SEPARATION AND WEAK KÖNIG'S LEMMA

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Abstract. We continue the work of [14, 3, 1, 19, 16, 4, 12, 11, 20] investigating the strength of set existence axioms needed for separable Banach space theory. We show that the separation theorem for open convex sets is equivalent to WKL_0 over RCA_0. We show that the separation theorem for separably closed convex sets is equivalent to ACA_0 over RCA_0. Our strategy for proving these geometrical Hahn–Banach theorems is to reduce to the finite-dimensional case by means of a compactness argument.

1. Introduction

Let A and B be convex sets in a Banach space X. We say that A and B are *separated* if there is a bounded linear functional $F : X \to \mathbb{R}$ and a real number α such that $F(x) < \alpha$ for all $x \in A$, and $F(x) \geq \alpha$ for all $x \in B$. We say that A and B are *strictly separated* if in addition $F(x) > \alpha$ for all $x \in B$.

There are several well-known theorems of Banach space theory to the effect that any two disjoint convex sets satisfying certain conditions can be separated or strictly separated. A good reference for such theorems is Conway [6]. The purpose of this paper is to consider the question of which set existence axioms are needed to prove such theorems. We study this question in the context of subsystems of second order arithmetic.

The subsystems of second order arithmetic that are relevant here are ACA_0, RCA_0, and above all WKL_0. ACA_0 is the system with arithmetical comprehension and arithmetical induction; it is conservative over first-order Peano arithmetic. RCA_0 is the much weaker system with only Δ^0_1 comprehension and Σ^0_1 induction; it may be viewed as a formalized version of recursive mathematics. WKL_0 consists of RCA_0 plus an additional set existence axiom known as Weak König's Lemma. WKL_0 and

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RCA_0 are conservative over arithmetic with Σ^0_1 induction [9, 18, 20], hence much weaker than ACA_0 in terms of proof-theoretic strength. Moreover, WKL_0 and RCA_0 are conservative over primitive recursive arithmetic for Π^0_2 sentences [7, 18, 20]. The foundational significance of this result is that any mathematical theorem provable in WKL_0 is finitistically reducible [19].

The main new result of this paper is that the basic separation theorem for convex sets in separable Banach spaces is provable in WKL_0; see Theorem 3.1 below. It follows that the basic separation theorem is finitistically reducible. This provides further confirmation of the well-known significance of WKL_0 with respect to Hilbert's program of finitistic reductionism [19].

As a byproduct of our work on separation theorems in WKL_0, we present new proofs of the closely related Hahn–Banach and extended Hahn–Banach theorems in WKL_0; see Section 4 below. These new proofs are more transparent than the ones that have appeared previously [3, 16, 20, 12].

We also obtain reversals in the sense of Reverse Mathematics. We show that the basic separation theorem is logically equivalent to WKL_0 over RCA_0; see Theorem 4.4 below. Thus Weak König's Lemma is seen to be logically indispensable for the development of this portion of functional analysis. In addition, we show that another separation theorem requires stronger set existence axioms, in that it is equivalent to ACA_0 over RCA_0; see Theorem 5.1 below.

One aspect of this paper may be of interest to readers who are familiar with Banach spaces but do not share our concern with foundational issues. Namely, we present a novel and elegant proof of the various separation and Hahn–Banach theorems. Our approach is to reduce to the finite-dimensional Euclidean case by means of a straightforward compactness argument. A similar proof strategy has been used previously (see Łoś/Ryll-Nardzewski [13]) but is apparently not widely known. We thank Ward Henson for bringing [13] to our attention.

2. Preliminaries

The prerequisite for a thorough understanding of this paper is familiarity with the basics of separable Banach space theory as developed in RCA_0. This material has been presented in several places: [3, §§2-5], [4, §1], [12, §4], [20, §II.10]. We briefly review some of the concepts that we shall need.

Within RCA_0, a (code for a) *complete separable metric space* $X = \widehat{A}$ is defined to be a countable set $A \subseteq \mathbb{N}$ together with a function $d : A \times A \to \mathbb{R}$ satisfying $d(a,a) = 0$, $d(a,b) = d(b,a)$, and $d(a,b)+d(b,c) \geq$

$d(a, c)$. A (code for a) *point* of X is defined to be a sequence $x = \langle a_n \rangle_{n \in \mathbb{N}}$ of elements of A such that $\forall m \forall n (m < n \to d(a_m, a_n) \leq 1/2^m)$. We extend d from A to X in the obvious way. For $x, y \in X$ we define $x = y$ to mean that $d(x, y) = 0$.

