Pseudojump inversion in special r. b. Π^0_1 classes

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Abstract

The Jump Inversion Theorem says that for every real $A \geq_{\mathrm{T}} 0'$ there is a real B such that $A \equiv_{\mathrm{T}} B' \equiv_{\mathrm{T}} B \oplus 0'$. A known refinement of this theorem says that we can choose B to be a member of any special Π^0_1 subclass of $\{0,1\}^{\mathbb{N}}$. We now consider the possibility of analogous refinements of two other well-known theorems: the Join Theorem – for all reals A and Z such that $A \geq_{\mathrm{T}} Z \oplus 0'$ and $Z >_{\mathrm{T}} 0$, there is a real B such that $A \equiv_{\mathrm{T}} B' \equiv_{\mathrm{T}} B \oplus 0' \equiv_{\mathrm{T}} B \oplus Z$ – and the Pseudojump Inversion Theorem – for all reals $A \geq_{\mathrm{T}} 0'$ and every $e \in \mathbb{N}$, there is a real B such that $A \equiv_{\mathrm{T}} B \oplus W^B_e \equiv_{\mathrm{T}} B \oplus 0'$. We show that in these theorems, B can be found in some special Π^0_1 subclasses of $\{0,1\}^{\mathbb{N}}$ but not in others.

Contents

1 Introduction

We begin with a well-known theorem.

Theorem 1.1 (Jump Inversion Theorem, due to Friedberg, see [11, Theorem 13.3.IX])**.** Given a real $A \geq_{\mathrm{T}} 0'$ we can find a real B such that $A \equiv_{\mathrm{T}} B' \equiv_{\mathrm{T}} B \oplus 0'$.

Here $0'$ denotes the Halting Problem, B' denotes the Turing jump of B, and $\equiv_{\mathrm{T}}$ denotes Turing equivalence. There are two important related theorems, which read as follows.

Theorem 1.2 (Join Theorem, due to Posner and Robinson [10, Theorem 1], see also [3, Theorem 2.1])**.** Given reals A and Z such that $A \geq_{\mathrm{T}} Z \oplus 0'$ and $Z >_{\mathrm{T}} 0$, we can find a real B such that $A \equiv_{\mathrm{T}} B' \equiv_{\mathrm{T}} B \oplus 0' \equiv_{\mathrm{T}} B \oplus Z$.

Theorem 1.3 (Pseudojump Inversion Theorem, due to Jockusch and Shore [3, Theorem 2.1])**.** Given a real $A \geq_{\mathrm{T}} 0'$ and an integer $e \in \mathbb{N}$, we can find a real B such that $A \equiv_{\mathrm{T}} J_e(B) \equiv_{\mathrm{T}} B \oplus 0'$.

Here J_e denotes the eth pseudojump operator, defined by $J_e(B) = B \oplus W_e^B$ where $\langle W_e^B \mid e \in \mathbb{N}\rangle$ is a fixed standard recursive enumeration of the B-recursively enumerable subsets of $\mathbb{N}$.

Our results in this paper concern special Π^0_1 classes. Following [4] we define a *special* Π^0_1 *class* to be a nonempty Π^0_1 subset of $\mathbb{N}^{\mathbb{N}}$ with no recursive elements. Here $\mathbb{N}^{\mathbb{N}}$ is the Baire space, but as in [4, 5] we focus on special Π^0_1 subclasses of $\{0,1\}^{\mathbb{N}}$, the Cantor space. On the other hand, because of [13, Theorem 4.10] our results concerning special Π^0_1 subclasses of $\{0,1\}^{\mathbb{N}}$ will also apply to *recursively bounded* special Π^0_1 subclasses of $\mathbb{N}^{\mathbb{N}}$. Following [5] we use "r. b." as an abbreviation for "recursively bounded."

We now recall another known theorem, which says that we can find the B for Theorem 1.1 in any special Π^0_1 subclass of the Cantor space.

Theorem 1.4 (due to Jockusch and Soare [5, just after the proof of Theorem 2.1])**.** Let P be a special Π^0_1 subclass of $\{0,1\}^{\mathbb{N}}$. Given a real $A \geq_{\mathrm{T}} 0'$, we can find a real $B \in P$ such that $A \equiv_{\mathrm{T}} B' \equiv_{\mathrm{T}} B \oplus 0'$.

This refinement of Theorem 1.1 suggests a question as to the existence or nonexistence of analogous refinements of Theorems 1.2 and 1.3. In order to state our results concerning this question, we make the following definition.

Definition 1.5. Let $P \subseteq \mathbb{N}^{\mathbb{N}}$ be a subclass of the Baire space.

1. P has the *Join Property* if for all reals A and Z such that $A \geq_{\mathrm{T}} Z \oplus 0'$ and $Z >_{\mathrm{T}} 0$, there exists $B \in P$ such that $A \equiv_{\mathrm{T}} B' \equiv_{\mathrm{T}} B \oplus 0' \equiv_{\mathrm{T}} B \oplus Z$.

2. P has the *Pseudojump Inversion Property* if for all reals $A \geq_{\mathrm{T}} 0'$ and all $e \in \mathbb{N}$, there exists $B \in P$ such that $A \equiv_{\mathrm{T}} J_e(B) \equiv_{\mathrm{T}} B \oplus 0'$.

The question that we are asking is, which special Π^0_1 subclasses of the Cantor space $\{0,1\}^{\mathbb{N}}$ have one or both of the properties in Definition 1.5? The purpose of this paper is to present some partial results in this direction, as follows. Let CPA be the special Π^0_1 subclass of $\{0,1\}^{\mathbb{N}}$ consisting of all complete, consistent extensions of Peano Arithmetic. We show that CPA has both the Join Property and the Pseudojump Inversion Property. More generally, we show that any special Π^0_1 class which is Turing degree isomorphic to CPA has both of these properties. For example, this holds for the Π^0_1 class CZF consisting of all complete, consistent extensions of Zermelo-Fraenkel Set Theory (assuming that CZF is nonempty), and for the Π^0_1 class DNR_2 consisting of all $\{0,1\}$-valued diagonally nonrecursive functions. On the other hand, let P be a special Π^0_1 subclass of $\{0,1\}^{\mathbb{N}}$ which is of positive measure. Citing a theorem of Nies and an observation of Patrick Lutz, we note that P need not have the Join Property, but P and indeed any special Π^0_1 class which is Turing degree isomorphic to P has the Pseudojump Inversion Property. Finally, we construct a special Π^0_1 subclass of $\{0,1\}^{\mathbb{N}}$ which has neither the Join Property nor the Pseudojump Inversion Property. We do not know whether there exists a special Π^0_1 subclass of $\{0,1\}^{\mathbb{N}}$ which has the Join Property but not the Pseudojump Inversion Property.

Here is an outline of the paper. In §2 we present the classical proofs of Theorems 1.2 and 1.3. In §3 we use the Gödel-Rosser Incompleteness Theorem [8] to prove that CPA has the Join Property and the Pseudojump Inversion Property. In §4 we present some results about Turing degree isomorphism of Π^0_1 classes, and we apply these results to show that any Π^0_1 class which is Turing degree isomorphic to CPA has the Join Property and the Pseudojump Inversion Property. In §5 we show that special Π^0_1 classes consisting of Martin-Löf random reals have the Pseudojump Inversion Property but not the Join Property. In §6 we use a priority argument to construct a special Π^0_1 subclass $Q \subseteq \{0,1\}^{\mathbb{N}}$ which has neither the Join Property nor the Pseudojump Inversion Property.

Remark 1.6. The lattice $\mathcal{E}_{\mathrm{w}}$ of all Muchnik degrees of nonempty Π^0_1 subclasses of $\{0,1\}^{\mathbb{N}}$ is of great interest; see for instance the references in [19]. Therefore, it is natural to ask about Muchnik degrees in $\mathcal{E}_{\mathrm{w}}$ corresponding to the properties in Definition 1.5. Our results in this paper shed some light on this question. First, it is known [13, Theorem 6.8] that any Π^0_1 subclass of $\{0,1\}^{\mathbb{N}}$ which is Muchnik equivalent to CPA – i.e., of Muchnik degree $\mathbf{1}$, where $\mathbf{1}$ denotes the top degree in $\mathcal{E}_{\mathrm{w}}$ – is Turing degree isomorphic to CPA. Therefore, our results in §4 below imply that all Π^0_1 classes of Muchnik degree $\mathbf{1}$ have both of the properties in Definition 1.5. Second, it is easy to see that each degree in $\mathcal{E}_{\mathrm{w}}$ contains a Π^0_1 subclass of $\{0,1\}^{\mathbb{N}}$ which includes CPA and hence again has both properties. On the other hand, by §5 below we have a nonzero degree $\mathbf{r} \in \mathcal{E}_{\mathrm{w}}$ containing a special Π^0_1 subclass of $\{0,1\}^{\mathbb{N}}$ which has one property but not the other. Moreover, in §6 below we construct a nonzero degree in $\mathcal{E}_{\mathrm{w}}$ containing a special Π^0_1 subclass of $\{0,1\}^{\mathbb{N}}$ which does not have either property.

In the remainder of this section, we fix some notation and terminology.

We write $f :\subseteq A \to B$ to mean that f is a *partial function* from A to B, i.e., a function with domain $\subseteq A$ and range $\subseteq B$. For $a \in A$ we write $f(a)\downarrow$ and say that $f(a)$ *converges* or $f(a)$ is *defined*, if $a \in$ the domain of f. Otherwise we write $f(a)\uparrow$ and say that $f(a)$ *diverges* or $f(a)$ is *undefined*. For $a \in A$ and $f, g :\subseteq A \to B$, we write $f(a) \simeq g(a)$ to mean that either ($f(a)\downarrow$ and $g(a)\downarrow$ and $f(a) = g(a)$) or ($f(a)\uparrow$ and $g(a)\uparrow$). We write $f(a)\downarrow = b$ to mean that $f(a)\downarrow$ and $f(a) = b$.

