THE JOURNAL OF SYMBOLIC LOGIC
Volume 47, Number 2, June 1982

ON THE ROLE OF RAMSEY QUANTIFIERS IN FIRST ORDER ARITHMETIC[1]

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§0. Introduction. The purpose of this paper is to study a formal system $PA(Q^2)$ of *first order Peano arithmetic*, PA, augmented by a *Ramsey quantifier* Q^2 which binds two free variables. The intended meaning of $Q^2xx'\varphi(x, x')$ is that there exists an infinite set X of natural numbers such that $\varphi(a, a')$ holds for all $a, a' \in X$ such that $a \neq a'$. Such an X is called a *witness set* for $Q^2xx'\varphi(x, x')$. Our results would not be affected by the addition of further Ramsey quantifiers $Q^3, Q^4, \ldots$. Here of course the intended meaning of $Q^kx_1 \cdots x_k\varphi(x_1, \ldots, x_k)$ is that there exists an infinite set X such that $\varphi(a_1, \ldots, a_k)$ holds for all k-element subsets $\{a_1, \ldots, a_k\}$ of X.

Ramsey quantifiers were first introduced in a general model theoretic setting by Magidor and Malitz [13]. The system $PA(Q^2)$, or rather, a system essentially equivalent to it, was first defined and studied by Macintyre [12]. Some of Macintyre's results were obtained independently by Morgenstern [15]. The present paper is essentially self-contained, but all of our results have been directly inspired by those of Macintyre [12].

After some preliminaries in §1, we begin in §2 by giving a new completeness proof for $PA(Q^2)$. A by-product of our proof is that for every regular uncountable cardinal κ, every consistent extension of $PA(Q^2)$ has a κ-like model in which all classes are definable. (By a *class* we mean a subset of the universe of the model, every initial segment of which is finite in the sense of the model.)

Macintyre's original completeness proof for $PA(Q^2)$ relied on the general Magidor/Malitz results which in turn depended on Jensen's combinatorial principle $\diamondsuit$. Our new completeness proof is quite simple and uses only well-known properties of the MacDowell/Specker construction. Thus, an additional by-product of our proof is that Macintyre's use of $\diamondsuit$ is avoided.

When Macintyre introduced the system $PA(Q^2)$, his stated purpose [12] was to strengthen PA in a natural way so as to eliminate the well-known incompleteness of PA with respect to finite combinatorics. He pointed out that, for instance, the Paris/Harrington [17] combinatorial principle $\forall klm\ \exists n\ n \xrightarrow{*} (m)^k_l$, although independent of PA, is a theorem of $PA(Q^2)$. Macintyre then went on to express

Received March 17, 1980; revised May 18, 1980.

[1]This research was performed in the fall of 1979 while Simpson was a visiting faculty member at the University of Connecticut. Schmerl and Simpson were partially supported by NSF grants MCS 79-05028 and MCS 77-13935, respectively.

0022-4812/82/4702-0014/$02.30

his opinion [12] that no "second generation" of combinatorial principles, independent of $PA(Q^2)$, is likely to arise.

In the light of Macintyre's concerns as just described, it would be of considerable interest to determine exactly the limits of $PA(Q^2)$ with respect to finite combinatorics. In particular, can we characterize the arithmetical consequences of $PA(Q^2)$ in a useful way? (By arithmetical we mean: expressible in the language of PA.)

In §3 we obtain an affirmative answer to this question. Namely, we show that the arithmetical consequences of $PA(Q^2)$ are the same as the arithmetical consequences of a certain well-known ([8], [5]) system of second order arithmetic, viz. Π^1_1-CA_0, i.e. Π^1_1 comprehension with restricted induction.

From this result, it is obvious that not only the Paris/Harrington principle, but also the stronger combinatorial principles considered by Friedman, McAloon and Simpson [9] are all theorems of $PA(Q^2)$. This is because the usual proofs of these principles are straightforwardly formalizable in Π^1_1-CA_0. It is not at all clear how to formalize such proofs directly in $PA(Q^2)$.

At the same time, our result in §3 reduces the problem of finding a transparent combinatorial principle independent of $PA(Q^2)$ to the more manageable problem of finding such a principle independent of Π^1_1-CA_0.

A further consequence of our result in §3 is of course the determination of the provable ordinals and provably recursive functions of $PA(Q^2)$.

In §4 we show that Ramsey quantifiers are effectively eliminable from Presburger arithmetic, i.e., the theory of the natural numbers under addition. This answers a question raised by L.P.D. van den Dries.

We would like to thank Angus Macintyre for letting us see a draft of his paper [12].

§1. The system $PA(Q^2)$. The language of PA contains an infinite supply of number variables $u, v, w, x, y, z, \ldots$ intended to range over ω, the set of natural numbers. The numerical terms are 0, 1, number variables, $t_1 + t_2$ and $t_1 \cdot t_2$ where t_1 and t_2 are numerical terms. The atomic formulae are $t_1 = t_2$ and $t_1 < t_2$ where t_1 and t_2 are numerical terms. The formulae of PA are built up from atomic formulae by means of propositional connectives $\wedge, \vee, \sim, \rightarrow, \leftrightarrow$ and number quantifiers $\forall x$ and $\exists x$. A *sentence* is a formula with no free variables. The axioms of PA consist of the usual ordered semiring axioms for $+, \cdot, 0, 1, <$ together with the *induction scheme*

$$\varphi(0) \wedge \forall x[\varphi(x) \rightarrow \varphi(x + 1)] \rightarrow \forall x \varphi(x)$$

for all formulae $\varphi(x)$ in the language of PA.