Within RCA_0, (a code for) an *open set* in X is defined to be a sequence of ordered pairs $U = \langle (a_m, r_m) \rangle_{m \in \mathbb{N}}$ where $a_m \in A$ and $r_m \in \mathbb{Q}$, the rational numbers. We write $x \in U$ to mean that $d(a_m, x) < r_m$ for some $m \in \mathbb{N}$. A *closed set* $C \subseteq X$ is defined to be the complement of an open set U, *i.e.*, $\forall x \in X (x \in C \leftrightarrow x \notin U)$.

It will sometimes be necessary to consider a slightly different notion. A (code for a) *separably closed set* $K = \overline{S} \subseteq X$ is defined to be a countable sequence of points $S \subseteq X$. We write $x \in K$ to mean that for all $\varepsilon > 0$ there exists $y \in S$ such that $d(x, y) < \varepsilon$. It is provable in ACA_0 (but not in weaker systems) that for every separably closed set K there exists an equivalent closed set C, *i.e.*, $\forall x \in X (x \in C \leftrightarrow x \in K)$. For further details on separably closed sets, see [2, 3, 4].

Within RCA_0, a *compact set* $K \subseteq X$ is defined to be a separably closed set such that there exists a sequence of finite sequences of points $x_{ni} \in K$, $i \leq k_n$, $n \in \mathbb{N}$, such that for all $n \in \mathbb{N}$ and all $x \in K$ there exists $i \leq k_n$ with $d(x, x_{ni}) < 1/2^n$. The sequence of positive integers k_n, $n \in \mathbb{N}$, is also required to exist. It is provable in RCA_0 that compact sets are closed and located [8]. It is provable in WKL_0 that compact sets have the Heine–Borel covering property, *i.e.*, any covering of K by a sequence of open sets has a finite subcovering.

Within RCA_0, a (code for a) *separable Banach space* $X = \widehat{A}$ is defined to be a countable pseudonormed vector space A over $\mathbb{Q}$. With $d(a, b) = \|a - b\|$, X is a complete separable metric space and has the usual structure of a Banach space over $\mathbb{R}$. A *bounded linear functional* $F : X \to \mathbb{R}$ may be defined as a continuous function which is linear. The equivalence of continuity and boundedness is provable in RCA_0. We write $\|F\| \leq \alpha$ to mean that $|F(x)| \leq \alpha \|x\|$ for all $x \in X$.

3. Separation in WKL_0

The purpose of this section is to prove the following theorem.

Theorem 3.1. *The following is provable in* WKL_0. *Let X be a separable Banach space. Let A be an open convex set in X, and let B be a separably closed convex set in X. If A and B are disjoint, then A and B can be separated.*

Remark 3.2. Theorem 3.1 verifies a conjecture that appeared in [11], page 61. A special case of this result had been conjectured earlier in

[12], page 4246. Corollary 5.1.2 of [11] (see also Lemma 4.10 of [12]) is essentially our present Theorem 3.1 with WKL_0 replaced by ACA_0.

Toward the proof of Theorem 3.1, we first prove a separation result for finite-dimensional Euclidean spaces.

Lemma 3.3. *The following is provable in* WKL_0. *Let A and B be compact convex sets in $\mathbb{R}^n$. If A and B are disjoint, then A and B can be strictly separated.*

Proof. For $x, y \in \mathbb{R}^n$ we denote by $x \cdot y$ the dot product of x and y. The norm on $\mathbb{R}^n$ is given by $\|x\|^2 = x \cdot x$. We imitate the argument of Lemma 3.1 of [5].

Put $C = B - A = \{y - x \mid x \in A, y \in B\}$. Then C is a compact convex set in $\mathbb{R}^n$. Since $A \cap B = \emptyset$, we have $0 \notin C$. The function $z \mapsto \|z\|$ is continuous on C, so it follows in WKL_0 that there exists $c \in C$ of minimum norm, *i.e.*, $0 < \|c\| \leq \|z\|$ for all $z \in C$.

