [8]

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independently obtained many results somewhat similar to ours. He works in finite type systems with weak Konig's lemma, and investigates particular examples of unique existence theorems, e.g., the best Chebysheff approximation. It is not so easy to translate his results into our terms, but from them, we can obtain more or less a solution to the conjecture for sentences of the form $\forall X \exists ! Y \varphi(X, Y)$ with $\varphi \in \Sigma^0_2$. Finally, Yamazaki [15] discusses variations of Tanaka's conjecture, generalizing a result of Brown and Simpson [2].

The origin of the present paper was a defective attack on this problem by the last two authors. Subsequently, by adducing a result of Pour-El/Kripke [9], the first author completed the proof, which launched a joint study on more elaborate results and techniques reported in this paper.

Let us note an application of our main result. The fundamental theorem of algebra, which asserts that any complex polynomial of any positive degree has a unique factorization into linear terms, can be stated in the form $\forall X \exists ! Y \varphi(X, Y)$ with φ arithmetical by using a canonical expression (i.e., the binary expansion) for the complex numbers. Most of popular proofs of the theorem use some analytical methods which can be easily formalized in WKL_0 but not in RCA_0. However, by our conservation result, it can be concluded without elaborating a computable solution that the fundamental theorem of algebra (for polynomials of positive standard degrees) is already provable in RCA_0.

By contrast, consider the statement that any continuous real function on the closed unit interval $[0, 1]$ has a maximum value. This sentence cannot be expressed in the form $\forall X \exists ! Y \varphi(X, Y)$ with φ arithmetical. The point is that we can not determine arithmetically whether or not a set encodes a *total* continuous function in the terms of Simpson [12].

Now, we recall some basic definitions about the systems RCA_0 and WKL_0. The language L_2 of second-order arithmetic is a two-sorted language with number variables $x, y, z, \ldots$ and set variables $X, Y, Z, \ldots$ Numerical terms are built up from numerical variables and constant symbols $0, 1$ by means of binary operations $+$ and $\cdot$. Atomic formulas are $s = t$, $s < t$ and $s \in X$, where s and t are numerical terms. *Bounded* (Σ^0_0 or Π^0_0) formulas are constructed from atomic formulas by propositional connectives and bounded numerical quantifiers $(\forall x < t)$ and $(\exists x < t)$, where t does not contain x. A Σ^0_n formula is of the form $\exists x_1 \forall x_2 \ldots x_n \theta$ with θ bounded, and a Π^0_n formula is of the form $\forall x_1 \exists x_2 \ldots x_n \theta$ with θ bounded. All the Σ^0_n and Π^0_n formulas are the *arithmetical* (Σ^1_0 or Π^1_0) formulas. A Σ^1_n formula is of the form $\exists X_1 \forall X_2 \ldots X_n \varphi$ with φ arithmetical, and a Π^1_n formula is of the form $\forall X_1 \exists X_2 \ldots X_n \varphi$ with φ arithmetical.

The system RCA_0 consists of

1. the ordered semiring axioms for $(\omega, +, \cdot, 0, 1, <)$,

2. Δ^0_1 comprehension scheme:

$$\forall x(\varphi(x) \leftrightarrow \psi(x)) \rightarrow \exists X \forall x(x \in X \leftrightarrow \varphi(x)),$$

where $\varphi(x)$ is Σ^0_1, $\psi(x)$ is Π^0_1, and X does not occur freely in $\varphi(x)$,

3. Σ^0_1 induction scheme:

$$\varphi(0) \wedge \forall x(\varphi(x) \rightarrow \varphi(x+1)) \rightarrow \forall x \varphi(x),$$

where $\varphi(x)$ is a Σ^0_1 formula.

Within RCA_0, we define $2^{<\mathbb{N}}$ to be the set of (codes for) finite sequences of 0's and 1's. A set $T \subseteq 2^{<\mathbb{N}}$ is said to be a *tree* (or precisely 0-1 *tree*) if any initial segment of a sequence in T is also in T. We say that $P \subseteq \mathbb{N}$ is a *path* through T if for each n, the sequence $P[n] = \langle \chi_P(0), \chi_P(1), \ldots, \chi_P(n-1) \rangle$ belongs to T, where χ_P is the characteristic function of P. The axioms of WKL_0 consists of those of RCA_0 plus *weak König's lemma*: every infinite 0-1 tree T has a path.

The interest of WKL_0 has been well established through an ongoing program, called *Reverse Mathematics*. H. Friedman, S. G. Simpson and others have shown that numerous well-known theorems in different fields of mathematics are provably equivalent to WKL_0 over RCA_0 [12].

An L_2-structure M is an ordered 7-tuple $(|M|, S_M, +_M, \cdot_M, 0_M, 1_M, <_M)$, where $|M|$ serves as the range of the number variables and S_M is a set of subsets of $|M|$, that is, the range of the set variables. The first order part of M is obtained from M by removing S_M. If its first order part is the structure of standard natural numbers, M is called an *ω-structure* or an *ω-model*. In particular, ω-models of WKL_0 are known as Scott systems and extensively studied by not a few people, e.g. Kaye [7].

In the next section, we use *tree forcing* to prove that for any countable model M of RCA_0, there exists a countable model M' of WKL_0 such that M and M' have the same first order part and $S_M \cap S_{M'}$ is the set of M-recursive subsets of $|M|$. This can be regarded as a non-ω extension of Kreisel's recursive hard core theorem, which asserts that the intersection of all ω-models of WKL_0 is the set of recursive sets. In Section 3, we introduce *universal tree forcing*. In Section 4, by combining the techniques in the preceding sections, we prove our main theorem that if WKL_0 proves $\forall X \exists ! Y \varphi(X,Y)$ with φ arithmetical, so does RCA_0. In Section 5, we use a forcing argument with uniformly pointed perfect trees to prove that if WKL_0 proves $\forall X \exists ! Y \varphi(X,Y)$ with φ Π^1_1, then RCA_0 proves $\forall X \exists Y \varphi(X,Y)$. In Section 6, we prove a stronger form of our main theorem, that is, if WKL_0^+ proves $\forall X \exists ! Y \varphi(X,Y)$ with φ arithmetical, so does RCA_0.

2 A non-ω hard core theorem

In this section, we first review the tree forcing argument which is originated by Jockusch/Soare [6] and used by L. Harrington for his conservation result on WKL_0. We then reinforce this argument with some other machinery to prove that for any countable model M of RCA_0, there exists a countable model M' of WKL_0 such that M' has the same first order part as M and $S_M \cap S_{M'}$ is the set of M-recursive subsets of $|M|$. The following exposition of the tree forcing argument is based on [12, Section IX.2]. See also [12, Section VIII.2] for an account of hard core theorems.

Let M be an L_2-structure which satisfies the axioms of ordered semirings and Σ^0_1 induction. We say that $X \subseteq |M|$ is Δ^0_1 definable over M, denoted $X \in \Delta^0_1$-def(M), if there exist Σ^0_1 formulas φ_1 and φ_2 with parameters from $|M| \cup S_M$ such that

$$X = \{n \in |M| : M \models \varphi_1(n)\} = \{n \in |M| : M \models \neg\varphi_2(n)\}.$$

If φ_1 and φ_2 have no set parameter (except $A \in S_M$), we say that X is M-

recursive (in A). By REC_M (or $\mathrm{REC}_M(A)$), we denote the set of subsets of $|M|$ which are M-recursive (M-recursive in A). If L_2-structures M and M' have the same first order part, $\mathrm{REC}_M = \mathrm{REC}_{M'}$. It is also easy to see that if M is a model of RCA_0, $\Delta^0_1\text{-def}(M) = S_M$.

Lemma 2.1 *Let M be an L_2-structure which satisfies the axioms of ordered semirings and Σ^0_1 induction. Let M' be the L_2-structure with the same first order part as M and $S_{M'} = \Delta^0_1\text{-def}(M)$. Then M' is a model of RCA_0.*

Proof. See the proof of [12, Lemma IX.1.8]. □

We now define basic notions of the tree forcing. Let M be a countable model of RCA_0. Let $\mathcal{T}_M$ be the set of all $T \in S_M$ such that $M \models T$ is an infinite 0-1 tree. For any $T \in \mathcal{T}_M$ and $P \subseteq |M|$, we say that P is a *path through* T if, for any $n \in |M|$, $P[n] \in T$. Here $P[n]$ is a sequence $\sigma \in (2^n)_M$ such that for all $m <_M n$, $m \in P$ if and only if $M \models \sigma(m) = 1$. Let $[T]$ be the set of paths through T. We put $\mathcal{P}_M = [(2^{<\mathbb{N}})_M]$. We say that $D \subseteq \mathcal{T}_M$ is *dense* if, for each $T \in \mathcal{T}_M$, there exists $T' \in D$ such that $T' \subseteq T$. A path G is said to be $\mathcal{T}_M$*-generic* if, for every M-definable dense set $D \subseteq \mathcal{T}_M$, there exists $T \in D$ such that $G \in [T]$.

Lemma 2.2 *Let M be a countable model of RCA_0. For any $T \in \mathcal{T}_M$, there exists a $\mathcal{T}_M$-generic G such that $G \in [T]$.*

Proof. Let $\langle D_i : i \in \omega \rangle$ be an enumeration of all M-definable dense sets. We can easily construct a sequence of trees T_i ($i \in \omega$) such that $T_0 = T, T_{i+1} \subseteq T_i$ and $T_{i+1} \in D_i$ for each $i \in \omega$. Then, a path $G = \bigcap T_i$ is what we want. □

Lemma 2.3 *Let M be a countable model of RCA_0 and suppose that $G \in \mathcal{P}_M$ is $\mathcal{T}_M$-generic. Let M' be the L_2-structure such that M' has the same first order part as M and $S_{M'} = S_M \cup \{G\}$. Then, $M' \models \Sigma^0_1$ induction.*

Proof. It suffices to prove that for any $m \in |M|$ and any Σ^0_1 formula $\varphi(x, G)$ with parameters from $|M| \cup S_{M'}$, the set $\{n \in |M| : n <_M m \wedge M' \models \varphi(n, G)\}$ is M-finite, since Σ^0_1 induction is provably equivalent to bounded Σ^0_1 comprehension (cf. [12, Remark II.3.11]). Without loss of generality, we may assume that $\varphi(x, G)$ is of the form $\exists y \theta(x, G[y])$, where $\theta(x, \tau)$ is Σ^0_0 with parameters from $|M| \cup S_M$. Let D_m be the set of all $T \in \mathcal{T}_M$ such that for any $n <_M m$, M satisfies either

1. $\forall\tau \in T \neg\theta(n,\tau)$, or

2. $\exists w \forall \tau \in T(lh(\tau) = w \to \exists \tau' \subseteq \tau \theta(n,\tau'))$,

where $lh(\tau)$ denotes the length of sequence τ. Since we can prove that D_m is dense (see[12, Lemma IX.2.4], there exists $T' \in D_m$ such that $G \in [T']$. Then, $\{n \in |M| : n <_M m \wedge M' \models \varphi(n,G)\}$ if and only if $\{n \in |M| : n <_M m \wedge \exists w \forall \tau \in T'(lh(\tau) = w \to \exists \tau' \subseteq \tau\theta(n,\tau'))\}$. Therefore, by Σ^0_1 induction over M, $\{n \in |M| : n <_M m \wedge M' \models \varphi(n,G)\}$ is M-finite. □

Let $B = \langle B_i : i \in \omega\rangle$ be a sequence from $\mathcal{P}_M = [(2^{<\mathbb{N}})_M]$. Δ^0_1-def$(M;B)$ is the set of all $X \subseteq |M|$ such that there exist Σ^0_0 formulas θ_1 and θ_2 with parameters from $|M| \cup S_M$ such that

$$X = \{n \in |M| : \forall m \in |M|(M \models \theta_1(n, B_1[m], \ldots, B_l[m]))\}$$

$$= \{n \in |M| : \exists m \in |M|(M \models \theta_2(n, B_1[m], \ldots, B_l[m]))\}$$

for some $l \in \omega$. $M[B]$ denotes the L_2-structure $(|M|, \Delta^0_1\text{-def}(M;B), +_M, \cdot_M, 0_M, 1_M, <_M)$. If $B = \langle P\rangle$, then we write $M[P]$ for $M[B]$.

Lemma 2.4 *Let M be a countable model of RCA_0. For any T_M-generic G, $M[G]$ is a countable model of RCA_0.*

Proof. It is obvious from Lemmas 2.1 and 2.3. □

Corollary 2.5 *Let M be a countable model of RCA_0. For any $T \in \mathcal{T}_M$, there exists a countable model M' of RCA_0 such that M' has the same first part as M, $S_M \subseteq S_{M'}$ and $M' \models T$ has a path.*

Proof. It is straightforward from Lemmas 2.2 and 2.4. □

Lemma 2.6 *Let M be a countable model of RCA_0. Then there exists a countable model M' of WKL_0 such that M' has the first part as M and $S_M \subseteq S_{M'}$.*

Proof. Use Corollary 2.5 repeatedly. □

Theorem 2.7 (*L. Harrington*) *For any Π^1_1 sentence φ, if φ is a theorem of WKL_0, then φ is already a theorem of RCA_0. In particular, the arithmetical part of WKL_0 is the same as that of RCA_0, or equivalently Σ^0_1-PA (first order Peano arithmetic with induction scheme restricted to the Σ^0_1-formulas).*

Proof. It easily follows from Lemma 2.6 by the help of Gödel's completeness theorem. □

We now recall another important characterization of models of WKL_0. Let M be a countable model of RCA_0. Let C be a countable subset of the set $P(|M|)$ of all subsets of $|M|$. $D \subseteq \mathcal{T}_M$ is $M \cup C$ *definable* if there exists a formula φ with parameters from $|M| \cup S_M \cup C$ such that for any $T \in \mathcal{T}_M$, $T \in D$ if and only if $M' \models \varphi(T)$, where $M' = (|M|, S_M \cup C, +_M, \cdot_M, 0_M, 1_M, <_M)$. A path G is said to be T_M-C-*generic* if, for every $M \cup C$ definable dense set $D \subseteq \mathcal{T}_M$, there exists $T \in D$ such that $G \in [T]$. If G is T_M-C-generic, then G is T_M-generic. The following lemma is a straightforward generalization of Lemma 2.2.