Let

$$M = \langle |M|, +, \cdot, 0, 1, < \rangle$$

be a *model of PA*, i.e. a structure in which all the axioms of PA are true. A set $X \subseteq |M|$ is said to be *bounded* in M if there exists $b \in |M|$ such that $a < b$ for all $a \in X$. Given an infinite cardinal κ, we say that M is *κ-like* if $|M|$ is of cardinality κ and every bounded subset of $|M|$ is of cardinality less than κ.

A set $X \subseteq |M|$ is said to be *M-finite* if it is bounded and definable over M by a formula with parameters from $|M|$. By a *class* of M we mean a set $X \subseteq |M|$ such that $X \cap \{a: a < b\}$ is M-finite for all $b \in |M|$. The set of all classes of M is denoted Class(M).

The language of $PA(Q^2)$ consists of the language of PA augmented by a quantifier Q^2 which binds two number variables. Thus, if φ is a formula and x and x' are number variables, then $Q^2xx'\varphi$ is a formula in which x and x' do not occur freely. The intended meaning of Q^2 is explained in the first paragraph of §0. The axioms of $PA(Q^2)$ consist of the axioms of PA, the induction scheme for all formulae of the language of $PA(Q^2)$, and the following schemes:

$$(1) \qquad \forall xx'[\varphi(x, x') \to \psi(x, x')] \wedge Q^2xx'\varphi(x, x') \to Q^2xx'\psi(x, x').$$

$$(2) \quad Q^2xx'[\theta(x) \wedge \varphi(x, x')] \\ \to \exists w\{\theta(w) \wedge Q^2xx'[\theta(x) \wedge \varphi(x, x') \wedge w \neq x \wedge \varphi(w, x) \wedge \varphi(x, w)]\}.$$

$$(3) \quad \forall y\, \exists x > y\theta(x) \wedge \forall xx'[x \neq x' \wedge \theta(x) \wedge \theta(x') \to \varphi(x, x')] \to Q^2xx'\varphi(x, x').$$

The idea behind axiom scheme (2) is that if X is a witness set for $Q^2xx'\varphi$, then X has a least element w and $X\setminus\{w\}$ is again a witness set for $Q^2xx'\varphi$. These axioms are somewhat simpler than those implicit in Macintyre [12].

Let Σ be a set of sentences in the language of $PA(Q^2)$. We say that Σ is *consistent with* $PA(Q^2)$ if $0 = 1$ is not deducible from Σ and the axioms of $PA(Q^2)$ by means of the usual Hilbert-style logical axioms and rules of inference (e.g. Enderton [4, p. 104]). By a *theorem* of $PA(Q^2)$ we mean of course a formula φ in the language of $PA(Q^2)$ such that the negation of the universal closure of φ is not consistent with $PA(Q^2)$.

The following lemma says intuitively that if $Q^2xx'\varphi$ holds then there exists a definable witness set for this fact. This lemma is essentially due to Macintyre [12].

1.1 LEMMA. *Given a formula $\varphi(x, x')$ in the language of $PA(Q^2)$, we can effectively write down a formula $W_\varphi(x)$ such that the following are theorems of $PA(Q^2)$*:

$$(4) \qquad W_\varphi(x) \wedge W_\varphi(x') \wedge x \neq x' \to \varphi(x, x');$$

$$(5) \qquad Q^2xx'\ \varphi(x, x') \leftrightarrow \forall y \exists x > y\ W_\varphi(x).$$

(Note that $\varphi(x, x')$ may contain free variables other than those displayed. If this is the case, then $W_\varphi(x)$ will also contain those free variables.)

PROOF. Using the $PA(Q^2)$ induction scheme, define a sequence $w_0, w_1, \ldots$ as follows: $w_0 =$ least w such that w is an element of some witness set for $Q^2xx'\varphi\ (x, x')$; $w_{n+1} =$ least w such that $\forall i \leq n\ (w \neq w_i)$ and the finite set $\{w_0, \ldots, w_n, w\}$ is extendible to a witness set for $Q^2xx'\varphi(x, x')$. Let $W_\varphi(x)$ say that $\exists n(x = w_n)$. For details of the construction of the formula $W_\varphi(x)$, see Macintyre [12]. The proofs of (4) and (5) from (1), (2), and (3) are straightforward.

We now discuss the semantics of $PA(Q^2)$. Let M be a model of PA and let Σ be a set of sentences in the language of $PA(Q^2)$. We say that M *strongly models* Σ if the sentences of Σ are true in M when $Q^2xx'\varphi(x, x')$ is interpreted to mean that there exists an unbounded set $X \subseteq |M|$ such that $\varphi(a, a')$ holds for all $a, a' \in X$ such that $a \neq a'$. Such an X is called a *witness set*. It is easy to see that any model M of PA strongly models all the axioms of $PA(Q^2)$ except possibly the $PA(Q^2)$ induction scheme. If, in addition, M strongly models the $PA(Q^2)$ induction scheme, we call M a *strong model of* $PA(Q^2)$. Macintyre [12] showed that any strong model of $PA(Q^2)$ is κ-like for some regular cardinal κ. In §2 below we shall prove the converse: any consistent set of sentences in the language of $PA(Q^2)$ has κ-like strong models for every regular uncountable cardinal κ. This result is essentially due to Macintyre [12] in the special case $\kappa = \aleph_1$, assuming $\diamondsuit$.

The following notion will be convenient for our purposes. A *weak model* is an ordered pair (M, q) where M is a model of PA and q is a subset of the power set of $|M|$. In a weak model, $Q^2xx'\varphi(x, x')$ is interpreted to mean that there exists an unbounded set $X \subseteq |M|$ such that $X \in q$ and $\varphi(a, b)$ whenever $a, b \in X$ and $a \neq b$. By a *weak model of* $PA(Q^2)$ we mean of course a weak model in which all the axioms of $PA(Q^2)$ are true. The completeness theorem of ordinary logic easily implies that any set of sentences consistent with the axioms of $PA(Q^2)$ has a countable weak model.