The set of all natural numbers is $\mathbb{N} = \{0, 1, 2, \ldots\}$. For any set A let $A^{\mathbb{N}}$ denote the set of all sequences $X : \mathbb{N} \to A$. Thus $\mathbb{N}^{\mathbb{N}}$ and $\{0,1\}^{\mathbb{N}}$ are the *Baire space* and the *Cantor space* respectively. The points $X \in \mathbb{N}^{\mathbb{N}}$ or $X \in \{0,1\}^{\mathbb{N}}$ are sometimes called *reals*. We sometimes identify $\{0,1\}^{\mathbb{N}}$ with the powerset of $\mathbb{N}$ in the usual manner. For $X, Y \in A^{\mathbb{N}}$ the *join* $X \oplus Y \in A^{\mathbb{N}}$ is given by $(X \oplus Y)(2n) = X(n)$, $(X \oplus Y)(2n+1) = Y(n)$, and for $P, Q \subseteq A^{\mathbb{N}}$ we write $P \times Q = \{X \oplus Y \mid X \in P \text{ and } Y \in Q\}$. Note in particular that $P \times Q \subseteq \mathbb{N}^{\mathbb{N}}$ whenever $P, Q \subseteq \mathbb{N}^{\mathbb{N}}$, and $P \times Q \subseteq \{0,1\}^{\mathbb{N}}$ whenever $P, Q \subseteq \{0,1\}^{\mathbb{N}}$.

For any set A let A^* be the set of *strings*, i.e., finite sequences, of elements of A. For $a_0, \ldots, a_{m-1} \in A$ we let $\langle a_0, \ldots, a_{m-1}\rangle$ denote the string $\sigma \in A^*$ of length m given by $\sigma(i) = a_i$ for all $i < m$. In this case we write $|\sigma| = m$, and for each $k \leq m$ we write $\sigma\restriction k = \langle a_0, \ldots, a_{k-1}\rangle$. We also write $A^m = \{\sigma \in A^* \mid |\sigma| = m\}$ and $A^{\geq m} = \{\sigma \in A^* \mid |\sigma| \geq m\}$ and $A^{<m} = \{\sigma \in A^* \mid |\sigma| < m\}$. For strings $\sigma = \langle a_0, \ldots, a_{m-1}\rangle$ and $\tau = \langle b_0, \ldots, b_{n-1}\rangle$, their *concatenation* is $\sigma^\frown\tau = \langle a_0, \ldots, a_{m-1}, b_0, \ldots, b_{n-1}\rangle$. Note that $|\sigma^\frown\tau| = |\sigma| + |\tau|$. We say that σ is an *initial segment* of τ, or equivalently τ is an *extension* of σ, written $\sigma \subseteq \tau$, if $\tau\restriction|\sigma| = \sigma$, i.e., if $\tau = \sigma^\frown\rho$ for some string ρ. We write $\sigma \subset \tau$ to mean that $\sigma \subseteq \tau$ and $\sigma \neq \tau$. We say that $\sigma \in A^*$ is an *initial segment* of $X \in A^{\mathbb{N}}$, or equivalently X is an *extension* of σ, written $\sigma \subset X$, if $X\restriction|\sigma| = \sigma$, i.e., if $\sigma = \langle X(0), \ldots, X(|\sigma| - 1)\rangle$.

Fix a standard recursive enumeration $\langle \varphi^{(k)}_e \mid e \in \mathbb{N}\rangle$ of the k-place partial recursive functions $\psi :\subseteq \mathbb{N}^k \to \mathbb{N}$. Here e is called an *index* of $\varphi^{(k)}_e$. Likewise, for $X \in \mathbb{N}^{\mathbb{N}}$ we have a standard recursive enumeration $\langle \varphi^{(k),X}_e \mid e \in \mathbb{N}\rangle$ of the k-place partial functions $\psi :\subseteq \mathbb{N}^k \to \mathbb{N}$ which are partial recursive relative to X. Here e is again called an *index* of $\varphi^{(k),X}_e$, while X is called an *oracle*. We write

$W_e^X = \{i \in \mathbb{N} \mid \varphi_e^{(1),X}(i)\downarrow\}$ and this gives a standard recursive enumeration of the subsets of $\mathbb{N}$ which are recursively enumerable relative to X. We also write $P_e = \{X \in \mathbb{N}^\mathbb{N} \mid \varphi_e^{(1),X}(0)\uparrow\}$ and this gives a standard recursive enumeration of the Π^0_1 subclasses of $\mathbb{N}^\mathbb{N}$.

For $X \in \mathbb{N}^\mathbb{N}$ let $X' \in \mathbb{N}^\mathbb{N}$ denote the *Turing jump* of X. For definiteness we take X' to be (the characteristic function of) the set

$$\{e \in \mathbb{N} \mid 0 \in W_e^X\} = \{e \in \mathbb{N} \mid X \notin P_e\}.$$

For each $e \in \mathbb{N}$ we have a *pseudojump operator* $J_e : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ given by $J_e(X) = X \oplus W_e^X$. We write $\leq_\mathrm{T}$ for *Turing reducibility* and $\equiv_\mathrm{T}$ for *Turing equivalence*. We write $\leq_\mathrm{tt}$ for *truth-table reducibility*[1] and $\equiv_\mathrm{tt}$ for *truth-table equivalence*.

2 The Join Theorem and the Pseudojump Inversion Theorem

In this section we present the classical proofs of the Join Theorem and the Pseudojump Inversion Theorem. Later we shall build on these proofs in order to obtain our refinements.

We begin with the Join Theorem, Theorem 1.2.

Proof of Theorem 1.2. Let A and Z be reals such that $A \geq_\mathrm{T} Z \oplus 0'$ and $Z >_\mathrm{T} 0$. It will suffice to find a real B such that $A \equiv_\mathrm{T} B' \equiv_\mathrm{T} B \oplus 0' \equiv_\mathrm{T} B \oplus Z$. We shall construct a sequence of strings $\sigma_0 \subset \sigma_1 \subset \cdots \subset \sigma_j \subset \sigma_{j+1} \subset \cdots$ in $\mathbb{N}^*$, and the desired real $B \in \mathbb{N}^\mathbb{N}$ will be obtained as the limit $\bigcup_j \sigma_j$ of this sequence.

Let us regard Z as a subset of $\mathbb{N}$. Since Z is not recursive, at least one of Z and its complement $\mathbb{N} \setminus Z$ is not Σ^0_1. And then, since Z is Turing equivalent to $\mathbb{N} \setminus Z$, we may safely assume that $\mathbb{N} \setminus Z$ is not Σ^0_1.

Stage $j = 0$. Let $\sigma_0 = \langle\rangle$.

Stage $j = 2n+1$. By induction we have σ_{2n}. Define

$$S_n = \{i \in \mathbb{N} \mid \exists \sigma\, (\sigma \supseteq \sigma_{2n}{}^\frown\langle \underbrace{0,\ldots,0}_{i}, 1\rangle \text{ and } \varphi^{(1),\sigma}_{n,|\sigma|}(0)\downarrow)\}$$

which is Σ^0_1. Since $\mathbb{N} \setminus Z$ is not Σ^0_1, there are infinitely many i such that $i \in Z \Leftrightarrow i \in S_n$. Choose the first such i, and use this i to define σ_{2n+1} as follows. If $i \in Z$, then $i \in S_n$ so let $\sigma_{2n+1} =$ the least $\sigma \supseteq \sigma_{2n}{}^\frown\langle \underbrace{0,\ldots,0}_{i}, 1\rangle$ such that $\varphi^{(1),\sigma}_{n,|\sigma|}(0)\downarrow$. If $i \notin Z$, let $\sigma_{2n+1} = \sigma_{2n}{}^\frown\langle \underbrace{0,\ldots,0}_{i}, 1\rangle$.

Stage $j = 2n+2$. Let $\sigma_{2n+2} = \sigma_{2n}{}^\frown\langle A(n)\rangle$.

This completes the definition of σ_j for all $j \in \mathbb{N}$. We now show that $B = \bigcup_j \sigma_j$ has the desired properties. We begin with the following observations.

[1] See for instance [11, §§8.3,9.6] and [13, §§4,5].

- If σ_{2n} is known, then the i in stage $2n+1$ can be found recursively in $Z \oplus 0'$ (because S_n is computable from $0'$), or recursively in B (because there is exactly one i such that $\sigma_{2n}{}^\frown\langle\underbrace{0,\ldots,0}_{i},1\rangle \subset B$). And then, using this i, σ_{2n+1} can be found recursively in Z (to test whether $i \in Z$), or recursively in $0'$ (to test whether $i \in S_n$).

- If σ_{2n+1} is known, then σ_{2n+2} can be found recursively in A (by the definition of σ_{2n+2}), or recursively in B (because $\sigma_{2n+2} = \sigma_{2n+1}{}^\frown\langle B(|\sigma_{2n+1}|)\rangle$), hence also recursively in $B \oplus Z$ or in $B \oplus 0'$.

Combining these observations, we see that the entire sequence $\langle \sigma_j \mid j \in \mathbb{N}\rangle$ is $\leq_{\mathrm{T}} B \oplus Z$, and $\leq_{\mathrm{T}} B \oplus 0'$, and $\leq_{\mathrm{T}} A$ (using the hypothesis $Z \oplus 0' \leq_{\mathrm{T}} A$). We also have:

- $B \oplus Z \leq_{\mathrm{T}} A$, because $B = \bigcup_j \sigma_j \leq_{\mathrm{T}} A$ and by hypothesis $Z \leq_{\mathrm{T}} A$.