Lemma 2.8 *Let M be a countable model of* RCA_0. *Let C be a countable subset of $P(|M|)$. For any $T \in \mathcal{T}_M$, there exists T_M-C-generic G such that $G \in [T]$.*

Lemma 2.9 *Let M be a countable model of* RCA_0. *Suppose that C is a countable subset of $P(|M|)$ with $S_M \cap C = \emptyset$ and G is T_M-C-generic. Then $S_{M[G]} \cap C = \emptyset$.*

Proof. We want to prove that for any $A \in C$ and any Σ^0_1 formula φ_1 and φ_2 with parameters from $|M| \cup S_M \cup \{G\}$,

$$A \neq \{n \in |M| : M[G] \models \varphi_1(n, G)\} \text{ or } A \neq \{n \in |M| : M[G] \models \neg\varphi_2(n, G)\}.$$

So fix any $A \in C$. Suppose that $\varphi_i(x, G)$ is of the form $\exists y \theta_i(x, G[y])$ where $\theta_i(x, \tau)$ is Σ^0_0 with parameters from $|M| \cup S_M$, for $i = 0, 1$. Then let D_A be the set of all $T \in \mathcal{T}_M$ such that one of the followings holds for some $m \in |M|$:

A1. $m \in A \wedge M \models \forall \tau \in T \neg\theta_1(m, \tau)$,

A2. $m \notin A \wedge M \models \exists w \forall \tau \in T(lh(\tau) = w \to \exists \tau' \subseteq \tau \theta_1(m, \tau'))$,

A3. $m \in A \wedge M \models \exists w \forall \tau \in T(lh(\tau) = w \to \exists \tau' \subseteq \tau \theta_2(m, \tau'))$,

A4. $m \notin A \wedge M \models \forall \tau \in T \neg\theta_2(m, \tau)$.

We show that D_A is dense. Then, there exists an M-tree in D_A such that G is a path through it. Hence, by the definition of D_A, the proof is completed.

To see that D_A is dense, let $T \in \mathcal{T}_M$ be given. We first claim that there exists $P \in [T]$ such that $A \neq \{n \in |M| : M[P] \models \varphi_1(n, P)\}$ or $A \neq \{n \in |M| : M[P] \models \neg\varphi_2(n, P)\}$. By way of contradiction, deny the claim. By Lemma 2.6, we can construct a countable model M' of WKL_0 such that M' has the same first order part as M and $S_M \subseteq S_{M'}$. Then,

$$n \in A \Leftrightarrow M' \models \forall Z(Z \text{ is a path through } T \rightarrow \varphi_1(n, Z)).$$

Since "Z is a path through T" is expressed as a Π^0_1 formula, "Z is a path through $T \rightarrow \varphi_1(n, Z)$" is Σ^0_1, and so the whole formula $\forall Z(Z$ is a path through $T \rightarrow \varphi_1(n, Z))$ is logically equivalent in M' to a Σ^0_1 formula $\varphi'_1(n)$ with parameters from $|M| \cup S_M$ by virtue of compactness of the Cantor space (cf. [12, Lemma V.III.2.4]). Since for any $n \in |M|$, $M' \models \varphi'(n)$ if and only if $M \models \varphi'(n)$, we finally have $n \in A \Leftrightarrow M \models \varphi'(n)$. Similarly, we have $n \in A \Leftrightarrow M' \models \exists Z(Z$ is a path through $T \wedge \neg\varphi_2(n, Z))\}$, and so by compactness, there exists a Π^0_1 formula $\psi'(x)$ with parameters from $|M| \cup S_M$ such that $n \in A \Leftrightarrow M \models \psi'(n)$ for all $N \in |M|$. Therefore, A is in S_M since M is a model of RCA_0. This contradicts with our assumption. Thus the claim is proved.

By the above claim, there exist $P \in [T]$ and $m \in |M|$ such that one of the following conditions holds:

B1. $m \in A \wedge M[P] \models \forall y \neg\theta_1(m, P[y])$,

B2. $m \notin A \wedge M[P] \models \exists y \theta_1(m, P[y])$,

B3. $m \in A \wedge M[P] \models \exists y \theta_2(m, P[y])$,

B4. $m \notin A \wedge M[P] \models \forall y \neg\theta_2(m, Z[y])$.

First suppose that condition B1 holds. Let $T' = \{\tau \in T : \forall \tau' \subseteq \tau \neg\theta_1(m, \tau')\}$. Then, $T' \in \mathcal{T}_M$. It is also clear that A1 holds with T' (instead of T). Thus $T' \in D_A$. Next suppose that condition B2 holds. Take $\sigma \in (2^{<\mathbb{N}})_M$ with $\sigma = P[lh(\sigma)]$ and $\theta_1(m, \sigma)$. Set $T' = \{\tau \in T : \tau$ is compatible with $\sigma\}$. T' clearly satisfies A2, hence $T' \in D_A$. The other two cases can be treated similarly. Hence, in any case, there exists a subtree T' of T such that $T' \in D_A$, which means that D_A is dense. □

Corollary 2.10 *Let M be a countable model of* RCA_0. *Let C a countable* *subset of $P(|M|)$ such that $S_M \cap C = \emptyset$. For any $T \in \mathcal{T}_M$, there exists a countable model M' of* RCA_0 *such that the following four conditions hold*:

(1) *M' has the same first part as M,*

(2) *$S_M \subseteq S_{M'}$,*

(3) *$S_{M'} \cap C = \emptyset$,*

(4) *$M' \models T$ has a path.*

Proof. It is straightforward from Lemmas 2.4, 2.8 and 2.9. □

Lemma 2.11 *Let M be a countable model of* RCA_0*, and C a countable subset of $P(|M|)$ such that $S_M \cap C = \emptyset$. Then there exists a countable model M' of* WKL_0 *such that M' has the same first order part as M, $S_M \subseteq S_{M'}$ and $S_{M'} \cap C = \emptyset$.*

Proof. Use Corollary 2.10 repeatedly. □

The next theorem is a generalized version of Kreisel's hard core theorem.

Theorem 2.12 *Let M be a countable model of* RCA_0. *Then there exists a countable model M' of* WKL_0 *such that M' has the same first order part as M and $S_M \cap S_{M'} = \mathrm{REC}_M$.*

Proof. Let M be a countable model of RCA_0. Then $(|M|, \mathrm{REC}_M, +_M, \cdot_M, 0_M$ $1_M, <_M)$ is a countable model of RCA_0. Set $C = S_M \setminus \mathrm{REC}_M$. By Lemma 2.11, there exists a countable model M' of WKL_0 such that M' has the same first order part as M, $S' \subseteq S_{M'}$ and $S_{M'} \cap C = \emptyset$. That is, $S_M \cap S_{M'} = \mathrm{REC}_M$. □

Corollary 2.13 *Let N be a countable model of* Σ^0_1-PA. *Then there exist uncountably many countable models M of* WKL_0 *such that N is the first order part of M.*

Proof. Suppose that $\mathcal{A} = \{M : M$ is a countable model of WKL_0 with the first order part $N\}$ is countable. Let C be the set $(\cup\{S_M : M \in \mathcal{A}\}) \setminus \mathrm{REC}_{M_0}$, where $M_0 = (|N|, \emptyset, +_N, \cdot_N, 0_N, 1_N, <_N)$. By Lemma 2.11, we obtain another model M' of WKL_0 such that N is the first order part of M' and $S_{M'} \cap C = \emptyset$. This is a contradiction. □

3 Forcing with universal trees

In this section, we introduce the notion of M-universal trees and prove that all M-universal trees are homeomorphic to one another over M, where M is a countable model of RCA_0. Then, we show that all M-universal trees weakly force the same $\mathrm{L}_2(|M| \cup S_M)$-sentences.

Definition 3.1 *Let M be a countable model of RCA_0. Let φ be a sentence in $\mathrm{L}_2(|M| \cup S_M \cup \{G\})$. For any $T \in \mathcal{T}_M$, φ is said to be weakly forced by T (denoted $T \Vdash \varphi$) if $M[G] \models \varphi$ for all $\mathcal{T}_M$-generic $G \in [T]$.*

Lemma 3.2 *Let M be a countable model of RCA_0. Let φ be a sentence in $\mathrm{L}_2(|M| \cup S_M \cup \{G\})$. Then we have*

(1) *$T \Vdash \varphi$ is definable over M. Indeed, there exists an L_2-formula φ' such that $T \Vdash \varphi(n_1, \ldots, n_k, A_1, \ldots A_l)$ if and only if $M \models \varphi'(n_1, \ldots, n_k, A_1, \ldots A_l, T)$, where $n_1, \ldots n_k$ are from $|M|$ and $A_1, \ldots A_l$ from S_M.*

(2) *For any $\mathcal{T}_M$-generic $G \in [T]$, if $M[G] \models \varphi$ then there exists $T' \in \mathcal{T}_M$ such that $T' \subseteq T$, $G \in [T']$ and $T' \Vdash \varphi$.*

Proof. We need to prove (1) and (2) of Lemma 3.2 simultaneously by induction on φ. However, we here only show (1) since (2) can be treated in an obvious way.

Case 1: Suppose that φ is atomic. When φ is $t \in G$, $T \Vdash \varphi$ if and only if

$$M \models \exists m(\forall \sigma \in T(lh(\sigma) = m \to \sigma(t) = 1)).$$

For other atomic φ, $T \Vdash \varphi$ if and only if $M \models \varphi$. Thus $T \Vdash \varphi$ is definable over M.

Case 2: Suppose that $\varphi \equiv \neg\psi$. We clearly have $\forall T' \in \mathcal{T}_M(T' \subseteq T \to T' \nVdash \psi)$ if $T \Vdash \varphi$. Conversely, assume that $T \nVdash \varphi$. Then, there exists $G \in [T]$ such that $M[G] \models \psi$. By the induction hypothesis of (2), there exists $T' \in \mathcal{T}_M$ such that $T' \subseteq T$, $G \in [T']$ and $T' \Vdash \psi$. Thus, $T \Vdash \varphi$ if and only if $\forall T' \in \mathcal{T}_M(T' \subseteq T \to T' \nVdash \psi)$. Therefore, $T \Vdash \varphi$ is definable.

Case 3: Suppose that $\varphi \equiv (\psi_1 \wedge \psi_2)$. Then, $T \Vdash \varphi$ if and only if $T \Vdash \psi_1 \,\wedge\, T \Vdash \psi_2$. So $T \Vdash \varphi$ is definable.

Case 4: Suppose that $\varphi \equiv \exists x\psi(x)$. We show that

$$T \Vdash \varphi \Leftrightarrow \forall T' \in \mathcal{T}_M(T' \subseteq T \to \exists T'' \in T_M \exists n \in |M|(T'' \subseteq T' \wedge T'' \Vdash \psi(n))).$$

First assume that the right hand side. Let $D = \{T' \in \mathcal{T}_M : \exists n \in |M|(T' \Vdash \psi(n))$ or $[T] \cap [T'] = \emptyset\}$. Then it is easy to see that D is dense. Fix any $\mathcal{T}_M$-generic path G through T. Since D is dense, there exists T' such that $G \in [T']$ and $T' \Vdash \psi(n)$ for some $n \in |M|$. Therefore $M[G] \Vdash \varphi$, and hence $T \Vdash \varphi$.

Conversely, assume that $T \Vdash \varphi$. Fix any $T' \in \mathcal{T}_M$ with $T' \subseteq T$. Let G be a generic path through T'. Then $M[G] \Vdash \varphi$. Therefore, $M[G] \Vdash \psi(n)$ for some $n \in |M|$. By the induction hypothesis of (2), there exists $T'' \in \mathcal{T}_M$ such that $T'' \subseteq T'$ and $T'' \Vdash \psi(n)$.

Case 5: Suppose that $\varphi \equiv \exists X\psi(X)$. Let Y be a triple $\langle A, \psi_1, \psi_2\rangle$ where $A \in S_M$ and, ψ_1 and ψ_2 are (codes for) Σ^0_1 and Π^0_1 formulas with parameters from $|M| \cup \{A, G\}$. Let $Tr_{\Sigma^0_1}$ and $Tr_{\Pi^0_1}$ be appropriate universal lightface formulas. $Name(Y)$ is defined to be $\forall x(Tr_{\Sigma^0_1}(\psi_1, x, A, G) \leftrightarrow Tr_{\Pi^0_1}(\psi_2, x, A, G))$. For any $T' \in T_M$ and any $\mathcal{T}_M$-generic $G \in [T']$, if $T' \Vdash Name(Y)$, then $\{n \in |M| : M[G] \models Tr_{\Sigma^0_1}(\psi_1, n, A, G)\} \in S_{M[G]}$. Conversely, for any $Z \in S_{M[G]}$, there exists a triple $W = \langle B, \psi'_1, \psi'_2\rangle$ such that $M[G] \models Name(W)$ and $Z = \{n \in |M| : M[G] \models Tr_{\Sigma^0_1}(\psi'_1, n, B, G)\} \in S_{M[G]}$.