We claim that $\|c\|^2 \leq c \cdot z$ for all $z \in C$. Suppose not. Let $z \in C$ be such that $\|c\|^2 - c \cdot z = \varepsilon > 0$. Consider $w = tz + (1-t)c$ where $0 < t \leq 1$. Since C is convex, we have $w \in C$, hence $0 < \|c\| \leq \|w\|$. Expansion of $\|w\|^2 = w \cdot w$ gives

$$\|w\|^2 \ = \ \|c\|^2 + t^2(\|z\|^2 - \|c\|^2) - 2t(1-t)\varepsilon$$

and from this it follows that $t(\|z\|^2 - \|c\|^2) \geq 2(1-t)\varepsilon$. Now set

$$t \ = \ \frac{\varepsilon}{\|z\|^2 - \|c\|^2 + 2\varepsilon}$$

and note that $0 < t \leq 1/2$. With this t we have

$$t(\|z\|^2 - \|c\|^2) \ = \ \varepsilon - 2\varepsilon t \ < \ 2\varepsilon - 2\varepsilon t \ = \ 2(1-t)\varepsilon\,,$$

a contradiction. This proves our claim.

Define $F : \mathbb{R}^n \to \mathbb{R}$ by $F(z) = c \cdot z$. Since $c \in C = B - A$, we may fix $a \in A$ and $b \in B$ such that $c = b - a$. Using our claim, it is easy to show that $F(x) \leq F(a) < F(b) \leq F(y)$ for all $x \in A$ and $y \in B$. Thus A and B are strictly separated. $\square$

In order to reduce Theorem 3.1 to the finite-dimensional Euclidean case, we need some technical lemmas.

Lemma 3.4. *The following is provable in* WKL_0. *Let X and K be complete separable metric spaces. Assume that K is compact. If $C \subseteq X \times K$ is closed, then*

$$\{x \in X \mid (x, y) \in C \text{ for some } y \in K\}$$

is closed.

Proof. Reasoning in WKL_0, put $V = (X \times K) \setminus C$ and

$$U \;=\; \{x \in X \mid (x, y) \in V \text{ for all } y \in K\}\,.$$

We shall prove that U is open.

Since V is open, there is a sequence of open balls $B((a_m, b_m), r_m)$, $(a_m, b_m) \in X \times K$, $r_m \in \mathbb{Q}$, $m \in \mathbb{N}$, such that

$$V \;=\; \bigcup_{m=0}^{\infty} B((a_m, b_m), r_m)\,.$$

Since K is compact, there is a sequence of points $y_{ni} \in K$, $i \le k_n$, $n \in \mathbb{N}$, such that $K = \bigcup_{i \le k_n} B(y_{i,n}, 1/2^n)$ for each n.

We claim that

$$\exists n \;\; \forall i \le k_n \;\; \exists m \;\; d((a_m, b_m), (x, y_{ni})) + 1/2^n \;<\; r_m \tag{1}$$

is a necessary and sufficient condition for $x \in U$. Obviously (1) is sufficient since it implies $\{x\} \times K \subseteq \bigcup_{i \le k_n} B((x, y_{ni}), 1/2^n) \subseteq V$ whence $x \in U$. For the necessity, let $x \in U$ be given. Then $\{x\} \times K \subseteq \bigcup_{m=0}^{\infty} B((a_m, b_m), r_m)$. By Heine–Borel compactness of K in WKL_0 it follows that $\{x\} \times K \subseteq \bigcup_{m=0}^{k} B((a_m, b_m), q_m)$ for some $k \in \mathbb{N}$ and finite sequence $q_m \in \mathbb{Q}$, $m \le k$, $q_m < r_m$. Let n be such that $1/2^n < \min_{m \le k}(r_m - q_m)$. Then for each $i \le k_n$ there exists $m \le k$ such that $d((a_m, b_m), (x, y_{ni})) < q_m$, hence $d((a_m, b_m), (x, y_{ni})) + 1/2^n < r_m$. This gives condition (1) and our claim is proved.