- $B' \leq_{\mathrm{T}} \langle \sigma_j \mid j \in \mathbb{N}\rangle$, because $n \in B' \Leftrightarrow \varphi^{(1),\sigma_{2n+1}}_{n,|\sigma_{2n+1}|}(0)\downarrow$.

- $A \leq_{\mathrm{T}} \langle \sigma_j \mid j \in \mathbb{N}\rangle$, because $A(n) = \sigma_{2n+2}(|\sigma_{2n+1}|)$ for all n.

Thus $A \leq_{\mathrm{T}} \langle \sigma_j \mid j \in \mathbb{N}\rangle \leq_{\mathrm{T}} B \oplus Z \leq_{\mathrm{T}} A$, and $B' \leq_{\mathrm{T}} \langle \sigma_j \mid j \in \mathbb{N}\rangle \leq_{\mathrm{T}} B \oplus 0' \leq_{\mathrm{T}} B'$, hence $A \equiv_{\mathrm{T}} B' \equiv_{\mathrm{T}} B \oplus 0' \equiv_{\mathrm{T}} B \oplus Z$, Q.E.D. $\square$

We now turn to Pseudojump Inversion, Theorem 1.3.

Proof of Theorem 1.3. Fix $e \in \mathbb{N}$, and let A be a real such that $A \geq_{\mathrm{T}} 0'$. It will suffice to find a real B such that $A \equiv_{\mathrm{T}} J_e(B) \equiv_{\mathrm{T}} B \oplus 0'$. We shall construct a sequence of strings $\sigma_0 \subseteq \sigma_1 \subseteq \cdots \subseteq \sigma_j \subseteq \sigma_{j+1} \subseteq \cdots$ in $\mathbb{N}^*$, and the desired real $B \in \mathbb{N}^{\mathbb{N}}$ will be obtained as the limit $\bigcup_j \sigma_j$ of this sequence.

Stage 0. Let $\sigma_0 = \langle\rangle$.

Stage $2n+1$. Let σ_{2n+1} = the least string $\sigma \supseteq \sigma_{2n}$ for which $n \in W^{\sigma}_{e,|\sigma|}$ if such a σ exists. Otherwise let $\sigma_{2n+1} = \sigma_{2n}$.

Stage $2n+2$. Let $\sigma_{2n+2} = \sigma_{2n+1}{}^\frown\langle A(n)\rangle$.

We now show that $B = \bigcup_j \sigma_j$ has the desired properties. We begin with the following observations.

- If σ_{2n} is known, then in stage $2n+1$ the question of whether σ exists can be answered recursively in $0'$ (because the question is Σ^0_1), or recursively in $J_e(B)$ (because σ exists if and only if $n \in W^B_e$), hence also recursively in A (because by hypothesis $0' \leq_{\mathrm{T}} A$), or recursively in $B \oplus 0'$. And once the existence or nonexistence of σ is known, σ_{2n+1} can be found recursively.

- If σ_{2n+1} is known, then σ_{2n+2} can be found recursively in A (by the definition of σ_{2n+2}), or recursively in B (because $\sigma_{2n+2} = \sigma_{2n+1}{}^\frown\langle B(|\sigma_{2n+1}|)\rangle$), hence also recursively in $J_e(B)$ (because $B \leq_{\mathrm{T}} J_e(B)$), or in $B \oplus 0'$.

Combining these observations, we see that the entire sequence $\langle \sigma_j \mid j \in \mathbb{N}\rangle$ is $\leq_{\mathrm{T}} J_e(B)$, and $\leq_{\mathrm{T}} B \oplus 0'$, and $\leq_{\mathrm{T}} A$. We also have:

- $B \oplus 0' \leq_{\mathrm{T}} A$, because $B = \bigcup_j \sigma_j \leq_{\mathrm{T}} A$ and by hypothesis $0' \leq_{\mathrm{T}} A$.
- $J_e(B) \leq_{\mathrm{T}} \langle \sigma_j \mid j \in \mathbb{N} \rangle$, because $n \in W_e^B$ if and only if $n \in W_{e,|\sigma_{2n+1}|}^{\sigma_{2n+1}}$.
- $A \leq_{\mathrm{T}} \langle \sigma_j \mid j \in \mathbb{N} \rangle$, because $A(n) = \sigma_{2n+2}(|\sigma_{2n+1}|)$ for all n.

Thus $A \leq_{\mathrm{T}} \langle \sigma_j \mid j \in \mathbb{N} \rangle \leq_{\mathrm{T}} B \oplus 0' \leq_{\mathrm{T}} A$ and $J_e(B) \leq_{\mathrm{T}} \langle \sigma_j \mid j \in \mathbb{N} \rangle \leq_{\mathrm{T}} J_e(B)$, hence $A \equiv_{\mathrm{T}} J_e(B) \equiv_{\mathrm{T}} B \oplus 0'$, Q.E.D. □

3 Completions of Peano Arithmetic

Let PA denote Peano Arithmetic, and let $\mathrm{Sent}_{\mathsf{PA}}$ denote the set of sentences of the language of PA. An *extension of* PA is a set $T \subseteq \mathrm{Sent}_{\mathsf{PA}}$ which includes the axioms of PA and is closed under logical consequence. For any such T, a *completion of* T is an extension X of PA which includes T and such that for each $\varphi \in \mathrm{Sent}_{\mathsf{PA}}$ exactly one of the sentences φ and $\neg\varphi$ belongs to X. Given an extension T of PA, let Comp_T denote the set of all completions of T. By Lindenbaum's Lemma, $\mathrm{Comp}_T \neq \emptyset$ if and only if T is consistent.

Fix a primitive recursive Gödel numbering $\# : \varphi \mapsto \#(\varphi) : \mathrm{Sent}_{\mathsf{PA}} \to \mathbb{N}$. For convenience we shall assume that $\#$ is a one-to-one correspondence between $\mathrm{Sent}_{\mathsf{PA}}$ and $\mathbb{N}$. This induces a one-to-one correspondence between subsets X of $\mathrm{Sent}_{\mathsf{PA}}$ and subsets of $\mathbb{N}$, namely $X \mapsto \{\#(\varphi) \mid \varphi \in X\}$. And of course subsets of $\mathbb{N}$ are identified with their characteristic functions in $\{0,1\}^{\mathbb{N}}$. Therefore, letting CPA be the subset of $\{0,1\}^{\mathbb{N}}$ corresponding to $\mathrm{Comp}_{\mathsf{PA}}$, we see that CPA is a special Π^0_1 subclass of $\{0,1\}^{\mathbb{N}}$.

The purpose of this section is to prove that CPA has the Join Property and the Pseudojump Inversion Property. The Join Property will follow easily from Theorem 1.2 plus other known results, but for the Pseudojump Inversion Property we shall use a construction involving the Gödel-Rosser Incompleteness Theorem [8].

Definition 3.1 (PA-degrees)**.** By a PA-*degree* we mean the Turing degree $\deg_{\mathrm{T}}(X)$ of some $X \in \mathrm{CPA}$. A set S of Turing degrees is said to be *upwardly closed* if for all Turing degrees $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a} \leq \mathbf{b}$, $\mathbf{a} \in S$ implies $\mathbf{b} \in S$.

Lemma 3.2. The set of all PA-degrees is upwardly closed.

Proof. This result is originally due to Robert M. Solovay. For a proof see [2, Theorem 2.21.3] or [13, Corollary 6.6]. □

Lemma 3.3. Let $P \subseteq \{0,1\}^{\mathbb{N}}$ be a nonempty Π^0_1 class such that $\{\deg_{\mathrm{T}}(X) \mid X \in P\}$ is upwardly closed. Then P has the Join Property.

Proof. Let A and Z be reals such that $A \geq_{\mathrm{T}} Z \oplus 0'$ and $Z >_{\mathrm{T}} 0$. By Theorem 1.2 there exists a real C such that $A \equiv_{\mathrm{T}} C' \equiv_{\mathrm{T}} C \oplus 0' \equiv_{\mathrm{T}} C \oplus Z$. By the Low Basis Theorem [5, Theorem 2.1] relativized to C, there exists $B_0 \in P$ such that $C' \equiv_{\mathrm{T}} (B_0 \oplus C)' \equiv_{\mathrm{T}} B_0 \oplus C \oplus 0'$. Since $B_0 \leq_{\mathrm{T}} B_0 \oplus C$, we can find $B \in P$ such that $B \equiv_{\mathrm{T}} B_0 \oplus C$, hence $C' \equiv_{\mathrm{T}} B' \equiv_{\mathrm{T}} B \oplus 0'$. We now have $C' \equiv_{\mathrm{T}}$

$C \oplus Z \leq_{\mathrm{T}} B \oplus Z \leq_{\mathrm{T}} C' \oplus Z \equiv_{\mathrm{T}} C'$, hence $A \equiv_{\mathrm{T}} C' \equiv_{\mathrm{T}} B' \equiv_{\mathrm{T}} B \oplus 0' \equiv_{\mathrm{T}} B \oplus Z$, Q.E.D. $\square$

Theorem 3.4. CPA has the Join Property.

Proof. This is immediate from Lemmas 3.2 and 3.3. $\square$

Next we shall prove that CPA has the Pseudojump Inversion Property. The proof will be presented in terms of completions of recursively axiomatizable theories. An extension T of PA is said to be *recursively axiomatizable* if there is a recursive set of sentences $S \subseteq \mathrm{Sent}_{\mathsf{PA}}$ such that T is the closure of S under logical consequence. In this case Comp_T is clearly a Π^0_1 subclass of CPA. The converse also holds:

Lemma 3.5. We have an effective one-to-one correspondence $T \mapsto \mathrm{Comp}_T$ between recursively axiomatizable extensions of PA and Π^0_1 subclasses of CPA.