By $\psi(Y)$, we denote the formula obtained from $\psi(X)$ by replacing $t \in X$ with $Tr_{\Sigma^0_1}(\psi_1, t, A, G)$. Then, by the same way as Case 4, we can prove that $T \Vdash \varphi$ if and only if $\forall T' \in \mathcal{T}_M(T' \subseteq T \to \exists T'' \in T_M(T'' \subseteq T' \wedge \exists Y(T'' \Vdash Name(Y) \wedge \psi(Y))))$ $\square$

Let $\mathcal{B}(X)$ be the set of Boolean expressions built from atoms in X by means of the usual set operations $\cup$, $\cap$ and ^{c}. For $\sigma \in (2^{<\mathbb{N}})_M$, let $[\sigma] = \{P \in \mathcal{P}_M : P[lh(\sigma)] = \sigma\}$. Then for any expression $b \in \mathcal{B}((2^{<\mathbb{N}})_M)$, $[b]$ is defined to be the subset of $\mathcal{P}_M$ which b denotes in the obvious way. For simplicity, we often write $\mathcal{B}$ for $\mathcal{B}((2^{<\mathbb{N}})_M)$.

For any two $T, T' \in \mathcal{T}_M$, a mapping F from $[T]$ to $[T']$ is said to be M-*continuous* or simply *continuous* if S_M contains a function $f : \mathcal{B} \to \mathcal{B}$ (called a *code* for F) such that for any $b \in \mathcal{B}$,

$$[f(b)] \cap [T] = F^{-1}([b] \cap [T']).$$

Then, we can easily see that $F(P) \in \Delta^0_1$-def$(M;P)$.

Definition 3.3 *Let M be a countable model of RCA_0. A tree $T \in \mathcal{T}_M$ is said to be (M-)universal if for any $T' \in \mathcal{T}_M$, there exists an M-continuous F from $[T]$ to $[T']$.*

Obviously, any subtree of a universal tree is also universal, whenever it belongs to $\mathcal{T}_M$. In the rest of this section, we only treat a countable model M of RCA_0 such that $S_M = \mathrm{REC}_M(A)$ for some A. Such a model M is said to be *principal* with a *generator* A.

Lemma 3.4 *Let M be a principal model of RCA_0. Then the following hold:*

(1) *There exists an M-universal tree.*

(2) *If T is a universal tree, then for any $T' \in \mathcal{T}_M$, there exists an M-continuous function F from $[T]$ onto $[T']$.*

(3) *If T and T' are universal trees, then there exists an M-homeomorphism F from $[T]$ to $[T']$.*

Proof. Let M be a principal model of RCA_0 with a generator A. For any $n \in |M|$ and $i = 0, 1$, let b^i_n be a Boolean expression $\bigcup\{\tau : \tau(n) = i \wedge lh(\tau) = n+1\}$. Then, $[T] \subseteq [b^i_n]$ if and only if every $P \in [T]$ satisfies $P(n) = i$, $i = 0, 1$.

Since M is principal, there exists a universal Σ^0_1 formula $\varphi_\Sigma(e, x)$ with parameters from $|M| \cup S_M$. Then, we say that $g : |M| \times |M| \to |M|$ is a *productive function* for T if for any e and $d \in |M|$, supposing that $(\forall n \in M([T] \subseteq [b^1_n] \to \varphi_\Sigma(e, n))$, $\forall n \in M([T] \subseteq [b^0_n] \to \varphi_\Sigma(d, n))$ and $\neg\exists x(\varphi_\Sigma(e, x) \wedge \varphi_\Sigma(d, x))$, we have

$$\neg(\varphi_\Sigma(e, g(e, d)) \vee \varphi_\Sigma(d, g(e, d))).$$

Claim 1 *There exists a tree $T \in \mathcal{T}_M$ which has a productive function in S_M.*

Proof. For any consistent first-order theory Γ, let T_Γ be an infinite tree such that $[T_\Gamma]$ = the set of the characteristic functions of consistent, complete extensions of Γ which is closed under logical consequence. It is known that for any $T \in \mathcal{T}_M$, there exists a first-order theory Γ_T such that there exists an M-homeomorphism function from $[T_{\Gamma_T}]$ to $[T]$. (See [12, Section IV.3.2] for details.)

For any $X \in S_M$, let Q_X be an $\mathcal{L}_1(R)$-theory whose axioms consist of Robinson arithmetic Q plus $\{R(n) : n \in X\} \cup \{\neg R(n) : n \notin X\}$ with a new unary relation symbol R. Then Q_X is consistent since it has a weak model [12, Theorem II.8.4].

We show that T_{Q_A} has a productive function in S_M where A is a generator of M. Assume that $\neg\exists x(\varphi_\Sigma(e,x) \wedge \varphi_\Sigma(d,x))$. We can effectively find an $\mathcal{L}_1(R)$-formula $\Phi_{e,d}$ with only one free variable such that

$$\varphi_\Sigma(e,n) \to \mathsf{Q}_A \vdash \Phi_{e,d}(\underline{\mathrm{n}}), \qquad \varphi_\Sigma(d,n) \to \mathsf{Q}_A \vdash \neg\Phi_{e,d}(\underline{\mathrm{n}}),$$

where $\underline{\mathrm{n}}$ is the numeral for n (cf. Theorem III.1.23 [5]). By a diagonal argument [5, pp. 158], we can also effectively find an $\mathcal{L}_1(R)$-sentence ψ such that $\mathsf{Q}_A \vdash \psi_{e,d} \leftrightarrow \neg\Phi_{e,d}(\underline{\ulcorner\psi_{e,d}\urcorner})$, where $\ulcorner\psi_{e,d}\urcorner$ is the Gödel number of $\psi_{e,d}$. Let g be a function such that $g(e,d) = \ulcorner\psi_{e,d}\urcorner$. Then g is a productive function for T_{Q_A} [10, pp. 94]. $\square$

Let f be a function from $|M|$ to $\mathcal{B}$. Then we can extend f to $f' : \mathcal{B} \to \mathcal{B}$ such that for each $\sigma \in (2^{<\mathbb{N}})_M$,

$$f'(\sigma) = \bigcap_{i<lh(\sigma),\sigma(i)=1} f(i) \cap \bigcap_{j<lh(\sigma),\sigma(j)=0} f(j)^c,$$

and that f' preserves Boolean operations. For simplicity, we also write f for f'.

Claim 2 *Assume that $T \in \mathcal{T}_M$ has a productive function in S_M. Then, for any $T' \in \mathcal{T}_M$, there exists an M-continuous function F from $[T]$ onto $[T']$.*

Proof. Our proof is inspired with an argument due to Pour-El/Kripke [9, the proof of Lemma 1].

Assume that $T \in \mathcal{T}_M$ has a productive function g in S_M. Fix any $T' \in \mathcal{T}_M$. To construct an M-continuous function F from $[T]$ onto $[T']$, it suffices to show that there exists an $f : |M| \to \mathcal{B}$ in S_M such that for any $b \in \mathcal{B}$, $[T] \cap [f(b)] \neq \emptyset \Leftrightarrow [T'] \cap [b] \neq \emptyset$. For, letting $F(P)$ be a unique $P' \in [T']$ such that $P' \in \bigcap_{n\in M} f(P[n])$, F is an M-continuous function from $[T]$ onto $[T']$ with code f.

Let $\psi_1(a,u,v,x,y)$ be a Σ^0_1 formula saying that $[T] \cap [v] \subseteq [b^0_x]$ or $[T'] \cap [u] \subseteq [b^1_a] \wedge x = g((y)_0,(y)_1)$. Similarly, let $\psi_2(a,u,v,x,y)$ mean that $[T] \cap [v] \subseteq [b^1_x]$

or $[T'] \cap [u] \subseteq [b^0_a] \wedge x = g((y)_0, (y)_1)$. By the recursion theorem, there exist two functions t_1 and t_2 in S_M such that

$$\forall x(\varphi_\Sigma(t_1(a,u,v),x) \leftrightarrow \psi_1(a,u,v,x,\langle t_1(a,u,v), t_2(a,u,v)\rangle)) \text{ and,}$$

$$\forall x(\varphi_\Sigma(t_2(a,u,v),x) \leftrightarrow \psi_2(a,u,v,x,\langle t_1(a,u,v), t_2(a,u,v)\rangle)).$$

Finally, put $k(a,u,v) = g(t_1(a,u,v), t_2(a,u,v))$.

Assuming that for any $l <_M n$, $f(l)$ is defined, let $f(n) = \bigcup_{\sigma \in (2^n)_M} (f(\sigma) \cap b^1_{k(n,\sigma,f(\sigma))})$. Then, it is obvious that $f \in S_M$. We now want to show that

$$(1) \quad \forall b \in \mathcal{B}([T] \cap [f(b)] \neq \emptyset \Leftrightarrow [T'] \cap [b] \neq \emptyset).$$

Let $\varphi(n)$ be a $\Sigma^0_0(\Sigma^0_1)$ formula which means that $\forall \sigma \in 2^n((([T] \cap [f(\sigma)] \neq \emptyset \leftrightarrow [T'] \cap [\sigma] \neq \emptyset))$. Then, it suffices to show that $M \models \forall n \varphi(n)$. Obviously, $M \models \varphi(0)$. Suppose that $M \models \varphi(n)$. Then, we will show that $M \models \varphi(n+1)$ holds, that is, for any $\sigma \in (2^n)_M$,

$$[T] \cap [f(\sigma)] \cap [f(n)] \neq \emptyset \Leftrightarrow [T'] \cap [\sigma] \cap [b^1_n] \neq \emptyset$$

and

$$[T] \cap [f(\sigma)] \cap [f(n)]^c \neq \emptyset \Leftrightarrow [T'] \cap [\sigma] \cap [b^0_n] \neq \emptyset.$$

We may suppose that $[T'] \cap [\sigma] \neq \emptyset$. By the hypothesis, $[T] \cap [f(\sigma)] \neq \emptyset$. We first prove that $[T'] \cap [\sigma] \cap [b^1_n] = \emptyset \Rightarrow [T_1] \cap [f(\sigma)] \cap [f(n)] = \emptyset$. Suppose that $[T'] \cap [\sigma] \cap [b^1_n] = \emptyset$. By the construction of k, M satisfies

$$(2)\ \forall x(\varphi_\Sigma(t_2(n,\sigma,f(\sigma)),x) \leftrightarrow (x = k(n,\sigma,f(\sigma)) \vee [T] \cap [f(\sigma)] \subseteq [b^1_x]))$$

and

$$(3)\ \forall x(\varphi_\Sigma(t_1(n,\sigma,f(\sigma)),x) \leftrightarrow [T] \cap [f(\sigma)] \subseteq [b^0_x]).$$

By way of contradiction, we assume that $[T] \cap [f(\sigma)] \cap [f(n)] \neq \emptyset$. Then, $[T] \cap [f(\sigma)] \cap [b^1_{k(n,\sigma,f(\sigma))}] \neq \emptyset$ since $f(n) = [f(\sigma)] \cap [b^1_{k(n,\sigma,f(\sigma))}]$. Therefore,

$$\neg \exists x(\varphi_\Sigma(t_1(n,\sigma,f(\sigma)),x) \wedge \varphi_\Sigma(t_2(n,\sigma,f(\sigma)),x).$$

Since g is a productive function for T,

$$\neg \varphi_\Sigma(t_2(n,\sigma,f(\sigma)), k(n,\sigma,f(\sigma))).$$

This contradicts with (2). Therefore, $[T']\cap[\sigma]\cap[b^1_n]=\emptyset \Rightarrow [T]\cap[f(\sigma)]\cap[f(n)]=\emptyset$. In a similar manner, we can prove that

$$[T']\cap[\sigma]\cap[b^0_n]=\emptyset \Rightarrow [T]\cap[f(\sigma)]\cap[f(n)]^c=\emptyset$$

and

$$[T']\cap[\sigma]\cap[b^1_n]\neq\emptyset \wedge [T']\cap[\sigma]\cap[b^0_n]\neq\emptyset \Rightarrow$$

$$[T]\cap[f(\sigma)]\cap[f(n)]\neq\emptyset \wedge [T]\cap[f(\sigma)]\cap[f(n)]^c\neq\emptyset.$$

Thus, $M \models \varphi(n+1)$. By $\Sigma^0_0(\Sigma^0_1)$-induction, then (1) holds. The proof is completed. □

Claim 3 *Let M be a countable model of RCA_0. Suppose that both T and T' have productive functions in S_M. Then there exists an M-homeomorphism H from $[T]$ to $[T']$.*

Proof. The proof is an obvious modification of the proof of Claim 2. (Cf. [9, the proof of Lemma 2].) Suppose that both T and T' have productive functions in S. To construct a homeomorphism H from $[T]$ to $[T']$, it suffices to show that there exist two functions h_1 and h_2 from M to $\mathcal{B}$ such that for any $b, b' \in \mathcal{B}$,

$$[T]\cap[b']\cap[h_1(b)]\neq\emptyset \text{ if and only if } [T']\cap[h_2(b')]\cap[b]\neq\emptyset.$$

For, letting $H(P)$ be a unique $P' \in [T']$ such that $P' \in \bigcap_{n\in M} h_1(P[n])$, H is an M-homeomorphism from $[T]$ to $[T']$ with code h_1 and H^{-1} has a code h_2.