For any weak model (M, q) we denote by $\mathrm{Def}(M, q)$ the set of all $X \subseteq |M|$ such that X is definable over (M, q) by a formula of $PA(Q^2)$ with parameters from $|M|$. Note that if (M, q) is a weak model of $PA(Q^2)$ then $\mathrm{Def}(M, q) \subseteq \mathrm{Class}(M)$. The following lemma is an immediate consequence of Lemma 1.1.

1.2 LEMMA. *If* (M, q) *is a weak model of* $PA(Q^2)$ *then so is* (M, q') *where* $q' = \mathrm{Def}(M, q)$. *Furthermore* (m, q) *and* (M, q') *satisfy the same* $PA(Q^2)$ *sentences with parameters from* $|M|$.

Any strong model M may of course be identified with the weak model (M, q) where q = power set of $|M|$. We shall routinely make this identification.

§2. Strong completeness theorem. The purpose of this section is to present our completeness proof for $PA(Q^2)$. We first need to discuss some properties of an iterated MacDowell/Specker construction.

If M and N are models of PA, we say that N is an *end extension* of M if M is a submodel of N and for all $a < b \in |N|$, if $b \in |M|$ then $a \in |M|$. The MacDowell/Specker theorem [11] says that every model of PA has a proper elementary end extension. We begin by sketching a proof of this theorem.

Let $\langle \varphi_i(x_0, \ldots, x_{n_i}): i < \omega \rangle$ be an enumeration of all formulae of the language of PA with exactly the free variables shown. By Ramsey's theorem [19] formalized within PA, we can find a sequence of formulae $\langle \theta_i(x): i < \omega \rangle$ with only the free variable x, such that for each $i < \omega$, PA proves

(i) $\forall x(\theta_{i+1}(x) \to \theta_i(x))$;

(ii) $\forall y \exists x > y \theta_i(x)$;

(iii) $\forall \bar{x}[x_0 < \cdots < x_{n_i} \wedge \theta_i(x_0) \wedge \cdots \wedge \theta_i(x_{n_i}) \to \varphi_i(\bar{x})]$
$\vee \forall \bar{x}[x_0 < \cdots < x_{n_i} \wedge \theta_i(x_0) \wedge \cdots \wedge \theta_i(x_{n_i}) \to \sim\varphi_i(\bar{x})]$.

From this it follows that for each formula $\varphi(u_1, \ldots, u_m, x_1, \ldots x_n)$ with exactly the free variables shown, there exists i such that PA proves

(iv) $\forall \bar{u} \exists v\{\forall \bar{x}[v < x_1 < \cdots < x_n \wedge \theta_i(x_1) \wedge \cdots \wedge \theta_i(x_n) \to \varphi(\bar{u}, \bar{x})]$
$\vee \forall \bar{x}[v < x_1 < \cdots < x_n \wedge \theta_i(x_1) \wedge \cdots \wedge \theta_i(x_n) \to \sim\varphi(\bar{u}, \bar{x})]\}$.

(To see this, let i be such that $n_i = 2n$ and $\varphi_i(x_0, \ldots, x_{2n})$ is the formula $\forall \bar{u} < x_0[\varphi(\bar{u}, x_1, \ldots, x_n) \leftrightarrow \varphi(\bar{u}, x_{n+1}, \ldots, x_{2n})]$.)

Now let M be a model of PA. We define $p_M(x)$ to be the complete 1-type over $|M|$ generated by the complete diagram of M and the formulae

$$\{\theta_i(x) : i < \omega\} \cup \{a < x : a \in |M|\}.$$

We can then construct an elementary extension $M(c)$ of M where c realizes $p_M(x)$ and $|M| \cup \{c\}$ generates $M(c)$. It is not hard to see that $M(c)$ is an end extension of M and is uniquely determined up to isomorphism over M. We call $M(c)$ *the canonical end extension of* M. The above construction is implicit in a proof of the MacDowell/Specker Theorem due to Morley (§7 of [16]). See also Theorem 3.9 of Mills [14].

Let κ be a regular uncountable cardinal which is greater than the cardinality of M. We can iterate the above construction by letting $\langle M_\xi : \xi \le \kappa \rangle$ be an elementary chain such that $M_0 = M$, $M_\delta = \bigcup_{\xi<\delta} M_\xi$ for all limit ordinals $\delta \le \kappa$, and for all $\xi < \kappa$, $M_{\xi+1} = M_\xi(c_\xi)$ is the canonical end extension of M_ξ. Clearly M_κ is a κ-like elementary end extension of M.

2.1 LEMMA. *Suppose that M_κ satisfies $Q^2xx'\psi(x, x')$ where ψ is a formula* of PA *with parameters from $|M_\kappa|$. Then there exists a witness set for this fact which is definable over M_κ by a formula of PA with parameters from $|M_\kappa|$.*

PROOF. Note first that, for each $\alpha < \kappa$, M_κ is generated by $|M_\alpha| \cup \{c_\xi : \alpha \le \xi < \kappa\}$. We may safely assume that the language of PA has been provided with a μ-operator, so that each element of M_κ is of the form $f(\bar{a}, c_{\xi_1}, \ldots, c_{\xi_n})$ where $a_1, \ldots, a_m \in |M_\alpha|$, $\alpha \le \xi_1 < \cdots < \xi_n$, and $f(\bar{u}, x_1, \ldots, x_n)$ is an $(m+n)$-ary μ-term.