Since the condition (1) is Σ^0_1, it follows by Lemma II.5.7 of [20] that $U \subseteq X$ is open. Therefore, the complementary set is closed. This proves our lemma. □

Lemma 3.5. *The following is provable in* WKL_0. *Let X be a separable Banach space. Fix $n \ge 1$ and let $Y = \bigcup_{m=1}^{n} X^m$ be the space of all finite sequences of elements of X of length $\le n$. Then*

$$\{s \in Y \mid s \text{ is linearly independent}\}$$

is an open set in Y.

Proof. Consider the compact space $K = \bigcup_{m=1}^{n} K_m$ where

$$K_m = \{\langle \alpha_1, \ldots, \alpha_m \rangle \mid |\alpha_1| + \cdots + |\alpha_m| = 1\}\,.$$

Here the α_i's are real numbers. Note that $s = \langle x_1, \ldots, x_m \rangle \in Y$ is linearly dependent if and only if $\alpha_1 x_1 + \cdots + \alpha_m x_m = 0$ for some $\langle \alpha_1, \ldots, \alpha_m \rangle \in K$. Hence by Lemma 3.4 the set of all such s is closed. It follows that the complementary set is open. □

Lemma 3.6. *The following is provable in* WKL_0. *Let* K *be a compact metric space, and let* $\langle C_j \rangle_{j\in\mathbb{N}}$ *be a sequence of nonempty closed sets in* K. *Then there exists a sequence of points* $\langle x_j \rangle_{j\in\mathbb{N}}$ *such that* $x_j \in C_j$ *for all* j.

Proof. Since K is compact, there is a sequence of points $x_{ni} \in K$, $i \le k_n$, $n \in \mathbb{N}$, such that $K = \bigcup_{i\le k_n} B(x_{in}, 1/2^n)$ for each $n \in \mathbb{N}$. Let S be the bounded tree consisting of all finite sequences $\sigma \in \mathbb{N}^{<\mathbb{N}}$ such that $\sigma(n) \le k_n$ for all $n <$ the length of σ. Construct a sequence of trees $T_j \subseteq S$, $j \in \mathbb{N}$, such that for each j there is a one-to-one correspondence between infinite paths g in T_j and points $x \in C_j$, the correspondence being given by $x = \lim_n x_{ng(n)}$. For details of the construction of the T_j's, see Section IV.1 of [20].

Let $(-,-) : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a primitive recursive pairing function which is onto and monotone in both arguments. Let $T = \bigoplus_{j\in\mathbb{N}} T_j$ be the interleaved tree, defined by putting $\tau \in T$ if and only if $\tau_j \in T_j$ for all j, where $\tau_j(n) = \tau((j,n))$. Note that T is a bounded tree, the bounding function $h : \mathbb{N} \to \mathbb{N}$ being given by $h((j,n)) = k_n + 1$. In order to show that T is infinite, we prove that for all m there exists $\tau \in T$ of length m such that for all j and all $n \ge$ length of τ_j, τ_j has at least one extension of length n in T_j. This Π^0_1 statement is easily proved by Π^0_1 induction on m, using the fact that each of the T_j's is infinite.

Since T is an infinite bounded tree, it follows by Bounded König's Lemma in WKL_0 (see Section IV.1 of [20]) that T has an infinite path, f. Then for each j we have an infinite path f_j in T_j given by $f_j(n) = f((j,n))$. Thus we obtain a sequence of points $\langle x_j \rangle_{j\in\mathbb{N}}$ where $x_j = \lim_n x_{nf_j(n)}$ is a point of C_j. □

Lemma 3.7. *The following is provable in* WKL_0. *Let* X *be a separable Banach space, and let* $x_1, \ldots, x_n$ *be a finite set of elements of* X. *Then there is a closed subspace* $X' = \mathrm{span}(x_1, \ldots, x_n) \subseteq X$ *consisting of all linear combinations of* $x_1, \ldots, x_n$. *Moreover, there exists a finite set*

$$x_{i_1}, \ldots, x_{i_m}, \quad 1 \le i_1 < \cdots < i_m \le n,$$

which is a basis *of* X', i.e., *each element of* X' *is uniquely a linear combination of* $x_{i_1}, \ldots, x_{i_m}$.