We construct h_1 and h_2 as follows. Assume that we have already defined $h_1(l)$ and $h_2(l)$ for any $l <_M n$. Then for any $b, b' \in \mathcal{B}(\{\sigma : lh(\sigma) \leq n\})$,

$$[T]\cap[b']\cap[h_1(b)]\neq\emptyset \text{ if and only if } [T']\cap[h_2(b')]\cap[b]\neq\emptyset.$$

As the proof of Claim 2, we can define $h_1(n)$ such that for any $b \in \mathcal{B}(\{\sigma : lh(\sigma) \leq n+1\})$ and $b' \in \mathcal{B}$, $[T]\cap[b']\cap[h_1(b)]\neq\emptyset$ if and only if $[T']\cap[h_2(b')]\cap[b]\neq\emptyset$. In a similar way, we can find $h_2(n)$ such that for any $b, b' \in \mathcal{B}(\{\sigma : lh(\sigma) \leq n+1\})$, $[T]\cap[b']\cap[h_1(b)]\neq\emptyset$ if and only if $[T']\cap[h_2(b')]\cap[b]\neq\emptyset$. The proof is completed. □

Claim 4 *Assume that T has a productive function in S_M and there exists an M-continuous function F from $[T']$ to $[T]$. Then $[T']$ is M-homeomorphic to $[T'']$ for some T'' which has a productive function in S_M.*

Proof. Our proof is just a formalization of a well-known fact on effectively inseparable sets (cf. [9, Lemma 3]). Let f be a code for F. Then, we have

$$[T] \subseteq [b_n^i] \Rightarrow [T'] \subseteq [f(b_n^i)],\ i = 0, 1.$$

Let Γ be a propositional theory $\{\bigvee\{\bigwedge\{a_i^{\tau(i)} : i <_M n\} : \tau \in T', lh(\tau) = n\} : n \in |M|\}$, where a_i's are atoms, and we set $a_i^1 = a_i$ and $a_i^0 = \neg a_i$. Let β be the natural interpretation of $\mathcal{B}$ into propositional formulas such that $\beta(b_n^1) = a_n$ for all $n \in |M|$. Then,

$$[T] \subseteq [b_n^i] \Rightarrow [T_\Gamma] \subseteq [b^i_{\beta(f(b_n^1))}],\ i = 0, 1.$$

By the S_n^m-theorem, there exists a function t in S_M such that

$$\forall x(\varphi_\Sigma(t(e), x) \leftrightarrow \varphi_\Sigma(e, \beta(f(x)))).$$

Let $h(e, d) = \beta(f(b^1_{g(t(e),t(d))}))$. Then h is a productive function for T_Γ, which is M-homeomorphic to T'. □

Claim 5 *$T \in \mathcal{T}_M$ is universal if and only if $[T]$ is M-homeomorphic to $[T']$ for some T' which has a productive function in S_M.*

Proof. It follows from Claims 2 and 4. □

It is straightforward from the above five claims to obtain (1) through (3) of Lemma 3.4. □

Lemma 3.5 *Let M be a principal model of RCA_0. If T_1 and T_2 are M-universal trees, then $T_1 \Vdash \varphi$ if and only if $T_2 \Vdash \varphi$ for any sentence φ in $\mathrm{L}_2(|M| \cup S_M)$.*

Proof. Let T_1 and T_2 be universal trees. Let H be an M-homeomorphism from $[T_1]$ to $[T_2]$. It is enough to show that for any sentence φ of $\mathrm{L}_2(|M| \cup S_M)$, if $T_2 \Vdash \varphi$ then $T_1 \Vdash_1 \varphi$. Assume that $T_2 \Vdash \varphi$. Fix any $\mathcal{T}_M$-generic G with $G \in [T_1]$. Since an M-homeomorphism preserves the genericity, $H(G)$ is a $\mathcal{T}_M$-generic path through T_2. Then $S[H(G)] \models \varphi$. Since $S[H(G)] = S[G]$,

$S[G] \models \varphi$. Then $T_1 \Vdash \varphi$. $\Box$

Fix a universal tree U. $\mathbb{P}^U_{1,M}$ be the set of all $T \in \mathcal{T}_M$ such that $T \subseteq U$. We always omit U unless there is a possibility of misunderstanding. G is said to be $\mathbb{P}_{1,M}$-*generic* if for any M-definable $\mathbb{P}_{1,M}$-dense set D, there exists $T \in D$ such that $G \in [T]$. G is $\mathbb{P}_{1,M}$-generic if and only if G is $\mathcal{T}_M$-generic with $G \in [U]$. Let φ be a sentence in $\mathrm{L}_2(|M| \cup S_M \cup \{G\})$. For any $T \in \mathbb{P}_{1,M}$, φ is said to be weakly forced by T (denoted $T \Vdash_1 \varphi$) if $M[G] \models \varphi$ for all $\mathbb{P}_{1,M}$-generic G with $G \in [T]$. That is, for any $T \in \mathbb{P}_{1,M}$, $T \Vdash_1 \varphi$ if and only if $T \Vdash \varphi$. We write $\Vdash_1 \varphi$ if $T \Vdash_1 \varphi$ for all $T \in \mathbb{P}_{1,M}$.

Lemma 3.6 *Let M be a principal model of* RCA_0*. Let φ be an $\mathrm{L}_2(|M| \cup S_M)$-sentence. If G is $\mathbb{P}_{1,M}$-generic, then $M[G] \models \varphi$ if and only if $\Vdash_1 \varphi$.*

Proof. Assume that $M[G] \models \varphi$. Then there exists $T \in \mathbb{P}_{1,M}$ such that $G \in [T]$ and $T \Vdash_1 \varphi$. By Lemma 3.5, $T' \Vdash \varphi$ for any $T' \in \mathbb{P}_{1,M}$. $\Box$

Corollary 3.7 *Let M be a principal model of* RCA_0*. If G and H are $\mathbb{P}_{1,M}$-generic, then $M[G]$ and $M[H]$ satisfy the same $\mathrm{L}_2(|M| \cup S_M)$-sentences.*

Proof. It is straightforward from Lemma 3.6. $\Box$

Let C be a countable subset of $P(|M|)$. G is said to be $\mathbb{P}_{1,M}$-C-*generic* if, for every $\mathbb{P}_{1,M}$-dense, $M \cup C$-definable set D, there exists $T \in D$ such that $G \in [T]$.

Lemma 3.8 *Let M be a countable model of* RCA_0*. Let C be a countable subset of $P(|M|)$ such that $S_M \cap C = \emptyset$. If G is $\mathbb{P}_{1,M}$-C-generic, then $M[G]$ is a countable model of* RCA_0 *with $S_{M[G]} \cap C = \emptyset$.*

Proof. Immediate from Lemma 2.9. $\Box$

4 A main result

We use *iterated forcing* to prove our main theorem that if WKL_0 proves $\forall X \exists ! Y \varphi(X, Y)$ with φ arithmetical, so does RCA_0. We first define the 2-forcing notion $\Vdash_2$. Let M be a principal model of RCA_0. A 2-condition is defined to be a pair $\langle T_1, T_2\rangle$ such that $T_1 \in \mathbb{P}_{1,M}$ and $T_1 \Vdash_1 (Name(T_2)$ and $T_2 \in \mathbb{P}_{1,M[G_1]})$. $\langle T_1, T_2\rangle \leq_2 \langle T_1', T_2'\rangle$ if $T_1 \subseteq T_1'$ and $T_1 \Vdash_1 T_2 \subseteq T_2'$. Let $\mathbb{P}_{2,M}$ be the set of 2-conditions. $D \subseteq \mathbb{P}_{2,M}$ is $\mathbb{P}_{2,M}$-dense if, for each $P \in \mathbb{P}_{2,M}$, there exists $P' \in D$ such that $P' \leq_2 P$. Let G be a generic filter of $\mathbb{P}_{2,M}$, i.e., a filter such that for all definable $\mathbb{P}_{2,M}$-dense set D, $G \cap D \neq \emptyset$. Then, $G_1 = \bigcap\{T_1 : \langle T_1, T_2\rangle \in G \text{ for some } T_2\}$ is $\mathbb{P}_{1,M}$-generic. Moreover, $G_2 = \bigcap\{i_{G_1}(T_2) : \langle T_1, T_2\rangle \in G \text{ for some } T_1 \text{ with } G_1 \in [T_1]\}$ is $\mathbb{P}_{1,M[G_1]}$-generic. Here $i_{G_1}(Y) = \{n \in |M| : \exists T' \in \mathbb{P}_{1,M}(G_1 \in [T'] \wedge T' \Vdash \psi_1(n)\}$, i.e., the evaluation of name $Y = \langle X, \varphi_1, \varphi_2\rangle$. Then, we regard G as a pair $\langle G_1, G_2\rangle$ and call it $\mathbb{P}_{2,M}$*-generic*. For any $\mathbb{P}_{2,M}$-generic $G = \langle G_1, G_2\rangle$ and any 2-condition $P = \langle T_1, T_2\rangle$, $G \in [P]$ means that $G_j \in [T_j]$ for $j = 1, 2$.

Definition 4.1 *Let M be a principal model of RCA_0. Let φ be a sentence of $\mathrm{L}_2(|M| \cup S_M \cup \{G_1, G_2\})$. For any $P \in \mathbb{P}_{2,M}$, φ is said to be weakly forced by P (denoted $P \Vdash_2 \varphi$) if $M[G] \models \varphi$ for all $\mathbb{P}_{2,M}$-generic $G \in [P]$.*

The next lemma can be proved in a standard way (cf. Lemma 3.2).

Lemma 4.2 *Let M be a principal model of RCA_0. Let φ be a sentence of $\mathrm{L}_2(|M| \cup S_M \cup \{G_1, G_2\})$. Then we have*

(1) *$P \Vdash_2 \varphi$ is definable over M.*

(2) *For any $\mathbb{P}_{2,M}$-generic $G \in [P]$, if $M[G] \models \varphi$ then there exists $P' \in \mathbb{P}_{2,M}$ such that $P' \leq_2 P$, $G \in [P']$ and $P' \Vdash_2 \varphi$.*

Lemma 4.3 *Let M be a principal model of RCA_0. For $G = \langle G_1, G_2\rangle \in \mathcal{P}_M \times \mathcal{P}_M$, G is $\mathbb{P}_{2,M}$-generic if and only if G_1 is $\mathbb{P}_{1,M}$-generic and G_2 is $\mathbb{P}_{1,M[G_1]}$-generic.*

Proof. Assume that G_1 is $\mathbb{P}_{1,M}$-generic and G_2 is $\mathbb{P}_{1,M[G_1]}$-generic. Set

$$G = \{\langle T_1, T_2\rangle : G_1 \in [T_1], G_2 \in [i_{G_1}(T_2)]\}.$$

Then, it is easy to see that G is generic filter of $\mathbb{P}_{2,M}$ with $G = \langle G_1, G_2\rangle$. So G is $\mathbb{P}_{2,M}$-generic. $\square$

Corollary 4.4 *Let M be a principal model of RCA_0. Let φ be a sentence of $\mathrm{L}_2(|M| \cup S_M \cup \{G_1, G_2\})$. Then, for any $\langle T_1, T_2\rangle \in \mathbb{P}_{2,M}$, $\langle T_1, T_2\rangle \Vdash_2 \varphi$ if and only if $T_1 \Vdash_1 T_2 \Vdash_1 \varphi$.*

Proof. Immediate from Lemma 4.3. □

Lemma 4.5 *Let M be a principal model of* RCA_0. *Let φ be an* $\mathrm{L}_2(|M| \cup S_M)$*-sentence. If P and P' are two 2-conditions, then $P \Vdash_2 \varphi$ if and only if $P' \Vdash_2 \varphi$.*

Proof. Let $P = \langle T_1, T_2 \rangle$ and $P' = \langle T_1', T_2' \rangle$ be 2-conditions. Suppose that $P' \Vdash_2 \varphi$. We shall show $P \Vdash_2 \varphi$. To see this, let $G = \langle G_1, G_2 \rangle \in [P]$ be $\mathbb{P}_{2,M}$-generic. Since T_1 and T_1' are M-universal, there exists an M-homeomorphism $H_1 : [T_1] \to [T_1']$. Then, $M[G_1] = M[H_1(G_1)]$. Therefore, $i_{G_1}(T_2)$ is $M[H_1(G_1)]$-universal. Similarly, there exists an $M[H_1(G_1)]$-homeomorphism $H_2 : [i_{G_1}(T_2)]$ $\to [i_{H_1(G_1)}(T_2)]$. Then, we have

$$M[\langle G_1, G_2 \rangle] = M[\langle H_1(G_1), H_2(G_2) \rangle] \models \varphi,$$

since $H(G) = \langle H_1(G_1), H_2(G_2) \rangle$ is $\mathbb{P}_{2,M}$-generic with $H(G) \in [P']$. Thus, $P \Vdash_2 \varphi$. The other direction can be proved in the same way. □

Lemma 4.6 *Let M be a principal model of* RCA_0. *Let φ be an* $\mathrm{L}_2(|M| \cup S_M)$*-sentence. If G is $\mathbb{P}_{2,M}$-generic, then $M[G] \models \varphi$ is equivalent to $\Vdash_2 \varphi$, i.e., $P \Vdash_2 \varphi$ for all $P \in \mathbb{P}_{2,M}$.*

Proof. Suppose that G is $\mathbb{P}_{2,M}$-generic and $M[G] \models \varphi$. Since $M[G] \models \varphi$, there exists $P \Vdash_2 \varphi$. By Lemma 4.5, for any $P' \in \mathbb{P}_{2,M}$, $P' \Vdash_2 \varphi$. □

Lemma 4.7 *Let M be a principal model of* RCA_0. *Let φ be an* $\mathrm{L}_2(|M| \cup S_M)$*-sentence. If G and H are $\mathbb{P}_{2,M}$-generic, then $M[G] \models \varphi$ is equivalent to $M[H] \models \varphi$.*

Proof. Immediate from Lemma 4.6. □

Let C be a countable subset of $P(|M|)$. A $\mathbb{P}_{2,M}$-generic G is said to be $\mathbb{P}_{2,M}$-*C-generic* if, for every $\mathbb{P}_{2,M}$-dense $M \cup C$ definable set D, there exists $P \in D$ such that $G \in [P]$. Then, $G = \langle G_1, G_2 \rangle \in \mathcal{P}_M^2$ is $\mathbb{P}_{2,M}$-C-generic if and only if G_1 is $\mathbb{P}_{1,M}$-C-generic and G_2 is $\mathbb{P}_{1,M[G_1]}$-C-generic.