Next, observe that for each $\alpha < \kappa$, $\langle c_\xi : \alpha \le \xi < \kappa \rangle$is a sequence of indiscernibles over $|M_\alpha|$. By (ii) and (iv) this indiscernibility takes on the following strong form: if $\bar{a} \in |M_\alpha|$ and M_κ satisfies $\varphi(\bar{a}, c_{\xi_1}, \ldots, c_{\xi_n})$, $\alpha \le \xi_1 < \cdots < \xi_n$, then there exist $b \in |M_\alpha|$ and a formula $\theta(x)$ such that M_κ satisfies $\forall y \exists x > y\, \theta(x)$ and

$$\forall \bar{x}[b < x_1 < \cdots < x_n \wedge \theta(x_1) \wedge \cdots \wedge \theta(x_n) \to \varphi(\bar{a}, \bar{x})].$$

Now, as in the hypothesis of the lemma, assume that M_κ satisfies $Q^2xx'\psi(x, x')$. Let $X \subseteq |M_\kappa|$ be an unbounded witness set for this fact. Since M_κ is κ-like, X has cardinality κ. A simple counting argument shows that, for some m and n, there exist an $(m+n)$-ary μ-term $f(\bar{u}, \bar{x})$, an m-tuple $\bar{a} \in |M_\kappa|$, and an increasing sequence $\langle \bar{c}^\alpha : \alpha < \kappa \rangle$ of increasing n-tuples of elements of $\langle c_\xi : \xi < \kappa \rangle$, such that $f(\bar{a}, \bar{c}^\alpha) \in X$ and $f(\bar{a}, \bar{c}^\alpha) < f(\bar{a}, \bar{c}^\beta)$ for all $\alpha < \beta < \kappa$. By the strong indiscernibility mentioned above, there exist $b \in |M_\kappa|$ and a formula $\theta(x)$ such that M_κ satisfies $\forall y \exists x > y\, \theta(x)$ and

$$\forall x_1 \cdots x_{2n}[b < x_1 < \cdots < x_{2n} \wedge \theta(x_1) \wedge \cdots \wedge \theta(x_{2n}) \\ \to \psi(f(\bar{a}, x_1, \ldots, x_n), f(\bar{a}, x_{n+1}, \ldots, x_{2n}))].$$

Let $g(b, z)$ be a μ-term denoting the zth element of $\{x : b < x \wedge \theta(x)\}$. Let Y be the set of all elements of $|M_\kappa|$ of the form $f(\bar{a}, g(b, nz+1), \ldots, g(b, nz+n))$. Clearly Y is a witness set for $Q^2xx'\psi(x, x')$. This proves the lemma.

We shall need the generalization of the above to a theory PA^* which is just PA with countably many extra predicates U_i, $i < \omega$. The language of PA^* is the language of PA augmented by new atomic formulae $t \in U_i$ where $i < \omega$ and t is a numerical term. The axioms of PA^* are those of PA plus the induction scheme for all formulae in the language of PA^*. The set of all formulae is still countable so the notion of canonical end extension can be defined for models of PA^*. Once this has been done, the statement of Lemma 2.1 and its proof go through unchanged for PA^*.

2.2 THEOREM. *Let κ be a regular uncountable cardinal. Let (M, q) be a weak model of $PA(Q^2)$ of cardinality less than κ. Then there exists a κ-like strong model N which is a $PA(Q^2)$ elementary end extension of (M, q).*

PROOF. Let $\langle U_i : i < \omega \rangle$ be an enumeration of the subsets of $|M|$ which are $PA(Q^2)$ definable over (M, q) without parameters. Let M^* be the model of PA^* obtained from M by adjoining the U_i as extra predicates. Let M^*_κ be the κth iterated canonical end extension of M^*. Thus M^*_κ is a κ-like elementary end extension of M^*. Let N be the reduct of M^*_κ to a model of PA.

To see that N is as desired, consider the theory $PA^*(Q^2)$ with countably many extra predicates. Clearly (M^*, q) is a weak model of $PA^*(Q^2)$. We claim that the quantifier Q^2 can be eliminated in the following sense: for each formula $\varphi(\bar{u})$ in the language of $PA^*(Q^2)$ with only the free variables shown, there exists a formula $\chi(\bar{u})$ in the language of PA^* with the same free variables, such that both (M^*, q) and M^*_κ satisfy $\forall \bar{u}[\varphi(\bar{u}) \leftrightarrow \chi(\bar{u})]$. Clearly this claim will complete the proof of the theorem.

We prove the claim by induction of the complexity of $\varphi(\bar{u})$. The only nontrivial case is when $\varphi(\bar{u})$ is of the form $Q^2 xx' \psi(\bar{u}, x, x')$. We may assume inductively that ψ is in the language of PA^*. By Lemma 1.1 applied to $PA^*(Q^2)$, there exists a formula $W_\psi(\bar{u}, x)$ such that

$$x \neq x' \wedge W_\psi(\bar{u}, x) \wedge W_\psi(\bar{u}, x') \rightarrow \psi(\bar{u}, x, x')$$

and

$$Q^2 xx' \psi(\bar{u}, x, x') \leftrightarrow \forall y \exists x > y \; W_\psi(\bar{u}, x)$$

are theorems of $PA^*(Q^2)$. Introduce a pairing function

$$(u, v) = \tfrac{1}{2}(u + v)(u + v + 1) + u$$

and nested pairing functions $(u_1) = u_1$, $(u_1, \ldots, u_m, u_{m+1}) = ((u_1. \ldots, u_m), u_{m+1})$. Let i be such that U_i is the set of all $(\bar{a}, d) \in |M^*|$ such that (M^*, q) satisfies $W_\psi(\bar{a}, d)$. Let $\chi(\bar{u})$ be the PA^* formula $\forall y \exists x > y \; (\bar{u}, x) \in U_i$. Trivially (M^*, q) satisfies $\forall \bar{u}[\varphi(\bar{u}) \leftrightarrow \chi(\bar{u})]$. It remains only to show that M^*_κ also satisfies this sentence.