Proof. We first prove that X' has a basis. By lemma 3.5, the set of all linearly independent $s \in \bigcup_{m=0}^n X^m$ is open. By bounded Σ^0_1 comprehension in WKL_0, it follows that

$$\mathcal{I} = \{I \subseteq \{1, \ldots, n\} \mid \{x_i \mid i \in I\} \text{ is linearly independent}\}$$

is a finite set of subsets of $\{1, \ldots, n\}$. Let $M = \{i_1, \ldots, i_m\}$ be a maximal element of $\mathcal{I}$. Then clearly each of $x_1, \ldots, x_n$ is a linear combination of $x_{i_1}, \ldots, x_{i_m}$. Moreover, we can apply Lemma 3.6 to obtain a double sequence of coefficients α_{ij}, $i = 1, \ldots, n$, $j = 0, 1, \ldots, m$, such that

$$|\alpha_{i0}| + |\alpha_{i1}| + \cdots + |\alpha_{im}| \;=\; 1$$

and

$$\alpha_{i0} x_i + \alpha_{i1} x_{i_1} + \cdots + \alpha_{im} x_{i_m} \;=\; 0$$

for each $i = 1, \ldots, n$. Obviously $\alpha_{i0} \neq 0$ so we may put $\beta_{ij} = -\alpha_{ij}/\alpha_{i0}$ to obtain

$$x_i \;=\; \beta_{i1} x_{i_1} + \cdots + \beta_{im} x_{i_m}$$

for each $i = 1, \ldots, n$. With this it is clear that every linear combination of $x_1, \ldots, x_n$ is uniquely a linear combination of $x_{i_1}, \ldots, x_{i_m}$.

It remains to prove that X' is a closed subspace of X. As a code for X' we may use $\mathbb{Q}^n$ identifying $\langle q_1, \ldots, q_n \rangle \in \mathbb{Q}^n$ with $q_1 x_1 + \cdots + q_n x_n \in X$. Thus X' is a subspace of X. The fact that X' is closed follows easily from Lemma 3.5. □

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. Reasoning within WKL_0, let X, A, B be as in the hypotheses of Theorem 3.1. We need to prove that A and B can be separated. Since A is open, we may safely assume that

$$\{x \in X \mid \|x\| \leq 1\} \subseteq A\,.$$

With this assumption, reasoning in WKL_0, our goal will be to prove the existence of a bounded linear functional $F : X \to \mathbb{R}$ such that $F(x) \leq 1$ for all $x \in A$, and $F(x) \geq 1$ for all $x \in B$; these properties easily imply that $F(x) < 1$ for all $x \in A$. Observe also that any such F will necessarily have $\|F\| \leq 1$.

Since X is a separable Banach space, there exists a countable vector space D over the rational field $\mathbb{Q}$ such that $D \subseteq X$ and D is dense in X. Since B is separably closed, there exists a countable sequence $S \subseteq B$ such that S is dense in B. We may safely assume that $S \subseteq D$. With this assumption, consider the compact product space

$$K \;=\; \prod_{d \in D} \left[-\|d\|, \|d\|\,\right].$$

Note that any bounded linear functional $F : X \to \mathbb{R}$ with $\|F\| \leq 1$ may be identified with a point of K in an obvious way, namely $F =$

$\langle F(d)\rangle_{d\in D}$. Thus our goal may be expressed as follows: to prove that there exists a point $\langle \alpha_d\rangle_{d\in D} \in K$ satisfying the conditions

1. $\alpha_d \leq 1$ for all $d \in D \cap A$;
2. $\alpha_d \geq 1$ for all $d \in S$;
3. $\alpha_d = q_1\alpha_{d_1} + q_2\alpha_{d_2}$ for all $d, d_1, d_2 \in D$ and $q_1, q_2 \in \mathbb{Q}$ such that $d = q_1d_1 + q_2d_2$.

Let Φ be this countable set of Π^0_1 conditions. By Heine–Borel compactness of K in WKL_0, it suffices to show that each finite subset of Φ is satisfied by some point of K.