Lemma 4.8 *Let M be a principal model of* RCA_0. *Let C be a countable subset of $P(|M|)$ such that $S_M \cap C = \emptyset$. If G is $\mathbb{P}_{2,M}$-C-generic, $S_{M[G]} \cap C = \emptyset$.*

Proof. Use Lemma 3.8 repeatedly. □

Now, by iterating 1-forcing notion, for any $i > 0$, we can define the i-forcing notion. Given the $(i-1)$-forcing notion, the i-forcing notion is defined as follows. An i-condition is defined to be a pair $\langle P, P' \rangle$ such that P is an $(i-1)$-condition and $P \Vdash_i (Name(P')$ and P' is a 1-condition). $\langle P, P' \rangle \leq_i \langle Q, Q' \rangle$ if $P \leq_{i-1} Q$ and $P \Vdash_{i-1} P' \subseteq Q'$. $\langle P, P' \rangle \Vdash_i Name(X)$ if $P \Vdash_{i-1} (P' \Vdash_1 Name(X))$. Let $\mathbb{P}_i$ be the set of i-conditions. $D \subseteq \mathbb{P}_{i,M}$ is $\mathbb{P}_{i,M}$-dense, if for each $P \in \mathbb{P}_{i,M}$, there exists $P' \in D$ such that $P' \leq_i P$. Let G be a generic filter of $\mathbb{P}_{i,M}$. Then, we can regard G as a sequence $\langle G_1, \ldots, G_i \rangle$ such that G_k is $\mathbb{P}_{1,M[\langle G_1,\ldots G_{k-1}\rangle]}$ for each $k = 1, \ldots, i$. We call it $\mathbb{P}_{i,M}$-*generic*. For any $\mathbb{P}_{i,M}$-generic $G = \langle G_1, \ldots G_i \rangle$ and any i-condition $P = \langle T_1, \ldots, T_i \rangle$, $G \in [P]$ means that $G_k \in [T_k]$ for $k = 1, \ldots i$. Let C be a countable subset of $P(|M|)$. A $\mathbb{P}_{i,M}$-generic G is said to be $\mathbb{P}_{i,M}$-C-*generic*, if for every $\mathbb{P}_{i,M}$-dense $M \cup C$ definable set D, there exists $P \in D$ such that $G \in [P]$.

Definition 4.9 *Let M be a principal model of* RCA_0. *Let φ be a sentence for* $\mathrm{L}_2(|M| \cup S_M \cup \{G_1, G_2, \ldots, G_i\})$. *For any $P \in \mathbb{P}_{i,M}$, φ is said to be weakly forced by P (denoted $P \Vdash_i \varphi$) if $M[G] \models \varphi$ for all $\mathbb{P}_{i,M}$-generic $G \in [P]$.*

The above properties on 2-forcing notion (Lemma 4.2 to Lemma 4.8) can be automatically extended to any i-forcing notion.

Next we define the ω-forcing notion. Fix a sequence $U = \langle U_i : i > 0 \rangle$ such that each U_i's are i-names and $\langle \ldots \langle U_1, U_2 \rangle, \ldots, U_{i-1} \rangle \Vdash_{i-1}$ "U_i is a universal tree". An ω-*condition* P is an i-condition such that $P \leq_i \langle \ldots \langle U_1, U_2 \rangle, \ldots, U_i \rangle$, for some $i > 0$. Let $\mathbb{P}_\omega$ be the set of ω-conditions. We may assume that ω is an initial segment of M closed under $+_M$ and $\cdot_M$ [7]. Then, $P \in \mathbb{P}_{\omega,M}$ is definable with parameters from $|M| \cup S_M \cup \{\omega\}$ over M. If $P \in \mathbb{P}_\omega$ is an i-condition and $j > i$, we can identify P with j-condition $\langle \ldots \langle \langle P, U_{i+1} \rangle, \ldots, U_j \rangle$. Then, for $P, P' \in \mathbb{P}_\omega$, we write $P \leq_\omega P'$ if P is an i-condition, P' is a j-condition, $j \leq i$ and $P \leq_i P'$. Let G be a generic filter of $\mathbb{P}_\omega$, i.e., a filter G meets all dense subsets of $\mathbb{P}_\omega$ definable with parameters from $|M| \cup S_M \cup \{\omega\}$ over M. Then, we can regard G as a sequence $\langle G_j : j > 0 \rangle$ such that the G_j's are $\mathbb{P}_{1,M[\langle G_1,\ldots G_{j-1}\rangle]}$. We call it $\mathbb{P}_{\omega,M}$-*generic*. For any $\mathbb{P}_{\omega,M}$-generic $G = \langle G_j : j > 0 \rangle$ and any ω-condition $P = \langle \ldots, \langle T_1, T_2 \rangle, \ldots, T_i \rangle$, $G \in [P]$ means that $G_k \in [T_k]$ for $k = 1, \ldots, i$.

Lemma 4.10 *Let M be a principal model of* RCA_0*. Let $G = \langle G_j : j > 0\rangle$ be $\mathbb{P}_{\omega,M}$-generic. Then, $M[G] \models \mathsf{WKL}_0$.*

Proof. For any $T \in \mathcal{T}_M$, if T' is an M-universal tree, there exists an M-continuous function $F : [T'] \to [T]$. Therefore, T has a path in $S_{M[G_1]}$ since G_1 is a path through some M-universal tree. Thus, for each $i \in \omega_{>0}$, any $T \in \mathcal{T}_{M[G_1,\ldots,G_{i-1}]}$ has a path in $S_{M[G_1,\ldots,G_i]}$. Then $M[G]$ is a model of WKL_0. $\square$

Definition 4.11 *Let M be a principal model of* RCA_0*. Let φ be a sentence in $\mathrm{L}_2(|M| \cup S_M \cup \{G_j : j > 0\})$. For any $P \in \mathbb{P}_{\omega,M}$, φ is said to be weakly forced by P (denoted $P \Vdash_\omega \varphi$) if $M[G] \models \varphi$ for all $\mathbb{P}_{\omega,M}$-generic $G \in [P]$.*

The next lemma can be proved in the same way as Lemma 3.2.

Lemma 4.12 *Let M be a principal model of* RCA_0*. Let φ be a sentence in $\mathrm{L}_2(|M| \cup S_M \cup \{G_j : j > 0\})$. Then we have*

(1) *$P \Vdash_\omega \varphi$ is definable with parameter from $|M| \cup S_M \cup \{\omega\}$ over M.*

(2) *For any $\mathbb{P}_{\omega,M}$-generic $G \in [P]$, if $M[G] \models \varphi$ then there exists $P' \in \mathbb{P}_{\omega,M}$ such that $P' \leq_\omega P$ and $P' \Vdash_\omega \varphi$.*

Lemma 4.13 *Let M be a principal model of* RCA_0*. If P_1 and P_2 are two ω-conditions, then $P_1 \Vdash_\omega \varphi$ if and only if $P_2 \Vdash_\omega \varphi$ for any sentence φ in $\mathrm{L}_2(|M| \cup S_M)$.*

Proof. The proof is an obvious modification of the proof of Lemma 4.5. Let P_1 and P_2 be two ω-conditions. Suppose that $P_2 \Vdash_\omega \varphi$. Fix any $\mathbb{P}_{\omega,M}$-generic $G = \langle G_j : j > 0\rangle \in [P_1]$. We assume that $P_1 = \langle \ldots \langle\langle T_1, T_2\rangle, T_3\rangle \ldots, T_j\rangle$ and $P_2 = \langle \ldots \langle\langle T'_1, T'_2\rangle, T'_3\rangle \ldots, T'_j\rangle$. Since T_1 and T'_1 are M-universal, there exists an M-homeomorphism $H_1 : [T_1] \to [T'_1]$. Then, $M[G_1] = M[H_1(G_1)]$. Therefore, $i_G(T_2)(= i_{G_1}(T_2))$ is $M[H_1(G_1)]$-universal. Then, there exists an $M[H_1(G_1)]$-homeomorphism $H_1 : [i_{G_1}(T_2)] \to [i_{H_1(G_1)}(T'_2)]$. Thus we have

$$M[G] = M[\langle H_1(G_1), H_2(G_2), G_3, \ldots, G_j, \ldots\rangle].$$

By iterating the above argument, let H be a sequence $\langle H_k : k \leq j\rangle$ such that each H_k is $M[\langle G_1, \ldots, G_k\rangle]$-homeomorphism from $[i_G(T_k)]$ to $[i_{G'}(T'_k)]$, where $G' = \langle H_1(G_1), \ldots, H_k(G_k)\rangle$. Then, we have a $\mathbb{P}_{\omega,M}$-generic $H(G)$ such that $H(G) = \langle H_1(G_1), \ldots, H_j(G_j), G_{j+1}, \ldots\rangle$. Therefore, $M[G] = M[H(G)] \models \varphi$

by $H(G) \in [P_2]$. Hence, $P_1 \Vdash_\omega \varphi \Rightarrow P_2 \Vdash_\omega \varphi$. Similarly, we can show that $P_2 \Vdash_\omega \varphi \Rightarrow P_1 \Vdash_\omega \varphi$. □

Lemma 4.14 *Let M be a principal model of* RCA_0*. Let φ be an $\mathrm{L}_2(|M| \cup S_M)$-sentence. If G is $\mathbb{P}_{\omega,M}$-generic, then $M[G] \models \varphi$ is equivalent to $\Vdash_\omega \varphi$, i.e., $P \Vdash_\omega \varphi$ for all $P \in \mathbb{P}_{\omega,M}$.*

Proof. Suppose that G is $\mathbb{P}_{\omega,M}$-generic and $M[G] \models \varphi$. Since $M[G] \models \varphi$, there exists $P \Vdash_\omega \varphi$. By Lemma 4.13, for any $P' \in \mathbb{P}_{\omega,M}$, $P' \Vdash_\omega \varphi$. □

Lemma 4.15 *Let M be a principal model of* RCA_0*. Let φ be an $\mathrm{L}_2(|M| \cup S_M)$-sentence. If G and H are $\mathbb{P}_{\omega,M}$-generic, then $M[G] \models \varphi$ is equivalent to $M[H] \models \varphi$.*

Proof. Immediate from Lemma 4.14. □

Let C be a countable subset of $P(|M|)$. A $\mathbb{P}_{\omega,M}$-generic G is said to be $\mathbb{P}_{\omega,M}$-*C-generic* if, for every $\mathbb{P}_{\omega,M}$-dense, $M \cup C$ definable set D, there exists $P \in D$ such that $G \in [P]$. Then, if $G = \langle G_j : j > 0 \rangle$, each G_j is $\mathbb{P}_{1,M[\langle G_1,\dots G_{j-1}\rangle]}$-$C$-generic.

Lemma 4.16 *Let M be a principal model of* RCA_0*. Let C be a countable subset of $P(|M|)$ such that $S_M \cap C = \emptyset$. If G is $\mathbb{P}_{\omega,M}$-C-generic, then $S_{M[G]} \cap C = \emptyset$.*

Proof. Suppose that $S_{M[G]} \cap C \neq \emptyset$. Then, there exists $A \in C$ such that $A \in S_{M[\langle G_1,\dots G_j\rangle]}$ for some $j > 0$. Since $\langle G_1, \dots, G_j \rangle$ is $\mathbb{P}_{j,M}$-C-generic (cf. Lemma 4.3), this is a contradiction. □

Lemma 4.17 *Let M be a principal model of* RCA_0*. Then there exist two countable models M_1, M_2 of* WKL_0 *such that:*

(1) *M_1 and M_2 have the same first order part as M,*

(2) $S_{M_1} \cap S_{M_2} = S_M$,

(3) *M_1 and M_2 satisfy the same $\mathrm{L}_2(|M| \cup S_M)$-sentences.*

Proof. Suppose that M is a principal model of RCA_0. Let G be $\mathbb{P}_{\omega,M}$-generic. Set $C = S_{M[G]} \setminus S_M$. By Lemma 4.16, there exists $\mathbb{P}_{\omega,M}$-generic H such that $M[H] \cap C = \emptyset$. By Lemma 4.15, $M[G]$ and $M[H]$ satisfy the same sentences with parameters from $|M| \cup S_M$. By Lemma 4.10, $M[G]$ and $M[H]$ are models of WKL_0. $\square$

Now, we have the main result of this paper.