Trivially M^* satisfies the PA^* sentence

$$\forall \bar{u} \forall xx'[x \neq x' \wedge (\bar{u}, x) \in U_i \wedge (\bar{u}, x') \in U_i \rightarrow \psi(\bar{u}, x, x')].$$

Since M^*_κ is a PA^* elementary extension of M^*, it follows that M^*_κ also satisfies it. Hence M^*_κ satisfies $\forall \bar{u}[\chi(\bar{u}) \rightarrow \varphi(\bar{u})]$.

Conversely, suppose that $\bar{a} \in |M^*_\kappa|$ and M^*_κ satisfies $\varphi(\bar{a})$, i.e. $Q^2 xx' \psi(\bar{a}, x, x')$. By Lemma 2.1 applied to PA^*, there exists a formula $\theta(\bar{v}, x)$ in the language of PA^* such that M^*_κ satisfies $\eta(\bar{a})$ where $\eta(\bar{u})$ is the formula

$$\exists \bar{v}\{\forall y \exists x > y \; \theta(\bar{v}, x) \wedge \forall xx'[x \neq x' \wedge \theta(\bar{v}, x) \wedge \theta(\bar{v}, x') \rightarrow \psi(\bar{u}, x, x')]\}.$$

Clearly (M^*, q) satisfies $\forall \bar{u}[\eta(\bar{u}) \rightarrow \varphi(\bar{u})]$. Hence M^* satisfies $\forall \bar{u}[\eta(\bar{u}) \rightarrow \chi(\bar{u})]$. The latter sentence belongs to the language of PA^*, so M^*_κ also satisfies it. Hence M^*_κ satisfies $\chi(\bar{a})$. This completes the proof of the theorem.

A slight modification of the proof shows that one can obtain 2^κ nonisomorphic models with the properties mentioned. This answers a question of Macintyre [12].

For the following corollaries, assume that Σ is a set of sentences in the language of $PA(Q^2)$ such that Σ includes the axioms of $PA(Q^2)$.

2.3 COROLLARY (COMPLETENESS). *Σ is consistent if and only if Σ has a strong model.*

2.4 COROLLARY (COMPACTNESS). *If every finite subset of Σ has a strong model, then Σ has a strong model; in fact, Σ has κ-like strong models for every regular uncountable cardinal κ.*

Both corollaries are immediate from Theorem 2.2 and the fact that Σ is consistent if and only if it has a countable weak model.

The particular strong models which were constructed above have a further property which may be of some interest:

2.5 THEOREM. *Let N be one of the κ-like strong models constructed in the proof of Theorem* 2.2. *Then* $\text{Class}(N) = \text{Def}(N)$, *i.e. every class of N is definable over N by a $PA(Q^2)$ formula with parameters from $|N|$.*

PROOF. The proof of Theorem 2.2 shows that $X \in \text{Def}(N)$ if and only if X is definable over M_κ^* by a PA^* formula with parameters from $|M_\kappa^*|$. Since trivially $\text{Class}(N) = \text{Class}(M_\kappa^*)$, the desired conclusion follows from Theorem 1.5 of Schmerl [20].

§3. Π_1^1 comprehension.

In this section we present our characterization of the arithmetical sentences which are provable in $PA(Q^2)$.

We need to consider formal systems in the language of second order arithmetic. This language contains number variables u, v, x, y, ... intended to range over ω, and set variables X, Y, ... intended to range over subsets of ω. The numerical terms of the language are as for PA. The atomic formulae are as for PA with the addition of a new kind of atomic formula $t \in X$ where t is a numerical term and X is a set variable. Formulae are built up from atomic formulae by means of propositional connectives, number quantifiers, and set quantifiers $\forall X$ and $\exists X$. A formula is said to be *arithmetical* if it contains no set quantifiers (but may contain free set variables). A *sentence* is a formula with no free variables.

The axioms of our basic system, ACA_0, are as follows: the usual ordered semiring axioms for $+$, $\cdot$, 0, 1, $<$; the induction axiom $0 \in X \wedge \forall u(u \in X \rightarrow u + 1 \in X) \rightarrow \forall u(u \in X)$; and comprehension axioms $\exists X \forall u(u \in X \leftrightarrow \varphi)$ where φ is any arithmetical formula in which X does not occur freely. Clearly every theorem of PA is a theorem of ACA_0.

A formula is said to be Π_1^1 if it is of the form $\forall Y \varphi$ where φ is arithmetical. The system Π_1^1-CA_0 consists of ACA_0 plus comprehension axioms $\exists X \forall u(u \in X \leftrightarrow \forall Y \varphi)$ where φ is any arithmetical formula in which X does not occur freely.

The purpose of this section is to show that there is a close relationship of mutual interpretability between the systems $PA(Q^2)$ and Π_1^1-CA_0. In particular, these superficially quite different looking systems will be seen to prove exactly the same arithmetical sentences. (An arithmetical sentence is of course just a sentence in the language of PA. Note that the language of PA is just the common part of the language of $PA(Q^2)$ and the language of second order arithmetic.)