Suppose we are given a finite set of conditions $\Phi' \subseteq \Phi$. Let $a_1, \ldots, a_m$ be the elements of $D \cap A$ that are mentioned in Φ'. Let $b_1, \ldots, b_n$ be the elements of S that are mentioned in Φ'. Let $d_1, \ldots, d_k$ be the nonzero elements of D that are mentioned in Φ'. By Lemma 3.7, let X' be the finite-dimensional subspace of X spanned by $d_1, \ldots, d_k$. Let A' be the convex hull of $a_1, \ldots, a_m, \pm d_1/\|d_1\|, \ldots, \pm d_k/\|d_k\|$. Let B' be the convex hull of $b_1, \ldots, b_n$. Note that $A' \subseteq A \cap X'$ and $B' \subseteq B \cap X'$; hence $A' \cap B' = \emptyset$. Moreover A' and B' are compact. By Lemmas 3.7 and 3.3, there exists a bounded linear functional $F' : X' \to \mathbb{R}$ such that $F'(x) \leq 1$ for all $x \in A'$, and $F'(x) \geq 1$ for all $x \in B'$. In particular $F'(\pm d_i/\|d_i\|) \leq 1$ for all $i = 1, \ldots, k$; hence $|F'(d_i)| \leq \|d_i\|$. Put $\alpha'_d = F'(d)$ for $d = d_1, \ldots, d_k$, and $\alpha'_d = 0$ for $d \in D \setminus \{d_1, \ldots, d_k\}$. Then $\langle \alpha'_d\rangle_{d\in D}$ is a point of K which satisfies Φ'. This completes the proof. □

Remark 3.8. Our proof of a separation theorem in WKL_0 (Theorem 3.1) was accomplished by means of a reduction to the finite-dimensional Euclidean case using a compactness argument. This proof technique is not entirely new (see [13]) but does not seem to be widely known.

4. Reversal via Hahn–Banach

Let X be a separable Banach space. Consider the following statements:

- SEP1**:** (First Separation) Let A be an open convex set in X, let B be a separably closed convex set in X, and assume $A \cap B = \emptyset$. Then A and B can be separated.
- SEP2**:** (Second Separation) Let A and B be open convex sets in X such that $A \cap B = \emptyset$. Then A and B can be strictly separated.
- SEP3**:** (Third Separation) Let A and B be separably closed, convex sets in X such that $A \cap B = \emptyset$. Assume also that A is compact. Then A and B can be strictly separated.
- HB**:** (Hahn–Banach) Let S be a subspace of X, and let $f : S \to \mathbb{R}$ be a bounded linear functional with $\|f\| \leq \alpha$ on S. Then there

exists a bounded linear functional $F : X \to \mathbb{R}$ such that F extends f and $\|F\| \leq \alpha$ on X.

EHB**:** (Extended Hahn–Banach) Let $p : X \to \mathbb{R}$ be a continuous sublinear functional. Let S be a subspace of X, and let $f : S \to \mathbb{R}$ be a bounded linear functional such that $f(x) \leq p(x)$ for all $x \in S$. Then there exists a bounded linear functional $F : X \to \mathbb{R}$ such that F extends f and $F(x) \leq p(x)$ for all $x \in X$.

It is known [3, 12] that EHB and HB are equivalent to WKL_0 over RCA_0. We are now going to prove that SEP1 and SEP2 are also equivalent to WKL_0 over RCA_0; see Theorem 4.4 below. In the next section we shall prove that SEP3 is equivalent to ACA_0, hence properly stronger than WKL_0, over RCA_0; see Theorem 5.1 below.

Lemma 4.1. *It is provable in* RCA_0 *that* SEP1 *implies* SEP2*.*

Proof. Reasoning in RCA_0, assume SEP1 and let A and B be disjoint, open, convex sets. Let B' be the separable closure of B. Clearly B' is convex and $A \cap B' = \emptyset$. By SEP1, let F and α be such that $F < \alpha$ on A and $F \geq \alpha$ on B'. It follows that $F > \alpha$ on B. Thus we have SEP2. This completes the proof. □

Lemma 4.2. *It is provable in* RCA_0 *that* SEP2 *implies* EHB*.*

Proof. Reasoning in RCA_0, assume SEP2 and let p, S, and f be as in the hypotheses of EHB. Let A be the convex hull of

$$\{x \in S \mid f(x) < 1\} \cup \{y \in X \mid p(y) < 1\} ,$$

and let B be the convex hull of

$$\{x \in S \mid f(x) > 1\} \cup \{y \in X \mid -p(-y) > 1\} .$$

Clearly A and B are open.