Theorem 4.18 *Let $\varphi(X,Y)$ be an arithmetical formula with only the free variables shown. If WKL_0 proves $\forall X \exists! Y \varphi(X,Y)$, then so does RCA_0. (Indeed, RCA_0 also proves $\forall X \exists Y (Y$ is recursive in $X \wedge \varphi(X,Y))$.)*

Proof. Let $\varphi(X,Y)$ be an arithmetical formula with only the free variables shown. Suppose that WKL_0 proves $\forall X \exists! Y \varphi(X,Y)$. By way of contradiction, we assume RCA_0 does not prove $\forall X \exists! Y (Y$ is recursive in $X \wedge \varphi(X,Y))$. Then by Gödel's completeness theorem, there exists a countable model M of RCA_0 in which $\neg\exists! Y (Y$ is recursive in $A \wedge \varphi(A,Y)$ holds for some $A \in S_M$. Let $S_0 = \{B \in S_M : M \models B$ is recursive in $A\}$ and $M_0 = (|M|, S_0, +_M, \cdot_M, 0_M, 1_M, <_M$). Then M_0 is a principal model of RCA_0 such that $M_0 \models \neg\exists! Y \varphi(A,Y)$.

First suppose that $\exists Y \varphi(A,Y)$ holds in M_0. Then there exist more than one sets in S_0 which satisfy φ. By Lemma 2.6, there exists a model M' of WKL_0 such that M' has the same first order part as M and $S_{M_0} \subseteq S_{M'}$. Therefore, WKL_0 does not prove $\forall X \exists! Y \varphi(X,Y)$, which is a contradiction.

Next assume that $\forall Y \neg\varphi(A,Y)$ holds within M_0. By Lemma 4.17, there exists two countable models M_1 and M_2 of WKL_0 such that:

(1) M_1 and M_2 have the same first order part as M_0,

(2) $S_{M_1} \cap S_{M_2} = S_{M_0}$,

(3) M_1 and M_2 satisfy the same sentences with parameters from $|M| \cup S_M$.

Let $Y_i \in S_{M_i}$ be such that M_i satisfies $\varphi(A,Y_i)$ $(i = 1,2)$. Then, for each $n \in |M|$ and each $i = 1,2$,

$$n \in Y_i \Leftrightarrow M_i \models \exists Y (\varphi(A,Y) \wedge n \in Y).$$

By (3), for each n in $|M|$,

$$M_1 \models \exists Y (\varphi(A,Y) \wedge n \in Y) \Leftrightarrow M_2 \models \exists Y (\varphi(A,Y) \wedge n \in Y).$$

Therefore, $Y_1 = Y_2$. Then, by (2), $Y_1 \in S_{M_0}$. Therefore, by (1) and (2), M_0 satisfies $\varphi(A, Y_1)$ since φ is arithmetical and $M \models \varphi(A, Y_1)$. This is a contradiction. $\square$

Remark 4.19 We can also show that if M is a principal model of $\mathsf{RCA}_0 + \Sigma^0_k$ induction ($k = 1, 2, \ldots, \infty$), then $M[G] \models \Sigma^0_k$ induction for any $\mathbb{P}_{\omega,M}$-generic G (Yamazaki [unpublished]). Therefore, Theorem 4.18 can be extended as follows: if $\mathsf{WKL}_0 + \Sigma^0_k$ induction proves $\forall X \exists! Y \varphi(X, Y)$, then $\mathsf{RCA}_0 + \Sigma^0_k$ induction also proves $\forall X \exists! Y (Y \text{ is recursive in } X) \land \varphi(X, Y))$, where $\varphi(X, Y)$ is an arithmetical formula with only the free variables shown. In case $k = 2$ and φ is Σ^0_3, the above result was already proved by A. M. Fernandes [3], where general cases were mentioned as an open problem. Simpson [13] gives a different proof to Theorem 4.18 with more sophisticated recursion-theoretic investigations.

The following theorem tends to show that our main theorem is optimal.

Theorem 4.20 (1) *There exists a* Π^1_1 *formula* $\varphi_1(Y)$ *such that* WKL_0 *proves the sentence* $\exists! Y \varphi_1(Y)$*, but* WKL_0 *does not prove* $\exists Y (Y \text{ is recursive} \land \varphi_1(Y))$.

(2) *There exists a* Π^1_1 *formula* $\varphi_2(Y)$ *such that* WKL_0 *proves the sentence* $\exists! Y \varphi_2(Y)$*, but* RCA_0 *does not prove it.*

(3) *There exists a* Σ^1_1 *formula* $\varphi_3(Y)$ *such that* WKL_0 *proves the sentence* $\exists! Y \varphi_3(Y)$*, but* RCA_0 *does not prove* $\exists Y \varphi_3(Y)$.

Proof. (1) Let $\varphi_1(Y)$ be the Π^1_1 formula

$$Y = K \text{ or } (K \text{ does not exist and } Y = \emptyset)$$

where K is a complete recursively enumerable set. Then, the ω-model $P(\omega)$ does not satisfy $\exists Y (Y \text{ is recursive} \land \varphi_1(Y))$.

(2) Let $\varphi_2(Y)$ be the Π^1_1 formula $Y = \emptyset \lor (T \text{ has no path})$, where T is a certain recursive infinite 0-1 tree with no recursive path.

The ω-model REC does not satisfy $\exists! Y \varphi_2(Y)$.

(3) Let $\varphi_3(Y)$ be the Σ^1_1 formula $Y = \emptyset \wedge (T \text{ has a path})$. Then REC does not satisfy $\exists Y \varphi_3(Y)$. □

Problems. The following are still unknown to our circle.

(1) Suppose $\mathsf{WKL}_0 \vdash \exists! X \varphi(X)$ where $\varphi(X)$ is a Σ^1_1 formula with no free set variables other than X. Is it the case that $\mathsf{WKL}_0 \vdash \exists X(X \text{ is recursive} \wedge \varphi(X))$?

(2) Suppose $\mathsf{WKL}_0 \vdash \exists X(X \text{ is not recursive} \wedge \varphi(X))$ where $\varphi(X)$ is a Σ^1_1 formula with no free set variables other than X. Is it the case that $\mathsf{WKL}_0 \vdash \exists X, Y(X \neq Y \wedge \varphi(X) \wedge \varphi(Y))$? A similar question has been asked by Friedman [4].

(3) In [1], Avigad constructed an effective translation of WKL_0-proofs of Π^1_1 sentences to RCA_0-proofs of the same sentences, by formalizing Harrington's forcing argument. In fact, his translation has at most a polynomial increase in the length of proofs. Unfortunately, we have not managed to find such an effective bound for our conservation result.

5 Forcing with uniformly pointed perfect trees

In this section, we introduce a forcing argument with *universal pointed perfect trees*, which is inspired by Sacks [11]. Then we show that for any countable model M of RCA_0, there exists a principal model M' of RCA_0 such that M' has the same first order part as M and $S_M \subseteq S_{M'}$.

Let M be a countable model of RCA_0. For any $T \in \mathcal{T}_M$, T is *M-perfect* if $M \models (T \text{ is perfect})$. An M-perfect tree T is said to be *uniformly pointed* if for all $X \in [T]$, T has the same index of M-recursiveness in X, that is, there exist $e, d \in |M|$ such that for all $X \in [T]$,

$$M[X] \models \forall m(m \in T \leftrightarrow \varphi_\Sigma(e, m, X)) \text{ and } \forall m(m \in T \leftrightarrow \neg\varphi_\Sigma(d, m, X)),$$

where $\varphi_\Sigma(e, m, X)$ is a fixed universal lightface Σ^0_1 formula.

Let $\mathbb{P}_{0,M}$ be the set of uniformly pointed M-perfect trees. Then, it is easy to show that $\mathbb{P}_{0,M}$ is M-definable. We say that $D \subseteq \mathbb{P}_{0,M}$ is *dense* if $\forall T \in \mathbb{P}_{0,M} \exists T' \in D(T' \subseteq T)$. G is a $\mathbb{P}_{0,M}$-*generic path* if for any $\mathrm{L}_2(|M| \cup S_M \cup \{\omega\})$-

definable dense set D, there exists $T \in D$ such that $G \in [T]$. The following lemma can be proved in the same way as Lemma 2.2

Lemma 5.1 *Let M be a countable model of* RCA_0*. For any $T \in \mathbb{P}_{0,M}$, there exists a $\mathbb{P}_{0,M}$-generic path $G \in [T]$.*

Lemma 5.2 *Let M be a countable model of* RCA_0*. If G is $\mathbb{P}_{0,M}$-generic, then $M[G] \models \mathsf{RCA}_0$.*

Proof. Let G be a $\mathbb{P}_{0,M}$-generic path. We only need to show that $M[G]$ satisfies Σ^0_1 induction. To see this, it suffices to prove the following. For any $m \in |M|$ and any Σ^0_1-formula $\varphi(x, G)$, the set $\{n : n <_M m \wedge \varphi(n, G)\}$ is M-finite.

We may assume that $\varphi(x, G) \equiv \exists y \theta(x, G[y])$ where $\theta(x, \tau)$ is Σ^0_0 with parameters from $|M| \cup S_M$. Let D_m be the set of $T \in \mathbb{P}_{0,M}$ such that there exists $\sigma \in (2^m)_M$ such that for each $n <_M m$, M satisfies either

(1) $\sigma(n) = 0$ and $\forall \tau \in T \neg\theta(n, \tau)$

or

(2) $\sigma(n) = 1$ and $\exists k \forall \tau \in T(lh(\tau) = k \to \exists \tau' \subseteq \tau \theta(n, \tau'))$.

Then, the set $\{n : n <_M m \wedge \varphi(n, P)\}$ is M-finite if $P \in [T]$ for some $T \in D_m$. Therefore, it remains to show that D_m is dense. Let $T \in \mathbb{P}_{0,M}$ be given. We say that $\sigma \in (2^m)_M$ is *good* if $M \models \exists \tau \in T \forall x < m(\sigma(x) = 1 \to \exists \tau' \subseteq \tau \theta(x, \tau'))$. Set $S_m = \{\sigma \in (2^m)_M : \sigma \text{ is good}\}$. Since M satisfies bounded Σ^0_1 comprehension, S_m is M-finite. Moreover, S_m is nonempty since $\langle 0, \ldots, 0\rangle$ (with m 0's) is an element of S_m. Let σ_m be the lexicographically largest element of S_m. Since σ_m is good, there exists $\tau_m \in T$ such that

$$M \models \forall x < m(\sigma_m(x) = 1 \to \exists \tau' \subseteq \tau_m \theta(x, \tau')).$$

Set $T' = \{\tau \in T : M \models \tau \text{ is compatible with } \tau_m\}$. We are going to show that $T' \in D_m$. To see this, let $n <_M m$ be given. If $\sigma_m(n) = 1$, then $M \models \exists \tau' \subseteq \tau_m \theta(n, \tau')$. Then

$$M \models \exists k \forall \tau \in T'(lh(\tau) = k \to \exists \tau' \subseteq \tau \theta(n, \tau')).$$

Suppose that $\sigma_m(n) = 0$. Let σ' be a 0-1 string such that $\sigma'(n) = 1$ and $\sigma'(x) = \sigma_m(x)$ for $x \neq n$. Then, by the definition of σ_m, $M \models \forall \tau \in T \neg\theta(n, \tau)$.

So $M \models \forall \tau \in T' \neg\theta(n,\tau)$. Therefore, T' belongs to D_m $\square$

Definition 5.3 *Let $T \in \mathbb{P}_{0,M}$. For any $\mathrm{L}_2(|M| \cup S_M \cup \{\omega, G\})$-sentence φ, we say that $T \Vdash_0 \varphi$ if $M[G] \models \varphi$ for all $\mathbb{P}_{0,M}$-generic path $G \in [T]$.*

The next lemma can be proved.

Lemma 5.4 *Let M be a countable model of RCA_0. Let φ be a sentence in $\mathrm{L}_2(|M| \cup S_M \cup \{\omega, G\})$. Then we have*

(1) *$T \Vdash_0 \varphi$ is definable with parameter from $|M| \cup S_M \cup \{\omega\}$ over M.*

(2) *For any $\mathbb{P}_{0,M}$-generic $G \in [T]$, if $M[G] \models \varphi$ then there exists $T' \in \mathbb{P}_{0,M}$ such that $T' \subseteq T$, $G \in [T']$ and $T' \Vdash_0 \varphi$.*

Lemma 5.5 *Let M be a countable model of RCA_0. Let T and T' be M-perfect trees. Then, there exists an M-homeomorphism H from $[T]$ to $[T']$ with its code M-recursive in $T \oplus T'$.*

Proof. We shall first prove Lemma 5.5 under the assumption that $T' = \mathcal{P}_M (= (2^{<\mathbb{N}})_M)$. Define a function h_T from $\mathcal{P}_M$ to T inductively as follows. $h_T(\langle\rangle) =$ the least $\tau \in T$ such that $\tau^\frown\langle i\rangle \in T$ for each $i = 0, 1$. For $j = 0$ or 1, $h_T(\sigma^\frown\langle j\rangle) =$ the least $\tau \in T$ such that $h_T(\sigma)^\frown\langle j\rangle \subseteq \tau$ and $\tau^\frown\langle i\rangle \in T$ for each $i = 0, 1$. Then, h_T is M-recursive in T. So, by the construction, Boolean-preserving extension h of h_T is a code for an M-homeomorphism from $[T]$ to $[\mathcal{P}_M]$ which is M-recursive in T.