This mutual interpretability result is interesting because the system Π^1_1-CA_0 has already been studied extensively by logicians. For instance, it is known that the minimum β-model of Π^1_1-CA_0 can be characterized as the smallest nonempty set of subsets of ω closed under relative recursiveness and hyperjump; in other words, as $P(\omega) \cap L_{\omega_\omega}$ where ω_ω is the limit of the first ω admissible ordinals (see [10, p. 26, Example (4)]). In addition, detailed information of a proof theoretical nature is available concerning Π^1_1-CA_0. Friedman [8] has shown that Π^1_1-CA_0 proves the same Π^1_1 sentences as the formal system $ID^{<\omega}$ of iterated inductive definitions. Feferman [5] and Zucker [22] have computed exactly the provable ordinals and provably recursive functions of $ID^{<\omega}$. See also the remarks after Corollary 3.8.

The system Π^1_1-CA_0 is also known to be extremely interesting from the viewpoint of the foundations of mathematics. For instance, Friedman [8] has shown that Π^1_1-CA_0 is just strong enough to formalize many of the usual arguments concerning Borel sets, etc., which depend on having a good theory of countable ordinals. For this reason, Π^1_1-CA_0 is one of the four or five systems of second order arithmetic considered by Friedman [7]. He proposes to use these systems as benchmarks by which to calibrate the intrinsic proof theoretical strength of mathematical texts. See also Feferman [6] and Simpson [21].

Our first goal is to show that every weak model of $PA(Q^2)$ gives rise to a model of Π^1_1-CA_0. By a *model* of any theory $T \supseteq ACA_0$ we mean of course an ordered pair (M, s) such that M is a model of PA, s is a subset of the powerset of $|M|$, and all the axioms of T are true when set variables are interpreted as ranging over s. Clearly every model M of PA can be expanded to a model (M, s) of ACA_0. (Just let s consist of the subsets of $|M|$ which are definable over M by formulae with parameters from $|M|$.) From this it follows that ACA_0 is a conservative extension of PA.

We assume that the usual pairing function $(u, v) = \frac{1}{2}(u + v)(u + v + 1) + u$ and the usual coding of finite sequences (e.g. via the Chinese remainder theorem) have been formalized within PA. Recall that a function $f: \omega \to \omega$ can be coded as a set $X \subseteq \omega$ where $X = \{(u, f(u)) : u \in \omega\}$. For $f: \omega \to \omega$ and $x \in \omega$ we denote by $f[x]$ the code of the finite sequence $\langle f(u) : u < x \rangle$. We assume that these codings have been formalized within ACA_0.

We shall need the following well-known formal version of the Kleene normal form theorem for Σ^1_1 relations.

3.1 LEMMA. *Given an arithmetical formula $\varphi(Y)$, we can find an arithmetical formula $\psi(y)$ such that*

$$\exists Y \varphi(Y) \leftrightarrow \exists f \forall x \psi(f[x])$$

is a theorem of ACA_0.

PROOF. Let

$$\forall x_1 \exists y_1 \cdots \forall x_k \exists y_k \, \theta(\bar{x}, \bar{y}, Y)$$

be the prenex normal form of $\varphi(Y)$. We say that $g_i: \omega^i \to \omega$, $1 \le i \le k$, are *Skolem functions* for Y if $\forall \bar{x}\, \theta(\bar{x}, \bar{g}(\bar{x}), Y)$. By arithmetical comprehension, $\varphi(Y)$ if and only if there exist Skolem functions for Y. Hence $\exists Y \varphi(Y)$ if and only if $\exists Y \exists \bar{g} \forall \bar{x}$ $\theta(\bar{x}, \bar{g}(\bar{x}), Y)$. Each of the quantifier strings $\exists Y \exists \bar{g}$ and $\forall \bar{x}$ can be collapsed into a single quantifier by means of the pairing function. This gives the desired result.

We are now ready to prove

3.2 LEMMA. *Let (M, q) be a weak model of $PA(Q^2)$. Then $(M, \mathrm{Def}(M, q))$ is a model of Π^1_1-CA_0.*

PROOF. By Lemma 1.2 we may safely assume that $q = \mathrm{Def}(M, q)$. Since the language of $PA(Q^2)$ includes the language of PA, it is clear that (M, q) is a model of ACA_0. In order to show that (M, q) is a model of Π^1_1-CA_0, suppose that

$$X = \{a \in |M| : (M, q) \text{ satisfies } \exists Y\varphi(a, Y)\}$$

where $\varphi(u, Y)$ is arithmetical with parameters from (M, q). It suffices to show that $X \in q$. By Lemma 3.1, let $\psi(u, y)$ be arithmetical with parameters from (M, q) such that ACA_0 proves

$$\exists Y\varphi(u, Y) \leftrightarrow \exists f\, \forall x\psi(u, f[x]).$$

Let us write $v \prec w$ to mean that the finite sequence coded by v is an initial segment of the finite sequence coded by w. Then, for each $a \in |M|$, we have the following chain of implications:

$$\begin{aligned} a \in X &\leftrightarrow (M, q) \text{ satisfies } \exists Y\varphi(a, Y) \\ &\leftrightarrow (M, q) \text{ satisfies } \exists f\, \forall x\psi(a, f[x]) \\ &\leftrightarrow (M, q) \text{ satisfies } Q^2vw[\psi(a, v) \wedge (v \prec w \vee w \prec v)]. \end{aligned}$$

The last formula can be construed as belonging to the language of $PA(Q^2)$. Hence $X \in \mathrm{Def}(M, q) = q$. This proves the lemma.