Proof. Given a Π^0_1 class $P \subseteq \mathrm{CPA}$, we shall exhibit a recursively axiomatizable extension T of PA such that $\mathrm{Comp}_T = P$. Namely, let $T = \{\varphi \in \mathrm{Sent}_{\mathsf{PA}} \mid X(\#(\varphi)) = 1 \text{ for all } X \in P\}$. To see that T is closed under logical consequence, it suffices to note that each $X \in P$ belongs to CPA and is therefore closed under logical consequence. We also have $\mathsf{PA} \subseteq T$, because $P \subseteq \mathrm{CPA}$. Moreover $\mathrm{Comp}_T = P$ by Lindenbaum's Lemma. From the effective compactness of P we see that T is recursively enumerable. Hence T is recursively axiomatizable, and from an index of P as a Π^0_1 class we can compute an index for a recursive axiomatization of T. This completes the proof. $\square$

Remark 3.6. The idea of Lemma 3.5 applies much more generally. Given any language L and any L-theory T, an *extension of* T is any set $\widetilde{T} \subseteq \mathrm{Sent}_L$ which includes T and is closed under logical consequence. Regarding Comp_T as a closed set in the product space $\{0,1\}^{\mathrm{Sent}_L}$, we have a one-to-one correspondence $\widetilde{T} \mapsto \mathrm{Comp}_{\widetilde{T}}$ between extensions of T and closed subsets of Comp_T.

Recall from §1 that $\langle P_i \mid i \in \mathbb{N}\rangle$ is a fixed standard recursive enumeration of the Π^0_1 subclasses of $\{0,1\}^{\mathbb{N}}$.

Lemma 3.7 (splitting property)**.** There is a primitive recursive function $s : \mathbb{N} \times \{0,1\} \to \mathbb{N}$ such that for all $i \in \mathbb{N}$, if P_i is a nonempty Π^0_1 subclass of CPA then $P_{s(i,0)}$ and $P_{s(i,1)}$ are nonempty disjoint Π^0_1 subclasses of P_i.

Proof. Lemma 3.5 tell us that, given $i \in \mathbb{N}$, we can effectively find a recursively axiomatizable extension T of PA such that Comp_T corresponds to $P_i \cap \mathrm{CPA}$. We then apply the Gödel-Rosser Incompleteness Theorem [8] to effectively find a sentence $\psi \in \mathrm{Sent}_{\mathsf{PA}}$ such that if T is consistent, then neither ψ nor $\neg\psi$ belongs to T. Let T_1 (respectively T_0) be the closure of $T \cup \{\psi\}$ (respectively $T \cup \{\neg\psi\}$) under logical consequence. Clearly Comp_{T_1} and Comp_{T_0} are disjoint, and they are nonempty if Comp_T is nonempty. Apply Lemma 3.5 to effectively find $s(i,1), s(i,0) \in \mathbb{N}$ such that $P_{s(i,1)}$ and $P_{s(i,0)}$ correspond to Comp_{T_1} and Comp_{T_0} respectively. This completes the proof. $\square$

We shall now use this splitting property to redo Theorem 1.3 within CPA.

Theorem 3.8. CPA has the Pseudojump Inversion Property.

Proof. Fix $e \in \mathbb{N}$, and let $A \in \{0,1\}^{\mathbb{N}}$ be a real such that $A \geq_{\mathrm{T}} 0'$. It will suffice to find a real $B \in \mathrm{CPA}$ such that $A \equiv_{\mathrm{T}} J_e(B) \equiv_{\mathrm{T}} B \oplus 0'$. To find B, we shall construct a sequence of Π^0_1 classes $Q_0 \supseteq Q_1 \supseteq \cdots \supseteq Q_j \supseteq Q_{j+1} \supseteq \cdots$, by induction on j starting with $Q_0 = \mathrm{CPA} \subseteq \{0,1\}^{\mathbb{N}}$. The intersection $\bigcap_j Q_j$ will be nonempty, and we shall have $B \in \bigcap_j Q_j$. At the same time, we shall construct a function $f : \mathbb{N} \to \mathbb{N}$ such that $Q_j = P_{f(j)}$ for all j.

Fix a primitive recursive splitting function s as in Lemma 3.7.

Stage 0. Fix $f(0) \in \mathbb{N}$ such that $P_{f(0)} = Q_0 = \mathrm{CPA}$.

Stage $2n+1$. By induction we have $Q_{2n} = P_{f(2n)}$. If $P_{f(2n)} \cap \{X \mid n \notin J_e(X)\} = \emptyset$ let $Q_{2n+1} = P_{f(2n)}$ and $f(2n+1) = f(2n)$. Otherwise, let $Q_{2n+1} = P_{f(2n)} \cap \{X \mid n \notin J_e(X)\}$ and choose $f(2n+1)$ so that $P_{f(2n+1)} = P_{f(2n)} \cap \{X \mid n \notin J_e(X)\}$. This $f(2n+1)$ is found primitive recursively from $f(2n)$ and n and e.

Stage $2n+2$. By induction we have $Q_{2n+1} = P_{f(2n+1)}$. Let $f(2n+2) = s(f(2n+1), A(n))$ and $Q_{2n+2} = P_{s(f(2n+1),A(n))}$, where s is our splitting function. This $f(2n+2)$ is found primitive recursively from $f(2n+1)$ and $A(n)$.

This completes the construction.

By construction each Q_j is nonempty, so by compactness of $\{0,1\}^{\mathbb{N}}$ their intersection $\bigcap_j Q_j$ is nonempty. It remains to show that $B \in \bigcap_j Q_j$ has the desired properties. We begin with the following observations.

- If $f(2n)$ is known then $f(2n+1)$ can be found recursively in $0'$ (by checking whether the Π^0_1 class $P_{f(2n)} \cap \{X \mid n \notin J_e(X)\}$ is empty or not), or recursively in $J_e(B)$ (by checking whether $n \in J_e(B)$ or not).
- If $f(2n+1)$ is known then $f(2n+2)$ can be found recursively in B (by finding the $k \in \{0,1\}$ such that $B \notin P_{s(f(2n+1),k)}$), or recursively in A (by evaluating $s(f(2n+1), A(n))$).

Combining these observations, we see that f is $\leq_{\mathrm{T}} A \oplus 0' \equiv_{\mathrm{T}} A$, and $\leq_{\mathrm{T}} J_e(B)$, and $\leq_{\mathrm{T}} B \oplus 0'$. Conversely, we also have:

- $A \leq_{\mathrm{T}} f$, because $A(n) = i$ if and only if $f(2n+2) = s(f(2n+1), i)$.
- $J_e(B) \leq_{\mathrm{T}} f$, because $n \in J_e(B)$ if and only if $f(2n+1) = f(2n)$.
- $B \oplus 0' \leq_{\mathrm{T}} f$, because $B \leq_{\mathrm{T}} J_e(B) \leq_{\mathrm{T}} f$ and $0' \leq_{\mathrm{T}} A \leq_{\mathrm{T}} f$.

Thus $f \equiv_{\mathrm{T}} A \equiv_{\mathrm{T}} J_e(B) \equiv_{\mathrm{T}} B \oplus 0'$ and the proof is complete. □

Remark 3.9. We have used the Splitting Lemma 3.7 to prove that CPA has the Pseudojump Inversion Property. Similarly, it would be possible to use the Splittng Lemma to prove directly that CPA has the Join Property. This direct proof would be in contrast to our shorter but indirect proof in Theorem 3.4 above. Note however that a version of the Splitting Lemma is used in our proof of Solovay's Lemma 3.2; see [13, §6] and [14, §3].

4 Turing degree isomorphism

Definition 4.1 (Turing degree isomorphism)**.** Following [4] we say that $P, Q \subseteq \mathbb{N}^\mathbb{N}$ are *Turing degree isomorphic* if $\{\deg_\mathrm{T}(X) \mid X \in P\} = \{\deg_\mathrm{T}(X) \mid X \in Q\}$.

The purpose of this section is to prove that the Join Property and the Pseudojump Inversion Property hold for all Π^0_1 subclasses of $\{0,1\}^\mathbb{N}$ which are Turing degree isomorphic to CPA. We also obtain some more general-looking results.

We begin with the Join Property.

Theorem 4.2. Let $P \subseteq \{0,1\}^\mathbb{N}$ be a Π^0_1 class. If P is Turing degree isomorphic to CPA, then P has the Join Property.

Proof. Let A and Z be reals such that $A \geq_\mathrm{T} Z \oplus 0'$ and $Z >_\mathrm{T} 0$. By Theorem 3.4 let $B \in \mathrm{CPA}$ be such that $A \equiv_\mathrm{T} B' \equiv_\mathrm{T} B \oplus 0' \equiv_\mathrm{T} B \oplus Z$. Since P is Turing degree isomorphic to CPA, let $C \in P$ be such that $B \equiv_\mathrm{T} C$. Then $A' \equiv_\mathrm{T} C' \equiv_\mathrm{T} C \oplus 0' \equiv_\mathrm{T} C \oplus Z$. Thus P has the Join Property, Q.E.D. $\square$

Remark 4.3. More generally, for $P, Q \subseteq \mathbb{N}^\mathbb{N}$ we say that P is *Turing degree embeddable* into Q if $\{\deg_\mathrm{T}(X) \mid X \in P\} \subseteq \{\deg_\mathrm{T}(X) \mid X \in Q\}$. The proof of Theorem 4.2 shows that if P has the Join Property and is Turing degree embeddable into Q, then Q has the Join Property.

We now turn to Pseudojump Inversion. Unfortunately, we cannot simply imitate the proof of Theorem 4.2. This is because pseudojump operators are not invariant under Turing equivalence, i.e., $X \equiv_\mathrm{T} Y$ typically does not imply $J_e(X) \equiv_\mathrm{T} J_e(Y)$. Consequently, we do not know whether the Pseudojump Inversion Property is invariant under Turing degree isomorphism of special Π^0_1 subclasses of $\{0,1\}^\mathbb{N}$. However, we shall settle some interesting special cases of this question. As a first step, consider the following notion, which is a special case of Turing degree isomorphism.