Let h_T (and $h_{T'}$) be the code for M-homeomorphisms from $[T]$ (and $[T']$) to $[\mathcal{P}_M]$. Then a function $h_T(h_{T'}^{-1} \lceil \mathrm{rng}(h_{T'}))$ can be extended to a code $h_{T,T'}$ for an M-homeomorphism $H_{T,T'}$ from $[T]$ to $[T']$, which is M-recursive in $T \oplus T'$. $\square$

$H_{T,T'} : [T] \to [T']$ and $h_{T,T'}$ in the proof of Lemma 5.5 are said be a canonical M-homeomorphism and a canonical code for $H_{T,T'}$, respectively.

Lemma 5.6 *Let T and T' be two M-trees in $\mathbb{P}_{0,M}$ such that $\mathrm{REC}_M(T) = \mathrm{REC}_M(T')$. Let $H : [T] \to [T']$ be a canonical M-homeomorphism. If $T_1 \in \mathbb{P}_{0,M}$ is a subtree of T, then there exists a subtree T_1' of T' such that $T_1' \in \mathbb{P}_{0,M}$ and $H([T_1]) = [T_1']$.*

Proof. Let $H : [T] \to [T']$ be a canonical M-homeomorphism. Fix $T_1 \in \mathbb{P}_{0,M}$ such that $T_1 \subseteq T$. Then, let $T_1' = \{\sigma \in T' : \exists \tau \in T'(\sigma \subseteq \tau \wedge h(\tau) \in T_1\}$, where h is a canonical code for H. Since T is M-recursive in T', h is M-recursive in T'. So T_1' is M-recursive in $T' \oplus T_1$. Obviously, T_1' is M-perfect and $H([T_1]) = [T_1']$.

It remains to show that T' is uniformly pointed. To see this, fix $X \in [T]$. Since $H^{-1}(X) \in [T_1]$ and T_1 is uniformly pointed, then

$$T_1' \leq_T T' \oplus T_1 \leq_T T' \oplus H^{-1}(X) \leq_T T' \oplus X.$$

Since T is M-recursive in T', h is M-recursive in T'. So T_1' is M-recursive in $T' \oplus T_1$. Since $X \in [T']$ and T' is uniformly pointed, then T_1' is M-recursive in X. In fact, by the above argument, for any $X \in [T]$, T_1' has the same index of M-recursiveness in X. $\square$

Lemma 5.7 *Let M be a countable model of* RCA_0. *Let $T \in \mathbb{P}_{0,M}$. Then, for any $A \in S_M$, if T is M-recursive in A, there exists a subtree T' of T such that $T' \in \mathbb{P}_{0,M}$ and* $\mathrm{REC}_M(A) = \mathrm{REC}_M(T')$.

Proof. Fix any $T \in \mathbb{P}_{0,M}$ and any $A \in S_M$. Let h be a canonical code for a canonical M-homeomorphism $H : [T] \to [\mathcal{P}_M]$. We work over M. Define $B \subseteq \mathcal{P}_M$ inductively as follows:

(1) $h(\langle\rangle) \in B$ and

(2) if $lh(\sigma) = 1$ is odd and $h(\sigma) \in B$, then $h(\sigma^\frown\langle i\rangle) \in B$ $(i = 0, 1)$ and

(3) if $lh(\sigma) = 1$ is even, $(lh(\sigma) - 1)/2 \in A$ and $h(\sigma) \in B$, then $h(\sigma^\frown\langle 0\rangle) \in B$

(4) if $lh(\sigma) = 1$ is even, $(lh(\sigma) - 1)/2 \notin A$ and $h(\sigma) \in B$, then $h(\sigma^\frown\langle 1\rangle) \in B$.

Set $T' = \{\sigma \in T : \exists \tau \in B(\sigma \subseteq \tau)\}$. By the construction, T' is perfect, and it is recursive in A since T is recursive A. Moreover, for all $m \in \mathbb{N}$,

$$m \in A \leftrightarrow \exists \sigma \in 2^{<\mathbb{N}}(lh(\sigma) = 2n + 1 \wedge h(\sigma^\frown\langle 0\rangle) \in T').$$

Therefore, A is recursive in $T \oplus T'$. Consider the leftmost path P through T'. Then, P is recursive in T'. Since T is uniformly pointed, T is recursive in T', so A is recursive in T'. Hence A and T' are recursive in each other.

It remains to prove that T' is uniformly pointed. To see this, fix $X \in [T']$.

Then, for all $m \in \mathbb{N}$,

$$m \in A \leftrightarrow \exists \sigma \in 2^{<\mathbb{N}}(lh(\sigma) = 2n+1 \wedge h(\sigma^\frown\langle 0\rangle)) = X[h(\sigma^\frown\langle 0\rangle)]).$$

Thus T' is uniformly pointed since T' is recursive in A. □

Lemma 5.8 *Let M be a countable model of* RCA_0. *Let $T_1, T_2 \in \mathbb{P}_{0,M}$. Then, $T_1 \Vdash_0 \varphi$ is equivalent to $T_2 \Vdash_0 \varphi$ for any sentence φ in $\mathrm{L}_2(|M| \cup S_M \cup \{\omega\})$.*

Proof. Let T_1 and T_2 be uniformly pointed M-perfect trees. Suppose that $T_1 \Vdash_0 \varphi$ and $T_2 \not\Vdash_0 \varphi$ for some $\mathrm{L}_2(|M| \cup S_M \cup \{\omega\})$-sentence φ. Then, by Lemma 5.4 (2), there exists $T_2' \in \mathbb{P}_{0,M}$ such that $T_2' \subseteq T_2$ and $T_2' \Vdash_0 \neg\varphi$. According to Lemma 5.7, there exists T_1' and T_2'' such that $T_1' \Vdash_0 \varphi$, $T_2'' \Vdash_0 \neg\varphi$ and $\mathrm{REC}_M(T_1') = \mathrm{REC}_M(T_2'')(= \mathrm{REC}_M(T_1 \oplus T_2'))$. Let G be a $\mathbb{P}_{0,M}$-generic path through T_1'. Then $M[G] \models \varphi$. Let $H : [T_1'] \to [T_2'']$ be a canonical M-homeomorphism. By Lemma 5.6, we can show that $H(G')$ is $\mathbb{P}_{0,M}$-generic through T_2''. Since $M[G] = M[H(G)]$, then $M[H(G)] \models \varphi$, so $T_2'' \not\Vdash_0 \neg\varphi$. This is a contradiction. □

Lemma 5.9 *Let M be a countable model of* RCA_0. *Let φ be an $\mathrm{L}_2(|M| \cup S_M \cup \{\omega\})$-sentence. If G is $\mathbb{P}_{0,M}$-generic, then $M[G] \models \varphi$ is equivalent to $\Vdash_0 \varphi$, i.e., $T \Vdash_0 \varphi$ for all $T \in \mathbb{P}_{0,M}$.*

Proof. Let $\mathbb{P}_{0,M}$-generic G be given. Suppose that $T \not\Vdash_0 \varphi$ for some $T \in \mathbb{P}_{0,M}$. Since $M[G] \models \varphi$, there exists $T' \in \mathbb{P}_{0,M}$ such that $T' \Vdash_0 \varphi$. By Lemma 5.8, this is a contradiction. □

Lemma 5.10 *Let M be a countable model of* RCA_0. *Then, $\Vdash_0 \exists X \forall Y(Y$ is recursive in $X)$. That is, for any $\mathbb{P}_{0,M}$-generic G, $M[G]$ is a principal model of* RCA_0.

Proof. It is sufficient to show that for any $Y \in S_M$, $\Vdash_0$ (Y is recursive in G). Fix $A \in S_M$. Set $D_A = \{T \in \mathbb{P}_{0,M} : T \Vdash_0 (A \text{ is recursive in } G)\}$. We want to show that D_A is dense. To see this, fix $T \in \mathbb{P}_{0,M}$. By Lemma 5.7, there exists $T' \in \mathbb{P}_{0,M}$ such that $T' \subseteq T$ and A is M-recursive in T'. Since T' is uniformly pointed, for any $\mathbb{P}_{0,M}$-generic G through T', T' is M-recursive in G, that is, A is M-recursive in G. Then $T' \in D_A$. □

Theorem 5.11 *Any countable model of* RCA_0 *is a submodel of a principal model of* RCA_0 *with the same first order part.*

Proof. This follows immediately from Lemma 5.10.

Let C be a countable subset of $P(M)$. Then, G is said to be $\mathbb{P}_{0,M}$-C-*generic* if there exists $T \in D$ such that $G \in [T]$ for all dense subset D of $\mathbb{P}_{0,M}$ definable with parameters from $|M| \cup S_M \cup C \cup \{\omega\}$. For any $T \in \mathbb{P}_{0,M}$, there exists a $\mathbb{P}_{0,M}$-C-generic path G through T.

Lemma 5.12 *Let M be a countable model of* RCA_0. *Let C be a countable subset of $P(|M|)$ such that $S_M \cap C = \emptyset$. If G is $\mathbb{P}_{0,M}$-C-generic, then $M[G] \cap C = \emptyset$.*

Proof. We want to show that for any $A \in C$ and any Σ^0_1 formulas $\varphi_1(x)$ and $\varphi_2(x)$ with parameters from $|M| \cup S \cup \{G\}$, either $A \neq \{n \in |M| : M[G] \models \varphi_1(n)\}$ or $A \neq \{n \in |M| : M[G] \models \neg\varphi_2(n)\}$.

We may assume that $\varphi_i(x)$ is of the form $\exists y \theta_i(x, G[y])$, where $\theta_i(x, \tau)$ is Σ^0_0 with parameter from $|M| \cup S_M$ $(i = 1, 2)$. Let D_A be the set of all $T \in \mathbb{P}_{0,M}$ such that one of the following holds for some $m \in |M|$:

A1. $m \in A \wedge M \models \forall \tau \in T \neg\theta_1(m, \tau)$,

A2. $m \notin A \wedge M \models \exists w \forall \tau \in T(\mathrm{lh}(\tau) = w \rightarrow \exists \tau' \subseteq \tau \theta_1(m, \tau'))$,

A3. $m \in A \wedge M \models \exists w \forall \tau \in T(\mathrm{lh}(\tau) = w \rightarrow \exists \tau' \subseteq \tau \theta_2(m, \tau'))$,

A4. $m \notin A \wedge M \models \forall \tau \in T \neg\theta_2(m, \tau)$.

Then it suffices to show that D_A is dense.

To see this, let $T \in \mathbb{P}_{0,M}$ be given. Case 1. Suppose that there exists $m \in |M|$ such that for all $\tau_1, \tau_2 \in T$, either $m \in A \wedge M \models \forall \tau' \subseteq \tau_1 \neg\theta_e(m, \tau')$ or $m \notin A \wedge M \models \forall \tau' \subseteq \tau_2 \neg\theta_d(m, \tau')$. Then T belongs to D_A.

Case 2. Suppose that there exists $m \in |M|$ and $\tau_1, \tau_2 \in T$ such that either $m \notin A \wedge M \models \exists \tau' \subseteq \tau_1 \theta_e(m, \tau')$ or, $m \in A \wedge M \models \exists \tau' \subseteq \tau_2 \theta_d(m, \tau')$. If $m \notin A$, let $T' = \{\sigma \in T : M \models \sigma \text{ is compatible with } \tau_1\}$. If $m \in A$, let $T' = \{\sigma \in T : M \models \sigma \text{ is compatible with } \tau_2\}$. Then, T' is M-perfect. It is also uniformly pointed since T' is M-recursive in T. So, T' belongs to D_A.

Case 3. Neither Case 1 nor Case 2. Then,

$$A = \{m : M \models \exists \tau \in T\theta_1(m, \tau)\} = \{m : M \models \forall \tau \in T\neg\theta_2(m, \tau)\}.$$

This is a contradiction. □

Let M be a countable model of RCA_0 and G be $\mathbb{P}_{0,M}$-generic. Then $M[G]$ is a principal model of RCA_0 by Theorem 5.11. Therefore there exists a $\mathbb{P}_{\omega,M[G]}$-generic G'.