We claim that A and B are disjoint. If not, then for some $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$, $x_1 \in S$, $y_1 \in X$, $x_2 \in S$, $y_2 \in X$ we have $f(x_1) < 1$, $p(y_1) < 1$, $f(x_2) > 1$, $-p(-y_2) > 1$, and

$$(1-\alpha)x_1 + \alpha y_1 \quad = \quad (1-\beta)x_2 + \beta y_2 .$$

Note that $\alpha y_1 - \beta y_2 \in S$. Hence

$$\begin{aligned} f(\alpha y_1 - \beta y_2) &\leq p(\alpha y_1 - \beta y_2) \\ &\leq \alpha p(y_1) + \beta p(-y_2) \\ &\leq \alpha - \beta , \end{aligned}$$

yet on the other hand we have

$$\begin{aligned} f(\alpha y_1 - \beta y_2) &= f((1-\beta)x_2 - (1-\alpha)x_1) \\ &= (1-\beta)f(x_2) - (1-\alpha)f(x_1) \\ &\geq (1-\beta) - (1-\alpha) \\ &= \alpha - \beta\,, \end{aligned}$$

hence $f(\alpha y_1 - \beta y_2) = \alpha - \beta$. Since at least one of the above inequalities must be strict, we obtain a contradiction. This proves our claim.

By SEP2, there exists a bounded linear functional $F : X \to \mathbb{R}$ such that $F(x) < 1$ for all $x \in A$, and $F(x) > 1$ for all $x \in B$. Clearly F extends f. It remains to show that $F(y) \leq p(y)$ for all $y \in X$. Suppose not, say $p(y) < F(y)$. If $F(y) > 0$, then for a suitably chosen $r > 0$ we have $p(ry) < 1 < F(ry)$, a contradiction. If $F(y) \leq 0$, then for a suitably chosen $r > 0$ we have $p(ry) < -1 < F(ry)$. Putting $z = -ry$ we get $-p(-z) > 1 > F(z)$, again a contradiction. This completes the proof. □

Lemma 4.3. *It is provable in* RCA_0 *that* EHB *implies* HB.

Proof. HB is a special case of EHB with $p(x) = \alpha\|x\|$. □

Theorem 4.4. *The following statements are pairwise equivalent over* RCA_0.

1. SEP1*, the first separation theorem.*
2. SEP2*, the second separation theorem.*
3. EHB*, the extended Hahn–Banach theorem.*
4. HB*, the Hahn–Banach theorem.*
5. WKL_0.

Proof. Lemmas 4.1, 4.2, 4.3 give the implications SEP1 ⇒ SEP2 and SEP2 ⇒ EHB and EHB ⇒ HB. The equivalence HB ⇔ WKL_0 is the main result of [3]; see also [14] and [16] and Chapter IV of [20]. Theorem 3.1 gives the implication WKL_0 ⇒ SEP1. This completes the proof. □

Corollary 4.5. *The extended Hahn–Banach theorem,* EHB*, is provable in* WKL_0.

Remark 4.6. Corollary 4.5 has been stated in the literature; see Theorem 4.9 of [12]. However, the proof given above is new. In addition, the proof given above contains full details, while the proof in [12] was presented in a very sketchy way.

Corollary 4.7. *The Hahn–Banach theorem,* HB*, is provable in* WKL_0.

Remark 4.8. Corollary 4.7 has been proved several times in the literature; see [3] and [16] and Chapter IV of [20]. The proof given here is new and, from some points of view, more perspicuous.

Remark 4.9. Hatzikiriakou [10] has shown that that an algebraic separation theorem for countable vector spaces over $\mathbb{Q}$ is equivalent to WKL_0 over RCA_0. This result may be compared to our Theorem 4.4. We do not see any easy way of deducing our result from that of [10] or vice versa, but the comparison is interesting.

5. Separation and ACA_0

Theorem 5.1. *The following statements are pairwise equivalent over* RCA_0.

1. ACA_0.
2. SEP3*, the third separation theorem.*
3. *Let A and B be disjoint, bounded, separably closed, convex sets in $\mathbb{R}^2$. Assume also that A is compact. Then A and B can be separated.*

Proof. Reasoning in ACA_0, let A and B satisfy the hypotheses of SEP3. In ACA_0, separably closed implies closed (see [2]), so B is closed. Hence we can use Heine–Borel compactness of A to find $\delta > 0$ such that $\|x - y\| > \delta$ for all $x \in A$ and $y \in B$. Let $B(0, \delta/2)$ be the open ball of radius $\delta/2$ centered at 0. Then $A' = A + B(0, \delta/2)$ and $B' = B + B(0, \delta/2)$ are disjoint open convex sets. By SEP2 we can strictly separate A' and B'. This proves SEP3 in ACA_0.

