Π^0_1 Sets and Models of WKL_0

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Abstract

We show that any two Medvedev complete Π^0_1 subsets of 2^ω are recursively homeomorphic. We obtain a Π^0_1 set $\widehat{Q}'$ of countable coded ω-models of WKL_0 with a strong homogeneity property. We show that if G is a generic element of $\widehat{Q}'$, then the ω-model of WKL_0 coded by G satisfies $\forall X\,\forall Y$ (if X is definable from Y, then X is Turing reducible to Y). We use a result of Kučera to refute some plausible conjectures concerning ω-models of WKL_0. We generalize our results to non-ω-models of WKL_0. We discuss the significance of our results for foundations of mathematics.

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Contents

1 Introduction

In this paper we apply recursion-theoretic methods to the study of ω-models of subsystems of second order arithmetic. Specifically, we present some results concerning Π^0_1 subsets of 2^ω, along with applications to countable ω-models of WKL_0. These results and applications may be regarded as an addendum or supplement to Simpson [31, §VIII.2]. We also present generalizations to countable non-ω-models of WKL_0. These generalizations may be regarded as an addendum to Simpson [31, §IX.2].

For background on subsystems of second order arithmetic, see Simpson [31]. We recall here that RCA_0 is the subsystem consisting of Δ^0_1 comprehension and Σ^0_1 induction, and WKL_0 is the subsystem consisting of RCA_0 plus *Weak König's Lemma*, i.e., the statement that every infinite tree of finite sequences of 0's and 1's has a path. These two systems play an important role in Reverse Mathematics [31]. Their ω-models are easy to understand in recursion-theoretic terms. An ω-model of RCA_0 is a set $S \subseteq P(\omega)$ such that (i) $S \neq \emptyset$, (ii) $X \oplus Y \in S$ for all $X, Y \in S$, and (iii) if $X \in S$ and $Y \leq_T X$ then $Y \in S$. An ω-model of WKL_0 has the additional property that if $T \in S$ and T is an infinite tree of finite sequences of 0's and 1's, then T has a path in S.

There is a large recursion-theoretic literature on Π^0_1 subsets of 2^ω and degrees of elements of such sets. An important paper in this area is Jockusch/Soare [17]. An extensive recent survey is Cenzer/Remmel [3]. This topic is well known to be closely related to ω-models of WKL_0. The connection is as follows: $P \subseteq 2^\omega$ is Π^0_1 if and only if there exists a recursive tree T of finite sequences of 0's and 1's such that $P = \{X \in 2^\omega : X \text{ is a path through } T\}$.

In the model-theoretic literature, ω-models of WKL_0 are known as *Scott systems*, after Scott [26], who proved that $S \subseteq P(\omega)$ is a countable ω-model of WKL_0 if and only if S is the set of sets representable in some complete extension of Peano arithmetic. This idea is important in the study of models of arithmetic. See also Kaye [18].

Here is an outline of the rest of this paper.

In §2 we discuss the significance of some of our results, in terms of foundations of mathematics.

In §3 we study and characterize the nonempty Π^0_1 subsets of 2^ω which are Medvedev complete. We prove that any two such sets are recursively homeomorphic (Theorem 3.21). This is related to a result of Pour-El/Kripke [23] concerning effectively inseparable theories.

In §4 we relativize and iterate the result of §3 to obtain a nonempty Π^0_1 set $\widehat{Q}'$ of codes for countable ω-models of WKL_0, with a strong homogeneity property: any two nonempty Π^0_1 subsets of $\widehat{Q}'$ are recursively homeomorphic, via a homeomorphism which preserves the ω-models (Theorem 4.11).

In §5 we combine the results of §§3,4 with Jockusch/Soare forcing, to obtain a countable ω-model of WKL_0 in which all definable elements are recursive (Theorem 5.11). This result is originally due to Friedman [11, unpublished]. In §6 we improve this result, to obtain a countable ω-model of WKL_0 satisfying $\forall X\, \forall Y$ (if X is definable from Y then $X \leq_T Y$) (Theorem 6.9).

In §7 we generalize the results of §§3,4,5,6 to non-ω-models. In this way we obtain a conservation result, showing that WKL_0 plus a strong relative nondefinability scheme is conservative over Σ^0_1-PA (Corollary 7.9).

In §8 we prove a recursion-theoretic result of Kučera [20]: There is a disjoint pair of recursively inseparable, recursively enumerable sets, such that any two separating sets which differ infinitely compute the complete recursively enumerable set (Theorem 8.3). In §9 we apply Kučera's result to the study of ω-models of WKL_0. It is well known that the intersection of all such models consists of the recursive sets. We now show that the intersection of all such models which are submodels of a given one may contain nonrecursive sets (Theorem 9.1).

In §10 we generalize Kučera's result, and we apply the generalization to the study of non-ω-models of WKL_0. We refute several plausible conjectures concerning the relationship between WKL_0 and RCA_0. See Remarks 10.4, 10.8, 10.9.