Definition 4.4 (recursive homeomorphism)**.** For $P, Q \subseteq \mathbb{N}^\mathbb{N}$, a *recursive homeomorphism* of P onto Q is a one-to-one onto mapping $\Phi : P \to Q$ such that both Φ and its inverse $\Phi^{-1} : Q \to P$ are the restrictions of partial recursive functionals to P and Q respectively. We say that P and Q are *recursively homeomorphic* if there exists a recursive homeomorphism of P onto Q.

Lemma 4.5. Let $P, Q \subseteq \mathbb{N}^\mathbb{N}$ be recursively homeomorphic. If P has the Pseudojump Inversion Property, then so does Q.

Proof. Assume that P has the Pseudojump Inversion Property. We shall prove that Q has the Pseudojump Inversion Property. Given $e \in \mathbb{N}$ and $A \geq_\mathrm{T} 0'$, it will suffice to find $C \in Q$ such that $A \equiv_\mathrm{T} J_e(C) \equiv_\mathrm{T} C \oplus 0'$.

Let $\Phi : P \to Q$ be a recursive homeomorphism. Let $i \in \mathbb{N}$ be such that for all $X \in P$ and all $n \in \mathbb{N}$ we have $\varphi^X_i(n) \simeq \varphi^{\Phi(X)}_e(n)$. Then for all $X \in P$ we have $W^X_i = \{n \in \mathbb{N} \mid \varphi^X_i(n)\downarrow\} = \{n \in \mathbb{N} \mid \varphi^{\Phi(X)}_e(n)\downarrow\} = W^{\Phi(X)}_e$. Since P has the Pseudojump Inversion Property, let $B \in P$ such that $A \equiv_\mathrm{T} J_i(B) \equiv_\mathrm{T} B \oplus 0'$. Let $C = \Phi(B)$. Because Φ is a recursive homeomorphism, we have $B \equiv_\mathrm{T} \Phi(B) = C$, hence $J_e(C) = C \oplus W_e^C = C \oplus W_e^{\Phi(B)} = C \oplus W_i^B \equiv_{\mathrm{T}} B \oplus W_i^B = J_i(B)$, hence $A \equiv_{\mathrm{T}} J_e(C) \equiv_{\mathrm{T}} C \oplus 0'$, Q.E.D. $\square$

As a bridge from Turing degree isomorphism to recursive homeomorphism, we have the following lemma.

Lemma 4.6. Let $P, Q \subseteq \{0,1\}^{\mathbb{N}}$ be nonempty Π^0_1 classes. If P is Turing degree embeddable into Q, then there exist nonempty Π^0_1 subclasses $\widetilde{P} \subseteq P$ and $\widetilde{Q} \subseteq Q$ such that $\widetilde{P}$ and $\widetilde{Q}$ are recursively homeomorphic.

Proof. We shall draw on some facts about *hyperimmune-freeness* and *truth-table reducibility*. For this background, see [11, §§8.3,9.6] and [13, §4].

By the Hyperimmune-Free Basis Theorem [5, Theorem 2.4] (see also [13, Theorem 4.19]), let $X_0 \in P$ be hyperimmune-free. Since P is Turing degree embeddable into Q, let $Y_0 \in Q$ be such that $X_0 \equiv_{\mathrm{T}} Y_0$. Then Y_0 is also hyperimmune-free, and by hyperimmune-freeness there exist truth-table functionals $\Phi, \Psi : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ such that $\Phi(X_0) = Y_0$ and $\Psi(Y_0) = X_0$. Let $\widetilde{P} = \{X \in P \mid \Phi(X) \in Q \text{ and } \Psi(\Phi(X)) = X\}$. Then $\widetilde{P}$ is a Π^0_1 subclass of P, and it is nonempty because it contains X_0. Moreover $\widetilde{Q} = \{\Phi(X) \mid X \in \widetilde{P}\}$ is also a Π^0_1 subclasss of Q, and we have a recursive homeomorphism $\Phi{\upharpoonright}\widetilde{P}$ of $\widetilde{P}$ onto $\widetilde{Q}$. $\square$

Lemma 4.7. Any nonempty Π^0_1 subclass of CPA is recursively homeomorphic to CPA.

Proof. See [14, §3]. $\square$

We can now prove the following theorem, which is parallel to Theorem 4.2.

Theorem 4.8. Let $P \subseteq \{0,1\}^{\mathbb{N}}$ be a Π^0_1 class. If P is Turing degree isomorphic to CPA, then P has the Pseudojump Inversion Property.

Proof. Let P be Turing degree isomorphic to CPA. By Lemma 4.6 there are nonempty recursively homeomorphic Π^0_1 classes $\widetilde{P} \subseteq P$ and $\widetilde{\mathrm{CPA}} \subseteq \mathrm{CPA}$. By Lemma 4.7 $\widetilde{\mathrm{CPA}}$ is recursively homeomorphic to CPA. By Theorem 3.8 CPA has the Pseudojump Inversion Property, so by Lemma 4.5 $\widetilde{\mathrm{CPA}}$ and hence $\widetilde{P}$ have the Pseudojump Inversion Property. But then, since $\widetilde{P} \subseteq P$, it follows that P has the Pseudojump Inversion Property, Q.E.D. $\square$

Next we prove a more general-looking result. Recall from [13, 16, 17, 18, 19] that a nonempty Π^0_1 class $P \subseteq \{0,1\}^{\mathbb{N}}$ is said to be *Muchnik complete* if every nonempty Π^0_1 class $Q \subseteq \{0,1\}^{\mathbb{N}}$ is *Muchnik reducible* to P, i.e., for all $X \in P$ there exists $Y \in Q$ such that $Y \leq_{\mathrm{T}} X$.

Lemma 4.9. A Π^0_1 class $P \subseteq \{0,1\}^{\mathbb{N}}$ is Muchnik complete if and only if it is Turing degree isomorphic to CPA. Moreover, for such a P the set of Turing degrees $\{\deg_T(X) \mid X \in P\}$ is upwardly closed.

Proof. See [13, §§3,6]. $\square$

Theorem 4.10. Let $P \subseteq \{0,1\}^{\mathbb{N}}$ be a nonempty Π^0_1 class. If P is Muchnik complete, then P has the Join Property and the Pseudojump Inversion Property.

Proof. This is immediate from Lemma 4.9 and Theorems 4.2 and 4.8. $\square$

An even more general-looking result reads as follows.

Theorem 4.11. Let $P \subseteq \{0,1\}^{\mathbb{N}}$ be a nonempty Π^0_1 class such that $\{\deg_T(Y) \mid Y \in P\}$ is upwardly closed. Then any $P_1 \subseteq \mathbb{N}^{\mathbb{N}}$ which includes P has the Join Property and the Pseudojump Inversion Property.

Proof. By Lemma 4.9 CPA is Turing degree embeddable into P. Our result then follows by Remark 4.3 and Theorems 4.2 and 4.8. $\square$

5 Π^0_1 classes of positive measure

For $\sigma \in \{0,1\}^*$ we write $[\![\sigma]\!] = \{X \in \{0,1\}^{\mathbb{N}} \mid \sigma \subset X\}$. Let μ be the *fair coin measure* on $\{0,1\}^{\mathbb{N}}$, defined by letting $\mu([\![\sigma]\!]) = 2^{-|\sigma|}$ for all $\sigma \in \{0,1\}^*$. This μ is a Borel probability measure on $\{0,1\}^{\mathbb{N}}$. A Π^0_1 class $P \subseteq \{0,1\}^{\mathbb{N}}$ is said to be *of positive measure* if $\mu(P) > 0$. In this section we note that such a P must have the Pseudojump Inversion Property but need not have the Join Property. We also obtain the same results for Π^0_1 subclasses of $\{0,1\}^{\mathbb{N}}$ which are Turing degree isomorphic to such a P.

To prove these results we need some basic facts about *Martin-Löf randomness* and LR-*reducibility*. We cite our semi-expository papers [13, 15, 17, 18] but one can also consult the treatises of Downey and Hirschfeldt [2] and Nies [9]. Let MLR $= \{X \in \{0,1\}^{\mathbb{N}} \mid X$ is Martin-Löf random$\}$.

Lemma 5.1. A Π^0_1 subclass of $\{0,1\}^{\mathbb{N}}$ is of positive measure if and only if it includes a nonempty Π^0_1 subclass of MLR.

Proof. This is because MLR $\subseteq \{0,1\}^{\mathbb{N}}$ is a Σ^0_2 class of full measure. See for instance [13, §8]. $\square$

Lemma 5.2 (due to Nies [9])**.** Let $P \subseteq \{0,1\}^{\mathbb{N}}$ be a Π^0_1 class of positive measure. Then P has the Pseudojump Inversion Property.

Proof. See [9, Theorem 6.3.9] or a simpler proof in [15, Theorem 5.1]. $\square$

Theorem 5.3. Let $P \subseteq \{0,1\}^{\mathbb{N}}$ be a Π^0_1 class of positive measure, and let $Q \subseteq \{0,1\}^{\mathbb{N}}$ be a Π^0_1 class which is Turing degree isomorphic to P. Then Q has the Pseudojump Inversion Property.