Conversely, we have

3.3 LEMMA. *Let (M, s) be a model of Π^1_1-CA_0. Then (M, s) is a weak model of $PA(Q^2)$.*

PROOF. In order to show that (M, s) is a weak model of $PA(Q^2)$, it suffices to show that $\mathrm{Def}(M, s) \subseteq s$. Let 0 be the empty set and for each $k \in \omega$ let $0^{(k+1)}$ be the hyperjump of $0^{(k)}$, i.e. the complete Π^1_1 set relative to $0^{(k)}$, as defined within (M, s). Thus $0^{(k)} \in s$ for all $k \in \omega$. We shall actually show that for any $PA(Q^2)$ formula $\varphi(u_1, \ldots, u_n)$ with exactly the free variables shown, the n-ary relation $R_\varphi \subseteq |M|^n$ defined by φ is satisfied in (M, s) to be arithmetical in $0^{(k)}$ for some $k \in \omega$. (In fact k may be taken to be the number of occurrences of Q^2 in $\varphi(\bar{u})$.) We proceed by induction on the complexity of $\varphi(\bar{u})$. The only nontrivial case is when $\varphi(\bar{u})$ is of the form $Q^2xx'\psi(\bar{u}, x, x')$. Clearly (M, s) satisfies $Q^2xx'\psi(\bar{a}, x, x')$ if and only if there exists $X \in s$ such that X is unbounded in M and $R_\psi(\bar{a}, c, d)$ for all $c, d \in X$ such that $c \neq d$. Thus R_φ is Σ^1_1 in R_ψ. Hence, if R_ψ is arithmetical in $0^{(k)}$, R_φ is recursive in $0^{(k+1)}$. This completes the proof.

Let us say in accordance with Lemma 1.2 that a weak model (M, q) of $PA(Q^2)$ is *reduced* if $q = \mathrm{Def}(M, q)$. Let us say that a model (M, s) of Π^1_1-CA_0 is *reduced* if for all $X \in s$ there exists $k \in \omega$ such that X is satisfied in (M, s) to be recursive in $0^{(k)}$. Combining the proofs of Lemmas 3.2 and 3.3, we obtain immediately:

3.4 THEOREM. *The reduced weak models of $PA(Q^2)$ are the same as the reduced models of Π^1_1-CA_0.*

From this point of view, a number of the results about $PA(Q^2)$ obtained by

Macintyre [12] are obvious. For instance, the existence of a truth definition for arithmetic in $PA(Q^2)$ is immediate from Lemma 3.2 since the usual construction of a satisfaction set for arithmetic is directly formalizable in Π^1_1-CA_0. Similarly, Macintyre's "Ramsey scheme" follows at once from Ramsey's theorem [19] formalized within Π^1_1-CA_0. The reader may have noticed also the close resemblance between the proof of Macintyre's key Lemma 1.1 and that of the Kleene basis theorem. Our original proof of Theorem 2.2 (not the proof presented in this paper) used the Kleene basis theorem in place of Lemma 1.1.

In another direction, the iterated MacDowell-Specker construction of §2 may be combined with Lemmas 3.2 and 3.3 to yield model theoretical information about Π^1_1-CA_0. Thus we have

3.5 COROLLARY. *For any regular uncountable cardinal κ, there exists a κ-like model N of PA such that* $(N, \text{Class}(N))$ *is a (reduced)model of* Π^1_1-CA_0.

PROOF. Let N be any one of the κ-like strong models constructed in the proof of Theorem 2.2. The desired conclusion is immediate from Theorems 2.5 and 3.4.

We do not know whether Corollary 3.5 continues to hold with Π^1_1-CA_0 replaced by, for instance, Δ^1_2-CA_0 or Π^1_2-CA_0 or full comprehension.

On the proof theoretical side, we have the following main result.

3.6 THEOREM. *Let σ be an arithmetical sentence. Then σ is a theorem of* $PA(Q^2)$ *if and only if it is a theorem of* Π^1_1-CA_0.

PROOF. Immediate from Lemmas 3.2 and 3.3.

3.7 COROLLARY. *The combinatorial principles of Paris and Harrington* [17] *and Friedman, McAloon and Simpson* [9] *are provable in* $PA(Q^2)$.

PROOF. Immediate from Theorem 3.6 since the usual proofs of these arithmetical sentences are directly formalizable in Π^1_1-CA_0.

3.8 COROLLARY. *The provably recursive ordinals of* $PA(Q^2)$ *are precisely the ordinals less than the Bachmann ordinal* $\theta\Omega_\omega 0$. *The provably recursive functions of* $PA(Q^2)$ *are precisely the* $\theta\Omega_\omega 0$-*recursive functions.*

PROOF. Immediate from Theorem 3.6 and the known results for Π^1_1-CA_0 and $ID^{<\omega}$ (Friedman [8], Feferman [5], Zucker [22]).

Note. Further proof-theoretic information concerning Π^1_1-CA_0 and theories of iterated inductive definitions may be found in a paper by S. Feferman and W. Sieg entitled: *Proof theoretic equivalences between classical and constructive theories for analysis.* This paper will appear in ***Iterated inductive definitions and subsystems of analysis: Recent Proof-theoretical Studies***, to be published as a volume of Springer Lecture Notes in Mathematics.

The earliest proof theoretical analysis of Π^1_1-CA_0 was given by Takeuti [28]. Perhaps the most accessible source of basic information on the proof theory of Π^1_1-CA_0 and the ordinal $\theta\Omega_\omega 0$ is Schütte [27].