Throughout this paper, we use recursion-theoretic concepts and notation from Rogers [25] and Soare [34]. We use $\omega = \{0, 1, 2, \ldots\}$ to denote the set of natural numbers. We identify points $X \in 2^\omega$ with functions $X : \omega \to \{0, 1\}$. For $e, n, s, k \in \omega$ and $X \in 2^\omega$, we write $\{e\}^X_s(n) = k$ to mean that the Turing machine with Gödel number e and oracle X and input n halts in $\leq s$ steps with output k. For $e, n, k \in \omega$ and $X \in 2^\omega$, we write $\{e\}^X(n) = k$ to mean that $\exists s\,(\{e\}^X_s(n) = k)$. Furthermore, $\{e\}^X(n)\downarrow$ means that $\{e\}^X(n)$ is *defined*, i.e., $\exists k\,(\{e\}^X(n) = k)$, and $\{e\}^X(n)\uparrow$ means that $\{e\}^X(n)$ is *undefined*, i.e., $\neg\,\exists k\,(\{e\}^X(n) = k)$. For $X, Y \in 2^\omega$, $X \leq_T Y$ means that X is *Turing reducible* to Y, i.e., $\exists e\,\forall n\,(X(n) = \{e\}^Y(n))$. The *Turing degree* of X, written $\deg_T(X)$, is the set of all Y such that $X \equiv_T Y$, i.e., $X \leq_T Y$ and $Y \leq_T X$. A predicate $R \subseteq 2^\omega \times \omega$ is said to be *recursive* if $\exists e\,\forall X\,\forall n\,(\{e\}^X(n) = 1$ if $R(X, n)$, and $\{e\}^X(n) = 0$ if $\neg\,R(X, n))$. A set $P \subseteq 2^\omega$ is said to be Π^0_1 if there exists a recursive predicate R such that $P = \{X \in 2^\omega : \forall n\,R(X, n)\}$. For Π^0_1 sets $P \subseteq 2^\omega$, we shall consider *recursive functionals* $\Phi : P \to 2^\omega$ given by $\Phi(X)(n) = \{e\}^X(n)$ for some $e \in \omega$ and all $X \in P$, $n \in \omega$.

2 Foundational Significance

In this section we explore the foundational significance of some of our results.

Foundations of mathematics is the study of the most basic concepts and logical structure of mathematics, with an eye to the unity of human knowledge. For general background in this area, the reader may turn to the van Heijenoort volume [38], where some of the most important modern papers have been carefully translated and reprinted. See also Gödel's collected works [8] and the Friedman volume [13].

As background for our work here, consider the well known foundational program of *computable analysis*, i.e., the development of mathematics in the computable world, $\mathrm{REC} = \{X : X \text{ is recursive}\}$. See Aberth [1] and Pour-El/Richards [24]. This program is obviously attractive from the viewpoint of Turing's analysis of computability. However, it is also known that the assumption "all real numbers are computable" conflicts with many basic, well known theorems of real analysis. For example, it is in conflict with the maximum principle for continuous real-valued functions on a closed bounded interval.

Clearly it would be desirable to strike a balance between these conflicting requirements. A fairly successful attempt in this direction is Theorem 5.11, below. In non-technical terms, the theorem asserts the existence of a world where the main theorems of real analysis hold, and the natural numbers are standard, yet each definable real number is computable. In technical terms, one obtains an ω-model S of WKL_0 in which all definable reals are recursive. The identification of the recursive reals with the computable reals is an outcome of Turing's foundational work on computable functions. Thus the computable reals play a large and important role in S, forming so to speak the "definable core" of S. On the other hand, from recent foundational work in Reverse Mathematics, we know that WKL_0 is just strong enough to prove many basic theorems of real analysis. See Simpson [31, Chapter IV]. Thus S contains just enough non-computable reals in order to satisfy the demands of real analysis.

Furthermore, in Theorem 6.9 below, we show that the same ω-model S satisfies a more general scheme:

For all reals X and Y, if Y is definable from X, then Y is *Turing reducible to* X, i.e., computable using X as an oracle.

We also show that WKL_0 plus the above scheme has the same first order part as WKL_0 alone. See Corollary 7.9, below.

The above scheme is foundationally interesting, for the following reason. Often in mathematics one has the situation that, under some assumptions on a real parameter X, there exists a unique real Y having some property which is stated in terms of X. In this kind of situation, our scheme allows us to conclude that Y is Turing reducible to X.

3 Medvedev Degrees of Π^0_1 Subsets of 2^ω

In this section we exposit the lattice of Medvedev degrees of nonempty Π^0_1 subsets of 2^ω. We prove an important result concerning nonempty Π^0_1 subsets of 2^ω which are Medvedev complete.