Proof. By Lemma 5.1 let P_0 be a nonempty Π^0_1 subclass of $P \cap$ MLR. Then P_0 is Turing degree embeddable in Q, so by Lemma 4.6 we can find nonempty Π^0_1 classes $\widetilde{P} \subseteq P_0$ and $\widetilde{Q} \subseteq Q$ which are recursively homeomorphic. By Lemma 5.1 $\widetilde{P}$ is of positive measure, so by Lemma 5.2 $\widetilde{P}$ has the Pseudojump Inversion Property. It then follows by Lemma 4.5 that $\widetilde{Q}$ and hence Q have the Pseudojump Inversion Property. $\square$

Relativizing the notion of Martin-Löf randomness, for any real Y we write $\mathrm{MLR}^Y = \{X \in \{0,1\}^{\mathbb{N}} \mid X$ is Martin-Löf random relative to $Y\}$. If $\mathrm{MLR}^Y \subseteq \mathrm{MLR}^Z$ we say that Z *is* LR-*reducible to* Y, abbreviated as $Z \leq_{\mathrm{LR}} Y$. Intuitively this means that Y has at least as much "derandomizing power" as Z. Clearly $Z \leq_{\mathrm{T}} Y$ implies $Z \leq_{\mathrm{LR}} Y$, but the converse does not hold:

Lemma 5.4. There exists a nonrecursive real Z such that $Z \leq_{\mathrm{LR}} 0$.

Proof. See [15, Theorem 6.1] or [2, 9]. $\square$

On the other hand, we have:

Lemma 5.5. If $Z \leq_{\mathrm{LR}} Y$ then $Z' \leq_{\mathrm{T}} Z \oplus Y'$.

Proof. See [15, Theorem 8.8] or [2, 9]. $\square$

Lemma 5.6 (Lutz [6])**.** If $Z \leq_{\mathrm{LR}} 0$ then $X \oplus Z \leq_{\mathrm{LR}} X$ for all $X \in \mathrm{MLR}$.

Proof. Assume $Z \leq_{\mathrm{LR}} 0$ and $X \in \mathrm{MLR}$. We must prove that $\mathrm{MLR}^X \subseteq \mathrm{MLR}^{X \oplus Z}$. Given $X_1 \in \mathrm{MLR}^X$, it follows by Van Lambalgen's Theorem ([15, Theorem 3.6], see also [2, Corollary 6.9.3]) that $X_1 \oplus X \in \mathrm{MLR}$. But then $X_1 \oplus X \in \mathrm{MLR}^Z$, so by Van Lambalgen's Theorem relative to Z we have $X_1 \in \mathrm{MLR}^{X \oplus Z}$, Q.E.D. $\square$

Theorem 5.7 (Lutz [6])**.** MLR does not have the Join Property.

Proof. By Lemma 5.4 let $Z >_{\mathrm{T}} 0$ be such that $Z \leq_{\mathrm{LR}} 0$. For any $X \in \mathrm{MLR}$ we have $X \oplus Z \leq_{\mathrm{LR}} X$ by Lemma 5.6, hence $(X \oplus Z)' \leq_{\mathrm{T}} X' \oplus Z$ by Lemma 5.5. If MLR had the Join Property, there would be an $X \in \mathrm{MLR}$ such that $X' \equiv_{\mathrm{T}} X \oplus Z$, hence $(X \oplus Z)' \leq_{\mathrm{T}} X \oplus Z$, a contradiction. $\square$

Theorem 5.8 (Lutz [6])**.** There is a Π^0_1 class $P \subseteq \{0,1\}^{\mathbb{N}}$ of positive measure which does not have the Join Property.

Proof. This is immediate from Theorem 5.7 and Lemma 5.1. $\square$

6 Π^0_1 classes constructed by priority arguments

In this section we construct a special Π^0_1 class $Q \subseteq \{0,1\}^{\mathbb{N}}$ which has neither the Join Property nor the Pseudojump Inversion Property. In addition, this Π^0_1 class Q has some other interesting features, which we also discuss.

We begin by stating a technical theorem which embodies our construction of Q. Recall from §1 that $\langle P_e \rangle_{e \in \mathbb{N}}$ is a fixed enumeration of all Π^0_1 classes.

Theorem 6.1. There is a nonempty perfect[2] Π^0_1 class $Q \subseteq \{0,1\}^{\mathbb{N}}$ with the following properties. For all $e, n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that for all pairwise distinct $\tau_1, \ldots, \tau_n \in \{0,1\}^*$ with $|\tau_1| = \cdots = |\tau_n| \geq m$, the Π^0_1 class

[2] A topological space is said to be *perfect* if it has no isolated points, i.e., there is no open set consisting of exactly one point. In particular, a set $P \subseteq \{0,1\}^{\mathbb{N}}$ is perfect if and only if there is no $\tau \in \{0,1\}^*$ such that $P \cap [\![\tau]\!] = \{X\}$ for some $X \in \{0,1\}^{\mathbb{N}}$.

$$(Q \cap [\![\tau_1]\!]) \times \cdots \times (Q \cap [\![\tau_n]\!])$$

is either disjoint from P_e or included in P_e. Moreover, this m can be computed from e using an oracle for $0'$.

Before proving Theorem 6.1, we spell out some of its consequences which are of more general interest. The next theorem summarizes these features of Q.

Theorem 6.2. Let $Q \subseteq \{0,1\}^{\mathbb{N}}$ be a nonempty Π^0_1 class as in Theorem 6.1.

1. Q is *thin*, i.e., for every Π^0_1 subclass P of Q, the complement $Q \setminus P$ is again a Π^0_1 subclass of Q. More generally, for any finite sequence $Q_1, \ldots, Q_n$ of pairwise disjoint[3] Π^0_1 subclasses of Q, the Π^0_1 class $Q_1 \times \cdots \times Q_n$ is thin.

2. Q is *special*, i.e., no $X \in Q$ is recursive. More generally, the Turing degrees of members of Q are *independent*, i.e., no $X \in Q$ is $\leq_{\mathrm{T}}$ the join of any finitely many other members of Q.

3. Every $X \in Q$ is of minimal truth-table degree. Consequently, every hyperimmune-free $X \in Q$ is of minimal Turing degree.

4. Every finite sequence $X_1, \ldots, X_n \in Q$ is *generalized low*, i.e., $(X_1 \oplus \cdots \oplus X_n)' \equiv_{\mathrm{T}} X_1 \oplus \cdots \oplus X_n \oplus 0'$.

Proof. We prove parts 1 through 4 in that order.

1. To see that Q is thin, let P be a Π^0_1 subclass of Q. Fix $e \in \mathbb{N}$ such that $P = P_e$, and let m be as in Theorem 6.1. Then for each $\tau \in \{0,1\}^m$ the Π^0_1 class $Q \cap [\![\tau]\!]$ is either disjoint from P or included in P. Hence $Q \setminus P = \bigcup_\tau (Q \cap [\![\tau]\!])$ where the union is taken over all $\tau \in \{0,1\}^m$ such that $Q \cap [\![\tau]\!]$ is disjoint from P. Thus $Q \setminus P$ is is a union of finitely many Π^0_1 classes, so it too is a Π^0_1 class.

For the generalization, let P be a Π^0_1 subclass of $Q_1 \times \cdots \times Q_n$ where $Q_1, \ldots, Q_n$ are pairwise disjoint Π^0_1 subclasses of Q. Fix $e \in \mathbb{N}$ such that $P = P_e$, and let m be as in Theorem 6.1. We then have

$$(Q_1 \times \cdots \times Q_n) \setminus P = \textstyle\bigcup_{\tau_1,\ldots,\tau_n} ((Q_1 \cap [\![\tau_1]\!]) \times \cdots \times (Q_n \cap [\![\tau_n]\!]))$$

where the union is taken over all pairwise distinct sequences $\tau_1, \ldots, \tau_n \in \{0,1\}^m$ such that $(Q_1 \cap [\![\tau_1]\!]) \times \cdots \times (Q_n \cap [\![\tau_n]\!])$ is disjoint from P. Thus $(Q_1 \times \cdots \times Q_n) \setminus P$ is a union of finitely many Π^0_1 classes, so it too is a Π^0_1 class, Q.E.D.

2. Assume for a contradiction that $X \in Q$ is recursive. Fix $e \in \mathbb{N}$ such that $P_e = \{X\}$, let $m \in \mathbb{N}$ be as in Theorem 6.1, and let $\tau = X{\upharpoonright}m$. Clearly $X \in Q \cap [\![\tau]\!]$, hence $Q \cap [\![\tau]\!]$ is not disjoint from P_e, hence $Q \cap [\![\tau]\!]$ is included in P_e, hence X is the unique member of $Q \cap [\![\tau]\!]$. Thus X is an isolated point of Q, contradicting the fact that Q is perfect.

For the generalization, it will suffice to show that $X \not\leq_{\mathrm{T}} X_1 \oplus \cdots \oplus X_n$ for all pairwise distinct $X, X_1, \ldots, X_n \in Q$. Let $\Psi : \subseteq \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ be a partial recursive functional, and assume for a contradiction that $X = \Psi(X_1 \oplus \cdots \oplus X_n)$. Let $e \in \mathbb{N}$ be such that

[3] The hypothesis of pairwise disjointness is essential. For instance, if $P = \{X \oplus X \mid X \in Q\}$ then $(Q \times Q) \setminus P$ is not Π^0_1, so $Q \times Q$ is not thin.

$$P_e = \{Y \oplus Y_1 \oplus \cdots \oplus Y_n \mid \forall i\, \forall j\, (\text{if } \Psi(Y_1 \oplus \cdots \oplus Y_n)(i){\downarrow} = j \text{ then } Y(i) = j)\}.$$

Let m be as in Theorem 6.1 and sufficiently large so that $\tau = X{\upharpoonright}m$, $\tau_1 = X_1{\upharpoonright}m$, $\ldots$, $\tau_n = X_n{\upharpoonright}m$ are pairwise distinct. The Π^0_1 class

$$(Q \cap [\![\tau]\!]) \times (Q \cap [\![\tau_1]\!]) \times \cdots \times (Q \cap [\![\tau_n]\!])$$

contains $X \oplus X_1 \oplus \cdots \oplus X_n$ and is therefore not disjoint from P_e, so it is included in P_e. In particular, for all $Y \in Q \cap [\![\tau]\!]$ and all $i \in \mathbb{N}$ we have $\Psi(X_1 \oplus \cdots \oplus X_n)(i){\downarrow} = X(i) = Y(i)$, hence $X = Y$. Thus $Q \cap [\![\tau]\!] = \{X\}$, so again X is an isolated point of Q, contradicting the fact that Q is perfect.