Recently the ordinal $\theta\Omega_\omega 0$ has emerged as being of interest in connection with subrecursive hierarchies. Namely, if $\langle\lambda(x): x < \omega\rangle$ is a canonically chosen fundamental sequence for a countable limit ordinal λ, one defines number-theoretic functions $G_0(x) = 0$, $G_{\alpha+1}(x) = G_\alpha(x) + 1$, $G_\lambda(x) = G_{\lambda(x)}(x)$ and $F_0(x) = 2^x$, $F_{\alpha+1}(x) = F^x_\alpha(x)$, $F_\lambda(x) = F_{\lambda(x)}(x)$. Then $\theta\Omega_\omega 0$ can be characterized as the first place where the "slow-growing" G_α hierarchy catches up to the "fast-growing" (Grzegorczyk/Wainer) F_α hierarchy; in other words, $\theta\Omega_\omega 0 =$ least $\alpha > 0$ such that

F_α is elementary recursive in G_α. This result is due to Girard [25]; see also Aczel [23], Cichon and Wainer [24], and Schmerl [26]. Thus one may characterize the provably recursive functions of $\Pi^1_1\text{-}CA_0$ as the computable functions with running time dominated by some F_α, $\alpha < \theta\Omega_\omega 0$, or equivalently by some G_α, $\alpha < \theta\Omega_\omega 0$.

§**4. Presburger arithmetic.** It is clear from the results of §3 that the theory of arithmetic with Ramsey quantifiers is far from decidable. In fact, the truth set for this language is recursively isomorphic to the set $0^{(\omega)} = \{(n, k): n \in 0^{(k)}\}$ of Jockusch and Simpson [10, p. 26, Example (4)]; here $0^{(k)}$ is the kth hyperjump of the empty set.

It turns out that for Presburger arithmetic the situation is quite different. By *Presburger arithmetic* we mean of course the complete first-order theory

$$PRA = \mathrm{Th}(\langle \omega, +, 0, 1, < \rangle)$$

of arithmetic without multiplication. It is well known that PRA is decidable; in fact, PRA has a very simple set of axioms all of which are provable in PA. The purpose of this section is to show that Ramsey quantifiers Q^2, Q^3, ... can be effectively eliminated from formulae of PRA. As a corollary we obtain decidability of Presburger arithmetic with Ramsey quantifiers.

In any model of PRA we introduce a *strong semantics* for the Ramsey quantifiers Q^2, Q^3, ... as before: $Q^k x_1 \cdots x_k \phi$ means that there exists an unbounded set X such that ϕ holds for all k-element subsets $\{x_1, \ldots, x_k\}$ of X.

4.1 THEOREM. *Given a formula θ in the language of PRA with Ramsey quantifiers, we can effectively find a first order formula ψ in the language of PRA, such that θ and ψ are equivalent in all models of PRA.*

PROOF. It will be convenient to work not with the Ramsey quantifiers Q^2, Q^3, ... but instead with the closely related *ordered Ramsey quantifiers* Q^2_0, Q^3_0, ... The semantics of the ordered Ramsey quantifiers are that $Q^k_0 x_1 \cdots x_k \phi$ holds if and only if there exists an unbounded set X such that ϕ holds for all ordered k-tuples $x_1 < \cdots < x_k$ of elements of X. Such an X is called a *witness set* for ϕ. Note that Q^k is uniformly definable in terms of Q^k_0. Thus it will suffice to prove the theorem for ordered Ramsey quantifiers in place of Ramsey quantifiers.

Let the language of PRA be augmented with binary relation symbols $\equiv_m$, $m \geq 2$. The intended meaning of $x \equiv_m y$ is that x is congruent to y modulo m. These new relations are of course definable in PRA. In view of the well-known quantifier elimination theorem for PRA (Presburger [18]; see also §3.2 of Enderton [4]), it suffices to prove the theorem only for θ of the form $Q^k_0 x_1 \cdots x_k \phi(x_1, \ldots, x_k, y_1, \ldots, y_n)$ where ϕ is a quantifier-free formula in the augmented language. Thus ϕ is a Boolean combination of atomic subformulae each of which looks like one of the following:

(1) $a_1x_1 + \cdots + a_kx_k = b_1y_1 + \cdots + b_ny_n + c$,
(2) $a_1x_1 + \cdots + a_kx_k > b_1y_1 + \cdots + b_ny_n + c$,
(3) $a_1x_1 + \cdots + a_kx_k \equiv_m b_1y_1 + \cdots + b_ny_n + c$,

where $a_1, \ldots, a_k, b_1, \ldots, b_n, c$ are *integers* (positive, negative, or zero).

Clearly any subformula of the form (1) may be eliminated in favor of subformulae of the form (2). Furthermore, any subformula having the form (2) with some

$a_i \neq 0$ may be eliminated in the following manner. Let j be such that $a_j \neq 0 = a_{j+1} = a_{j+2} = \cdots = a_k$. If $a_j > 0$ (respectively, $a_j < 0$) then any unbounded set X may be pared down to an unbounded witness set for (2) (respectively, the negation of (2)). Thus (2) may be replaced by a formula which is either identically true or identically false. Finally, any subformula in which $a_1 = \cdots = a_k = 0$ may safely be ignored.

The preceding paragraph permits us to assume that each nontrivial atomic subformula of ϕ is a congruence of the form (3). Let M be the least common multiple of the moduli m of these congruences. By the pigeonhole principle, any unbounded witness set for ϕ will have an unbounded subset all of whose elements belong to a single residue class modulo M. Hence

$$Q_0^k x_1 \cdots x_k\, \phi(x_1, \ldots, x_k, y_1, \ldots, y_n)$$

is equivalent to

$$\bigvee_{x=1}^{M} \phi(x, \ldots, x, y_1, \ldots, y_n).$$

The latter formula is quantifier free and so the proof is complete.

4.2 COROLLARY. *The theory of arithmetic without multiplication but with Ramsey quantifiers* $Q^2, Q^3, \ldots$ *is decidable.*

PROOF. Given a sentence σ in this language, we can effectively find an equivalent quantifier free sentence ψ in the augmented language of Presburger arithmetic. It is easy to decide the truth of ψ.

Other results concerning theories which admit elimination of Ramsey quantifiers may be found in Baldwin and Kueker [1], Baudisch [2], and Cowles [3].

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