Trivially SEP3 implies statement 5.1.3.

It remains to prove that statement 5.1.3 implies ACA_0 over RCA_0. Reasoning in RCA_0, assume that ACA_0 fails. Then there exists a bounded increasing sequence of rational numbers a_n, $n \in \mathbb{N}$, such that $\sup_n a_n$ does not exist. (See Chapter III of [20].) We may safely assume $0 < a_n < 1$. Let $A = [0, 1] \times \{0\}$, and let B be the separably closed convex hull of the points $(a_n, 1/n)$, $n \geq 1$. Note that A and B are bounded, separably closed, convex sets in $\mathbb{R}^2$. Moreover A is compact, and clearly A and B cannot be separated. Thus we have a counterexample to 5.1.3, once we show that A and B are disjoint.

To show that A and B are disjoint, let S be the countable set consisting of all rational convex combinations of points $(a_n, 1/n)$, $n \geq 1$. Thus B is the separable closure of S.

We claim: for all $n \geq 1$ there exists $\varepsilon_n > 0$ such that for all $(x, y) \in S$, if $x < a_n$ then $y > \varepsilon_n$. To see this, note that

$$(x, y) \;=\; \sum_{i=1}^{k} q_i \left(a_{n_i}, \frac{1}{n_i} \right)$$

where $\sum_{i=0}^{k} q_i = 1$, $q_i > 0$, $q_i \in \mathbb{Q}$. Thus $x = \sum_{i=0}^{k} q_i a_{n_i}$. Putting $r = \sum \{ q_i \mid n_i \leq n \}$ we have

$$a_n \;>\; x \;\geq\; r a_1 + (1 - r) a_{n+1} \;,$$

hence

$$r \;>\; \frac{a_{n+1} - a_n}{a_{n+1} - a_1} \;>\; 0 \;.$$

Furthermore

$$y \;=\; \sum_{i=0}^{k} q_i \frac{1}{n_i} \;\geq\; r \cdot \frac{1}{n} \;.$$

Therefore we put

$$\varepsilon_n \;=\; \frac{a_{n+1} - a_n}{a_{n+1} - a_1} \cdot \frac{1}{n}$$

and our claim is proved.

Now if $(x, 0) \in A \cap B$, we clearly have $x < a_n$ for some n. Since S is dense in B, let $(x', y') \in S$ be such that

$$|x - x'| \,,\, |y'| \;<\; \min(a_{n+1} - a_n, \varepsilon_{n+1}) \;.$$

Then $x' < a_{n+1}$ and $y' < \varepsilon_{n+1}$, a contradiction. Thus A and B are disjoint. This completes the proof. □

Remark 5.2. A modification of the above argument shows that ACA_0 is equivalent over RCA_0 to the following even weaker-sounding statement: if A and B are disjoint, bounded, separably closed, convex sets in $\mathbb{R}^2$, and if A is compact, then there exists an open set U such that $A \subseteq U$ and $U \cap B = \emptyset$.

Remark 5.3. In the functional analysis literature, separation results such as SEP1, SEP2, and SEP3 are sometimes referred to as "geometrical forms of the Hahn–Banach theorem." It is therefore of interest to perform a detailed comparison of these separation results with the (non-geometrical) Hahn–Banach and extended Hahn–Banach theorems. Our results in this paper shed some light on the logical or foundational aspect of such a comparison. We note that, although SEP1 and SEP2 are logically equivalent to HB and EHB over RCA_0 (Theorem 4.4), SEP3 is logically stronger (Theorem 5.1). Moreover, even though SEP2 and

EHB turn out to be equivalent in this sense, we were unable to find a direct proof of this fact; the proof that we found is highly indirect, via WKL_0. Thus we conclude that, from our foundational standpoint, it is somewhat inaccurate to view the separation theorems as trivial variants of the Hahn–Banach or extended Hahn–Banach theorems.

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