3. Let $X \in Q$ be given. We have already seen that X is not recursive. To prove that X is of miminal truth-table degree, it remains to show that $X \leq_{\mathrm{tt}} \Psi(X)$ for any truth-table functional Ψ such that $\Psi(X)$ is not recursive. Given such a functional Ψ, let e be such that

$$P_e = \{X_0 \oplus X_1 \in Q \times Q \mid \Psi(X_0) = \Psi(X_1)\}.$$

By Theorem 6.1 let m be such that for all $\tau \in \{0,1\}^{\geq m}$ the Π^0_1 class $(Q \cap [\![\tau^\frown\langle 0\rangle]\!]) \times (Q \cap [\![\tau^\frown\langle 1\rangle]\!])$ is either disjoint from P_e or included in P_e. If it is disjoint from P_e, let us say that τ is *splitting*.

Case 1: For all sufficiently large $\tau \subset X$, τ is splitting. Then for all sufficiently large $\tau \subset X$ and all $X_0 \in Q \cap [\![\tau^\frown\langle 0\rangle]\!]$ and $X_1 \in Q \cap [\![\tau^\frown\langle 1\rangle]\!]$, we have $\Psi(X_0) \neq \Psi(X_1)$. In particular we have $\Psi(X) \neq \Psi(Y)$ for all $Y \in Q \cap [\![\tau]\!]$ such that $X \neq Y$. From this it follows that X is truth-table reducible to $\Psi(X)$.

Case 2: There are arbitrarily large $\tau \subset X$ such that τ is not splitting. Let $\tau \subset X$ be non-splitting with $|\tau| \geq m$. Then $Q \cap [\![\tau^\frown\langle 0\rangle]\!]$ and $Q \cap [\![\tau^\frown\langle 1\rangle]\!]$ are nonempty, and for all $X_0 \in Q \cap [\![\tau^\frown\langle 0\rangle]\!]$ and $X_1 \in Q \cap [\![\tau^\frown\langle 1\rangle]\!]$ we have $\Psi(X_0) = \Psi(X_1)$. In particular, letting $i, j \in \{0,1\}$ be such that $X \in Q \cap [\![\tau^\frown\langle i\rangle]\!]$ and $i + j = 1$, we have $\Psi(X) = \Psi(Y)$ for all $Y \in Q \cap [\![\tau^\frown\langle j\rangle]\!]$. From this it follows that $\Psi(X)$ is recursive, Q.E.D.

A general property of hyperimmune-free reals X is that the Turing degrees $\leq \deg_{\mathrm{T}}(X)$ are the same as the truth-table degrees $\leq \deg_{\mathrm{tt}}(X)$. Since every $X \in Q$ is of minimal truth-table degree, it follows that every hyperimmune-free $X \in Q$ is of minimal Turing degree.

4. Up until now we have not used the "moreover" clause of Theorem 6.1, but now we shall use it. We may safely assume that $X_1, \ldots, X_n \in Q$ are pairwise distinct. Recall that we have defined the Turing jump $X' \in \{0,1\}^{\mathbb{N}}$ of $X \in \{0,1\}^{\mathbb{N}}$ to be (the characteristic function of) the set $\{e \in \mathbb{N} \mid X \notin P_e\}$. To compute $(X_1 \oplus \cdots \oplus X_n)'$ we proceed as follows. First we use our oracle for $X_1 \oplus \cdots \oplus X_n$ to find $l \in \mathbb{N}$ such that $X_1{\upharpoonright}l$, $\ldots$, $X_n{\upharpoonright}l$ are pairwise distinct. Then, given $e \in \mathbb{N}$, we use our oracle for $0'$ to compute $m \geq l$ as in Theorem 6.1. Letting $\tau_i = X_i{\upharpoonright}m$ for $i = 1, \ldots, n$, we know that the Π^0_1 class $(Q \cap [\![\tau_1]\!]) \times \cdots \times (Q \cap [\![\tau_n]\!])$ is either disjoint from P_e or included in P_e. Using our oracle for $0'$ again, we can decide whether this Π^0_1 class is disjoint from P_e or not. The answer to this question tells us whether $e \in (X_1 \oplus \cdots \oplus X_n)'$ or not. Thus $(X_1 \oplus \cdots \oplus X_n)'$ is computable from $X_1 \oplus \cdots \oplus X_n \oplus 0'$, Q.E.D. $\square$

Remark 6.3. A plausible generalization of part 3 of Theorem 6.2 would say that for every pairwise distinct finite sequence $X_1, \ldots, X_n \in Q$, the tt-degrees $\leq \deg_{\mathrm{tt}}(X_1 \oplus \cdots \oplus X_n)$ should form a lattice isomorphic to the powerset of $\{1, \ldots, n\}$. From this it would follow that the same holds for Turing degrees, provided $X_1 \oplus \cdots \oplus X_n$ is hyperimmune-free.

Theorem 6.4. The special Π^0_1 class $Q \subseteq \{0,1\}^{\mathbb{N}}$ of Theorem 6.1 has neither the Join Property nor the Pseudojump Inversion Property.

Proof. We rely mainly on part 4 of Theorem 6.2. To see that the Join Property fails for Q, fix $Z \in Q$. Then $Z >_{\mathrm{T}} 0$ but for all $B \in Q$ we have $(B \oplus Z)' \equiv_{\mathrm{T}} B \oplus Z \oplus 0'$, hence $0' \not\leq_{\mathrm{T}} B \oplus Z$. To see that Pseudojump Inversion fails for Q, consider a pseudojump operator J_e with the property[4] that $X <_{\mathrm{T}} J_e(X)$ and $(J_e(X))' \equiv_{\mathrm{T}} X'$ for all $X \in \mathbb{N}^{\mathbb{N}}$. Then for all $B \in Q$ we have $(J_e(B))' \equiv_{\mathrm{T}} B' \equiv_{\mathrm{T}} B \oplus 0'$, hence $0' \not\leq_{\mathrm{T}} J_e(B)$. $\square$

The rest of this section is devoted to the proof of Theorem 6.1. We use a priority construction in the vein of Martin/Pour-El [7] and Jockusch/Soare [5, Theorem 4.7].

Our construction will be presented in terms of treemaps. A *treemap* is a mapping $T : \{0,1\}^* \to \{0,1\}^*$ such that $T(\sigma^\frown\langle i\rangle) \supseteq T(\sigma)^\frown\langle i\rangle$ for all $\sigma \in \{0,1\}^*$ and all $i \in \{0,1\}$. Note that for any treemap T and $\sigma, \rho \in \{0,1\}^*$ we have $\sigma \subset \rho$ if and only if $T(\sigma) \subset T(\rho)$. Note also that there is a one-to-one correspondence between treemaps T and nonempty perfect closed subsets of $\{0,1\}^{\mathbb{N}}$, given by $T \mapsto [T] = \{T(X) \mid X \in \{0,1\}^{\mathbb{N}}\}$ where $T(X) = \bigcup_{n \in \mathbb{N}} T(X{\upharpoonright}n)$.

Recall that any Π^0_1 class $P \subseteq \{0,1\}^{\mathbb{N}}$ is closed. Therefore, if P is also nonempty and perfect, there is a unique treemap T_P such that $P = [T_P]$.

Lemma 6.5. Let $P \subseteq \{0,1\}^{\mathbb{N}}$ be a nonempty perfect Π^0_1 class. Then, the treemap $T = T_P$ corresponding to P is computable using $0'$ as an oracle.

Proof. Given $\tau \in \{0,1\}^*$ we can use our oracle for $0'$ to decide whether $P \cap [\![\tau]\!]$ is empty or not. Thus, if already know $T(\sigma)$ for some $\sigma \in \{0,1\}^*$, we can then compute $T(\sigma^\frown\langle i\rangle)$ for $i \in \{0,1\}$, because $T(\sigma^\frown\langle i\rangle)$ is the smallest $\tau \supseteq T(\sigma)^\frown\langle i\rangle$ such that $P \cap [\![\tau^\frown\langle j\rangle]\!]$ is nonempty for all $j \in \{0,1\}$. Thus $T \leq_{\mathrm{T}} 0'$, Q.E.D. $\square$

In our proof of Theorem 6.1, the treemap $T = T_Q$ corresponding to Q will be *uniform*, in the sense that $|T(\sigma)|$ will depend only on $|\sigma|$. This feature of T will be for convenience only, but T will also have another key property, which reads as follows. For all $e, l, n \in \mathbb{N}$ with $e \leq l$ and $n \leq 2^l$ and all pairwise distinct $\sigma_1, \ldots, \sigma_n \in \{0,1\}^l$, either $[\![T(\sigma_1)]\!] \times \cdots \times [\![T(\sigma_n)]\!] \cap P_e = \emptyset$ or $(Q \cap [\![T(\sigma_1)]\!]) \times \cdots \times (Q \cap [\![T(\sigma_n)]\!]) \subseteq P_e$. Thus by Lemma 6.5 the "moreover" clause of Theorem 6.1 will hold with $m = m_e = |T(\sigma)|$ for all $\sigma \in \{0,1\}^e$.