Lemma 5.13 *Let M be a countable model of RCA_0. Let φ be an $\mathrm{L}_2(|M| \cup S_M)$-sentence. If G is $\mathbb{P}_{0,M}$-generic and G' is $\mathbb{P}_{\omega,M[G]}$-generic, then $M[G][G'] \models \varphi$ is equivalent to $\Vdash_0 \Vdash_\omega \varphi$.*

Proof. Let G be $\mathbb{P}_{0,M}$-generic and G' be $\mathbb{P}_{\omega,M[G]}$-generic. By Lemma 4.14,

$$M[G][G'] \models \varphi \Leftrightarrow M[G] \models \Vdash_\omega \varphi.$$

By Lemma 5.9,

$$M[G] \models \Vdash_\omega \varphi \Leftrightarrow M \models \Vdash_0 \Vdash_\omega \varphi.$$

□

Theorem 5.14 *Let G and H be $\mathbb{P}_{0,M}$-generic. Let G' and H' be $\mathbb{P}_{\omega,M[G]}$-generic and $\mathbb{P}_{\omega,M[H]}$-generic, respectively. Then, $M[G][G']$ and $M[H][H']$ satisfy the same $\mathrm{L}_2(|M| \cup S_M)$-sentences.*

Proof. Immediate from Lemma 5.13. □

Lemma 5.15 *There exist $\mathbb{P}_{0,M}$-generic G, $\mathbb{P}_{0,M}$-generic H, $\mathbb{P}_{\omega,M[G]}$-generic G' and $\mathbb{P}_{\omega,M[H]}$-generic H' such that $S_{M[G][G']} \cap S_{M[H][H']} = S_M$.*

Proof. Fix any $\mathbb{P}_{0,M}$-generic G over M and any $\mathbb{P}_{\omega,M[G]}$-generic G'. Let $C = S_{M[G][G']} \setminus S$. Let H be $\mathbb{P}_{0,M}$-C-generic. Let H' be $\mathbb{P}_{\omega,M[H]}$-C-generic. Then, by Lemma 5.12 and Lemma 4.15, $S_{M[G][G']} \cap S_{M[H][H']} = S_M$. □

Corollary 5.16 *Let M be a countable model of RCA_0. Then there exist two countable models M_1 and M_2 of WKL_0 such that:*

(1) *M_1 and M_2 have the same first order part as M,*

(2) $S_{M_1} \cap S_{M_2} = S_M$,

(3) *M_1 and M_2 satisfy the same* $\mathrm{L}_2(|M| \cup S_M)$*-sentences.*

Proof. Take G, H, G' and H' as in Lemma 5.15. Let $M_1 = M[G][G']$ and $M_2 = M[H][H']$. By Lemma 4.10, M_1 and M_2 are models of WKL_0. Moreover, according to Theorem 5.14, they satisfy the same sentences with parameters from $|M| \cup S_M$. □

Theorem 5.17 *Let $\varphi(X, Y)$ be a Π^1_1 formula with exactly the free variables shown. If WKL_0 proves $\forall X \exists ! Y \varphi(X, Y)$, then RCA_0 proves $\forall X \exists Y \varphi(X, Y)$.*

Proof. Let $\varphi(X, Y)$ be a Π^1_1 formula with exactly the free variables shown. Suppose that WKL_0 proves $\forall X \exists ! Y \varphi(X, Y)$. By way of contradiction, we assume RCA_0 does not prove $\forall X \exists Y \varphi(X, Y)$. Then by Gödel's completeness theorem, there exists a countable model M of RCA_0 in which $\neg \exists Y \varphi(A, Y)$ holds for some $A \in S_M$. By Corollary 5.16, there exist two countable models M_1 and M_2 of WKL_0 such that (1) they have the same first order part as M, (2) $S_{M_1} \cap S_{M_2} = S_M$ and (3) they satisfy the same $\mathrm{L}_2(|M| \cup S_M)$-sentences. Let $Y_i \in S_{M_i}$ be such that M_i satisfies $\varphi(A, Y_i)$ $(i = 1, 2)$. Then, for each $n \in |M|$ and each $i = 1, 2$,

$$n \in Y_i \Leftrightarrow M_i \models \exists Y (\varphi(A, Y) \wedge n \in Y).$$

By (3), for each n in $|M|$,

$$M_1 \models \exists Y (\varphi(A, Y) \wedge n \in Y) \Leftrightarrow M_2 \models \exists Y (\varphi(A, Y) \wedge n \in Y).$$

Therefore, $Y_1 {=} Y_2$. Then, by (2), $Y_1 \in S_{M_0}$. Therefore, by (1) and (2), M satisfies $\varphi(A, Y_1)$ since φ is Π^1_1 and M_1 satisfies $\varphi(A, Y_1)$. This is a contradiction. □

6 A further conservation result

The system WKL_0^+ (RCA_0^+) is obtained from WKL_0 (RCA_0) by adding the following scheme:

$$\forall n \forall \sigma \in 2^{<\mathbb{N}} \exists \tau \in 2^{<\mathbb{N}} (\sigma \subseteq \tau \wedge \varphi(n, \tau)) \to \exists X \forall n \exists k \varphi(n, X[k]),$$

where $\varphi(x, y)$ is an arithmetical formula with no occurrence of X. We recall some backgrounds from Brown/Simpson [2]. There are two versions of the

Baire category theorem, BCT-I and BCT-II. A version of Urysohn's lemma for complete separable metric spaces follows from BCT-I, which is provable in RCA_0 (cf. [12, Lemma II.7.3]). By contrast, the Bounded Inverse Mapping Theorem for separable Banach spaces is usually deduced from BCT-II, which is not provable in RCA_0, but in RCA_0^+. It is unknown whether or not the Bounded Inverse Mapping Theorem is provable in RCA_0. Brown/Simpson [2] proved also that WKL_0^+ is conservative over RCA_0 with respect to Π^1_1 sentences.

In this section, we generalize our main theorem to show that if $\mathsf{WKL}_0^+ \vdash \forall X \exists! Y \varphi(X, Y)$, then so does RCA_0, where $\varphi(X, Y)$ is arithmetical. To prove it, for any principal model M of RCA_0, we will construct two countable models M_1, M_2 of WKL_0^+ with $S_{M_1} \cap S_{M_2} = S_M$ which have the same first order part, and satisfy the same sentences with parameters from $|M| \cup S_M$.

Let M be a countable model of RCA_0. For each $\sigma, \tau \in (2^{<\mathbb{N}})_M$, we write $\tau \leq_1^+ \sigma$ if τ extends σ. We say that $D \subseteq (2^{<\mathbb{N}})_M$ is *dense* if for each $\sigma \in (2^{<\mathbb{N}})_M$, there exists $\tau \in D$ such that $\tau \leq_1^+ \sigma$. A path G is said to be $(2^{<\mathbb{N}})_M$*-generic* if, for every M-definable dense set $D \subseteq (2^{<\mathbb{N}})_M$, there exists $\sigma \in D$ such that $G \in [\sigma]$.

Lemma 6.1 *Let M be a countable model of RCA_0 and suppose that $G \in \mathcal{P}_M$ is $(2^{<\mathbb{N}})_M$-generic. Then $M[G] \models \mathsf{RCA}_0$.*

Proof. See [2, Lemma 6.1]. □

Definition 6.2 *Let M be a countable model of RCA_0. Let φ be a sentence in $\mathrm{L}_2(|M| \cup S_M \cup \{G\})$. For any $\sigma \in (2^{<\mathbb{N}})_M$, φ is said to be weakly forced by σ (denoted $T \Vdash_1^+ \varphi$) if $M[G] \models \varphi$ for all $(2^{<\mathbb{N}})_M$-generic $G \in [\sigma]$.*

Lemma 6.3 *Let M be a countable model of RCA_0. Let φ be a sentence of $\mathrm{L}_2(|M| \cup S_M \cup \{G\})$. Then we have*

(1) *$\sigma \Vdash_1 \varphi$ is definable over M.*

(2) *For any $(2^{<\mathbb{N}})_M$-generic $G \in [\sigma]$, if $M[G] \models \varphi$ then there exists $\sigma \subseteq \tau$ such that $G \in [\tau]$ and $\tau \Vdash_1^+ \varphi$.*

Proof. Similar to Lemma 3.2. □

Lemma 6.4 *Let M be a countable model of RCA_0. If σ_1 and σ_2 are in $(2^{<\mathbb{N}})_M$,* *then* $\sigma_1 \Vdash_1^+ \varphi$ *if and only if* $\sigma_2 \Vdash_1^+ \varphi$ *for any sentence* φ *of* $\mathrm{L}_2(|M| \cup S_M)$.

Proof. For any $\sigma, \tau \in (2^{<\mathbb{N}})_M$, let F be a function from $[\sigma]$ to $[\tau]$ such that for each $X \in [\sigma]$, $F(X) = \{n : \tau(n) = 1\} \cup \{lh(\tau) + n : n \in X'\}$ where $X = \{n : \sigma(n) = 1\} \cup \{lh(\sigma) + n : n \in X'\}$. Obviously, F is an M-homeomorphism from $[\sigma]$ to $[\tau]$. Therefore, Lemma 6.4 can be proved in the same way as Lemma 3.5. □

Let C be a countable subset of $P(|M|)$. G is said to be $(2^{<\mathbb{N}})_M$-*C-generic* if for every $M \cup C$-definable dense D, there exists $\sigma \in D$ such that $G \in [\sigma]$.

Lemma 6.5 *Let* M *be a countable model of* RCA_0, *and* C *a countable subset of* $P(|M|)$ *such that* $C \cap S_M = \emptyset$. *If* G *is* $(2^{<\mathbb{N}})_M$-*C-generic, then* $M[G]$ *is a countable model of* RCA_0 *with* $S_{M[G]} \cap C = \emptyset$.

Proof. It suffices to show that for any $A \in C$ and any Σ_1^0 formulas $\varphi_1(x)$ and $\varphi_2(x)$ with parameters from $|M| \cup S \cup \{G\}$, either $A \neq \{n \in |M| : M[G] \models \varphi_1(n)\}$ or $A \neq \{n \in |M| : M[G] \models \neg\varphi_2(n)\}$. By way of contradiction, we suppose that $A = \{n \in |M| : M[G] \models \varphi_1(n)\} = \{n \in |M| : M[G] \models \neg\varphi_2(n)\}$.

We may assume that $\varphi_i(x)$ is of the form $\exists y \theta_i(x, G[y])$, where $\theta_i(x, \tau)$ is Σ_0^0 with parameter from $|M| \cup S_M$ $(i = 1, 2)$. Let E^A be the set of all $\sigma \in (2^{<\mathbb{N}})_M$ such that there exists $m \in |M|$ such that for any extension τ of σ, one of the following holds:

A1. $m \in A \wedge M \models \forall \tau' \subseteq \tau \neg\theta_1(m, \tau')$,

A2. $m \notin A \wedge M \models \exists \tau' \subseteq \tau \theta_1(m, \tau')$,

A3. $m \in A \wedge M \models \exists \tau' \subseteq \tau \theta_2(m, \tau')$,

A4. $m \notin A \wedge M \models \forall \tau' \subseteq \tau \neg\theta_2(m, \tau')$.

We define D_A by

$$\sigma \in D_A \text{ if and only if } \sigma \in E_A \vee \neg\exists \tau \in E_A(\sigma \subseteq \tau).$$

Then D_A is $(2^{<\mathbb{N}})_M$-dense, and $M \cup C$-definable. Take $\sigma_0 \in D_A$ with $G \in [\sigma_0]$. We first claim that $\sigma_0 \in E_A$. By way of contradiction, suppose that $\sigma_0 \notin E_A$.

Then for all $\tau \supseteq \sigma_0$,

$$\forall x \exists \tau' \supseteq \tau((x \in A \leftrightarrow \exists \tau'' \subseteq \tau' \theta_1(x, \tau'')) \wedge (x \in A \leftrightarrow \forall \tau'' \subseteq \tau' \neg\theta_2(x, \tau''))).$$

Therefore, for any $n \in |M|$,

$$n \in A \Leftrightarrow M \models \exists \tau \supseteq \sigma_0 \exists \tau' \subseteq \tau \theta_1(n, \tau') \Leftrightarrow M \models \forall \tau \supset \sigma_0 \forall \tau' \subseteq \tau \neg\theta_2(n, \tau').$$

Then $A \in S_M$. This is a contradiction with $C \cap S_M = \emptyset$. Thus, since σ_0 is in E_A, there exists $m_0 \in |M|$ such that for all $\tau \supset \sigma_0$, either

B1. $m_0 \in A \wedge M \models (\forall \tau' \subseteq \tau \neg\theta_1(m_0, \tau') \vee \exists \tau' \leq \tau \theta_2(m, \tau'))$, or

B2. $m_0 \notin A \wedge M \models (\forall \tau' \subseteq \tau \neg\theta_2(m_0, \tau') \vee \exists \tau' \leq \tau \theta_1(m, \tau'))$.

Assume that $m_0 \in A$. Fix an initial segment τ of G such that τ is an end-extension of σ_0 and $G[l]$ with $\theta_1(m_0, G[l])$. By B1, $\exists \tau' \leq \tau \theta_2(m, \tau'))$. Then, $\exists y \theta_2(m_0, G[y])$, that is, $m_0 \notin A$. This is a contradiction. The case of $x_0 \notin A$ can be treated similarly. This completes the proof. □

As in Section 4, by iterating the two forcing notions $\Vdash_1$ and $\Vdash_1^+$ alternatively, we can define the notion of $+$-ω-forcing $\Vdash_\omega^+$, which satisfies the following properties.

Lemma 6.6 *Let M be a principal model of RCA_0. Then the following hold.*

(1) *If G is generic for $\Vdash_\omega^+$, then $M[G] \models \mathsf{WKL}_0^+$.*

(2) *Any two $+$-ω-conditions weakly force the same $\mathrm{L}_2(|M| \cup S_M)$-sentences.*

(3) *Let C be a countable subset of $P(|M|)$ such that $S_M \cap C = \emptyset$. Then there exists a generic G for $\Vdash_\omega^+$ such that $S_{M[G]} \cap C = \emptyset$.*

Lemma 6.7 *Let M be a principal model of RCA_0. Then there exist two countable models M_1, M_2 of WKL_0^+ such that:*

(1) *M_1 and M_2 have the same first order part as M,*

(2) *$S_{M_1} \cap S_{M_2} = S_M$,*

(3) *M_1 and M_2 satisfy the same $\mathrm{L}_2(|M| \cup S_M)$-sentences.*

Proof. It is straightforward from Lemma 6.6 (3). □

Theorem 6.8 *Let $\varphi(X, Y)$ be an arithmetical formula with only the free variables shown. If WKL_0^+ proves $\forall X \exists ! Y \varphi(X, Y)$, then so does RCA_0. (Then, RCA_0 also proves $\forall X \exists Y(Y$ is recursive in $X \wedge \varphi(X, Y)))$.*

Proof. The proof is an obvious modification of the proof of Theorem 4.18 with the help of Lemma 6.7. □

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