[4] The existence of such pseudojump operators is well known [20, §VII.1]. One way to obtain such an operator is to combine part 4 of Theorem 6.2 with the R. E. Basis Theorem [4, Theorem 3] to get an $e \in \mathbb{N}$ such that $0 <_{\mathrm{T}} W_e$ and $(W_e)' \equiv_{\mathrm{T}} 0'$. The desired operator J_e is then obtained by uniform relativization to an arbitrary Turing oracle X.

Proof of Theorem 6.1. We shall construct $T = T_Q$ as the limit $T = \lim_s T_s$ of a recursive sequence of recursive uniform treemaps T_s, $s \in \mathbb{N}$. This will be a pointwise limit as $s \to \infty$, in the sense that for each $\sigma \in \{0,1\}^*$ we shall have $T(\sigma) = T_s(\sigma)$ for all sufficiently large s. Also, the treemaps T_s will be *nested*, in the sense that for all $s \in \mathbb{N}$ and all $\sigma \in \{0,1\}^*$ there will be a $\rho \in \{0,1\}^*$ such that $T_{s+1}(\sigma) = T_s(\rho)$. From this it follows that $[T_{s+1}] \subseteq [T_s]$ for all s, and that $[T] = \bigcap_s [T_s]$. Thus $Q = [T]$ will be a nonempty perfect Π^0_1 subclass of $\{0,1\}^{\mathbb{N}}$.

In presenting our construction of $T = \lim_s T_s$, we shall use the notation

$$P_{e,s} = \{X \in \{0,1\}^{\mathbb{N}} \mid \varphi^{(1),X\restriction s}_{e,s}(0)\uparrow\}.$$

Note that $P_{e,s} = \bigcup_\tau [\![\tau]\!]$ where the union is taken over a finite subset of $\{0,1\}^s$. Note also that $P_{e,s+1} \subseteq P_{e,s}$ for all $s \in \mathbb{N}$, and that

$$P_e = \{X \in \{0,1\}^{\mathbb{N}} \mid \varphi^{(1),X}_e(0)\uparrow\} = \textstyle\bigcap_s P_{e,s}.$$

We now offer a preliminary account of the construction and proof. To each $e, l, n \in \mathbb{N}$ with $e \le l$ and $n \le 2^l$ and each pairwise distinct sequence $\sigma_1, \ldots, \sigma_n \in \{0,1\}^l$, we associate a *requirement* $R(e, \sigma_1, \ldots, \sigma_n)$ *at level* l. Intuitively, the purpose of this requirement is to insure that $([\![T(\sigma_1)]\!] \times \cdots \times [\![T(\sigma_n)]\!]) \cap P_e = \emptyset$ "if possible." The strategy here will be to attempt to arrange that $([\![T_s(\sigma_1)]\!] \times \cdots \times [\![T_s(\sigma_n)]\!]) \cap P_{e,s} = \emptyset$ for all sufficiently large s, "if possible." If this attempt fails, we shall argue that $(Q \cap [\![T_s(\sigma_1)]\!]) \times \cdots \times (Q \cap [\![T_s(\sigma_n)]\!]) \subseteq P_{e,s}$ for all sufficiently large s, and hence $(Q \cap [\![T(\sigma_1)]\!]) \times \cdots \times (Q \cap [\![T(\sigma_n)]\!]) \subseteq P_e$.

We now give the detailed construction of T_s for all $s \in \mathbb{N}$. Begin by fixing a recursive linear ordering of all of the requirements, called the *priority ordering*. Arrange this ordering so that for each $l \in \mathbb{N}$, all requirements at level l are of lower priority than all requirements at level $< l$. The idea here is that requirements at level l may be "injured" by requirements at level $< l$ but will never be "injured" by requirements at level $\ge l$. Note that for each l there are only finitely many requirements at level $\le l$. We proceed by induction on s.

Stage 0. Let $T_0(\nu) = \nu$ for all $\nu \in \{0,1\}^*$. Clearly T_0 is a uniform treemap.

Stage $s+1$. Assume inductively that we have defined a uniform treemap T_s. Our task at this stage is to define T_{s+1}. A requirement $R(e, \sigma_1, \ldots, \sigma_n)$ at level l is said to be *requesting attention at stage* s if $([\![T_s(\sigma_1)]\!] \times \cdots \times [\![T_s(\sigma_n)]\!]) \cap P_{e,s} \neq \emptyset$ but there exist $k \le s$ and $\rho_1, \ldots, \rho_n \in \{0,1\}^k$ such that $\rho_i \supset \sigma_i$ for all $i = 1, \ldots, n$ and $([\![T_s(\rho_1)]\!] \times \cdots \times [\![T_s(\rho_n)]\!]) \cap P_{e,s} = \emptyset$. If no requirements are requesting attention, do nothing, i.e., let $T_{s+1}(\nu) = T_s(\nu)$ for all $\nu \in \{0,1\}^*$. Otherwise, let R_s be the requirement of highest priority which is requesting attention. For this requirement only, choose $k > l \ge e$ and $\rho_1, \ldots, \rho_n$ as above and define T_{s+1} as follows. For each $\nu \in \{0,1\}^{<l}$ let $T_{s+1}(\nu) = T_s(\nu)$. For each $i = 1, \ldots, n$ let $T_{s+1}(\sigma_i) = T_s(\rho_i)$. For each $\sigma \in \{0,1\}^l$ other than $\sigma_1, \ldots, \sigma_n$, let $T_{s+1}(\sigma) = T_s(\sigma^\frown\langle\underbrace{0, \ldots, 0}_{k-l}\rangle)$. For each $\sigma \in \{0,1\}^l$ and each $\nu \in \{0,1\}^*$, let $T_{s+1}(\sigma^\frown\nu) = T_s(\rho^\frown\nu)$ where ρ is such that $T_{s+1}(\sigma) = T_s(\rho)$. Clearly T_{s+1} is a treemap, and it is uniform because T_s is uniform and for each $\sigma \in \{0,1\}^l$ we have $T_{s+1}(\sigma) = T_s(\rho)$ for some $\rho \in \{0,1\}^k$.

We now have a recursive nested sequence of uniform treemaps T_s. As s goes to infinity, consider the history of a single requirement $R = R(e, \sigma_1, \ldots, \sigma_n)$ at level l. Let us say that R is *satisfied at stage* s if $([\![T_s(\sigma_1)]\!] \times \cdots \times [\![T_s(\sigma_n)]\!]) \cap P_{e,s} = \emptyset$, otherwise *unsatisfied at stage* s. By construction, if $R = R_s$ then R is unsatisfied at stage s but becomes satisfied at stage $s+1$. And if R is satisfied at stage s, it remains satisfied at stage $s+1$ unless R_s is at level $< l$. Let us say that R is *injured at stage* s if the latter case holds, i.e., if the level of R_s is less than the level of R. By induction along the priority ordering, we now see that there are only finitely many stages s at which R is injured, and only finitely many stages s at which R requests attention. This holds for all of the finitely many requirements at level $\leq l$, so for all $\sigma \in \{0,1\}^l$ and all sufficiently large s we have $T_{s+1}(\sigma) = T_s(\sigma)$. We now have a uniform treemap T defined by $T(\sigma) = \lim_s T_s(\sigma)$, so letting $Q = [T] = \bigcap_s [T_s]$ we have a nonempty perfect Π^0_1 class $Q \subseteq \{0,1\}^{\mathbb{N}}$.

We claim that for each requirement R at level l as above, either $([\![T(\sigma_1)]\!] \times \cdots \times [\![T(\sigma_n)]\!]) \cap P_e = \emptyset$ or $(Q \cap [\![T(\sigma_1)]\!]) \times \cdots \times (Q \cap [\![T(\sigma_n)]\!]) \subseteq P_e$. Suppose not. Then $([\![T(\sigma_1)]\!] \times \cdots \times [\![T(\sigma_n)]\!]) \cap P_e \neq \emptyset$ and

$$((Q \cap [\![T(\sigma_1)]\!]) \times \cdots \times (Q \cap [\![T(\sigma_n)]\!])) \setminus P_e \neq \emptyset.$$

For $i = 1, \ldots, n$ fix $X_i \in Q \cap [\![T(\sigma_i)]\!]$ so that $X_1 \oplus \cdots \oplus X_n \notin P_e$. Let m be so large that $([\![X_1 \restriction m]\!] \times \cdots \times [\![X_n \restriction m]\!]) \cap P_e = \emptyset$. Let $k > l$ and $\rho_1, \ldots, \rho_n \in \{0,1\}^k$ be such that $T(\rho_i) \subset X_i$ and $|T(\rho_i)| \geq m$ for all $i = 1, \ldots, n$. Thus we have $\rho_i \supset \sigma_i$ for all $i = 1, \ldots, n$, and $([\![T(\rho_1)]\!] \times \cdots \times [\![T(\rho_n)]\!]) \cap P_e = \emptyset$. It follows by compactness that for all sufficiently large s we have $([\![T(\rho_1)]\!] \times \cdots \times [\![T(\rho_n)]\!]) \cap P_{e,s} = \emptyset$. And we can also let s be so large that $e \leq l < k \leq s$ and $T_s(\sigma_i) = T(\sigma_i)$ and $T_s(\rho_i) = T(\rho_i)$ for all $i = 1, \ldots, n$. So now we see that $([\![T_s(\sigma_1)]\!] \times \cdots \times [\![T_s(\sigma_n)]\!]) \cap P_{e,s} \neq \emptyset$ and $([\![T_s(\rho_1)]\!] \times \cdots \times [\![T_s(\rho_n)]\!]) \cap P_{e,s} = \emptyset$ for all suffiently large s. Thus our requirement R is requesting attention at all sufficiently large stages s. This contradiction proves our claim.

The above claim gives us the principal conclusion of Theorem 6.1. And then, as we have seen, Lemma 6.5 gives us the "moreover" clause with $m = m_e$. The proof of Theorem 6.1 is now complete. □

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