For background on Medvedev degrees of subsets of the Baire space, ω^ω, see Rogers [25, §13.7] and Sorbi [35]. For background on Π^0_1 subsets of the Cantor space, 2^ω, and of the Baire space, see the survey by Cenzer/Remmel [3].

Definition 3.1. Let P and Q be nonempty Π^0_1 subsets of 2^ω. We say that P is *Medvedev reducible* to Q, written $P \leq_M Q$, if there exists a recursive functional $\Phi : Q \to P$. We say that Q is *Medvedev complete* if $P \leq_M Q$ for all nonempty Π^0_1 subsets P of 2^ω. We say that P and Q are *Medvedev equivalent*, written $P \equiv_M Q$, if $P \leq_M Q$ and $Q \leq_M P$. The *Medvedev degree* of P, written $\deg_M(P)$, is the set of all Q such that $P \equiv_M Q$. The Medvedev degrees are partially ordered by writing $\deg_M(P) \leq \deg_M(Q)$ if and only if $P \leq_M Q$. We write $\deg_M(P) < \deg_M(Q)$ if and only if $P <_M Q$, i.e., $P \leq_M Q$ and $Q \not\leq_M P$.

Theorem 3.2. *The Medvedev degrees of nonempty Π^0_1 subsets of 2^ω form a distributive lattice $\mathcal{P}_M$ with a bottom and a top element. The top element of $\mathcal{P}_M$ consists of the nonempty Π^0_1 subsets of 2^ω which are Medvedev complete.*

Proof. In this proof and throughout this paper, we use the following notation. For $X, Y \in 2^\omega$ we have $X \oplus Y \in 2^\omega$ where $(X \oplus Y)(2n) = X(n)$ and $(X \oplus Y)(2n+1) = Y(n)$. We use $2^{<\omega}$ to denote the set of *strings*, i.e., finite sequences of 0's and 1's. The length of $\sigma \in 2^{<\omega}$ is denoted $\mathrm{lh}(\sigma)$. For $X \in 2^\omega$ and $n \in \omega$, we have $X[n] = \langle X(0), \ldots, X(n-1) \rangle \in 2^{<\omega}$ and $\mathrm{lh}(X[n]) = n$. For $\sigma \in 2^{<\omega}$ and $X \in 2^\omega$, we have $\sigma^\frown X \in 2^\omega$ given by

$$(\sigma^\frown X)(n) = \begin{cases} \sigma(n) & \text{if } n < \mathrm{lh}(\sigma), \\ X(n - \mathrm{lh}(\sigma)) & \text{if } n \geq \mathrm{lh}(\sigma). \end{cases}$$

We fix a primitive recursive, one-to-one, onto function $(\cdot,\cdot) : \omega \times \omega \to \omega$. For $Y \in 2^\omega$ and $m \in \omega$, we have $(Y)_m \in 2^\omega$ where $(Y)_m(n) = Y((m, n))$.

To prove our theorem, let P and Q be nonempty Π^0_1 subsets of 2^ω. The supremum or least upper bound of $\deg_M(P)$ and $\deg_M(Q)$ is $\deg_M(P \times Q)$

where $P \times Q = \{X \oplus Y : X \in P \text{ and } Y \in Q\}$. The infimum or greatest lower bound of $\deg_M(P)$ and $\deg_M(Q)$ is $\deg_M(P + Q)$ where

$$P + Q = \{\langle 0\rangle^\frown X : X \in P\} \cup \{\langle 1\rangle^\frown Y : Y \in Q\}.$$

The distributive laws $P \times (Q + R) \equiv_M (P \times Q) + (P \times R)$ and $P + (Q \times R) \equiv_M (P + Q) \times (P + R)$ are easily verified. The bottom element of our lattice $\mathcal{P}_M$ is $\deg_M(2^\omega)$, or equivalently $\deg_M(P)$ where P is any Π^0_1 subset of 2^ω with a recursive element. The top element of $\mathcal{P}_M$ is $\deg_M(Q)$ where Q is any nonempty Π^0_1 subset of 2^ω which is Medvedev complete. See Lemma 3.3 and Remark 3.4 below. □

Lemma 3.3. *There exists a nonempty Π^0_1 subset Q of 2^ω which is Medvedev complete.*

Proof. Let $\{P_e : e \in \omega\}$ be a standard, recursive enumeration of the Π^0_1 subsets of 2^ω. (See Remark 3.9 below.) In particular, the predicate $U(e, X) \equiv (X \in P_e)$ is Π^0_1. By the Normal Form Theorem for Π^0_1 predicates, we have $U(e, X) \equiv \forall n\, U_1(e, X[n])$ where $U_1 \subseteq \omega \times 2^{<\omega}$ is primitive recursive. As in Simpson [31, Lemmas VIII.2.5 and VIII.2.9], put $U^+(e, X) \equiv$

$$\forall n\, (\forall \sigma \text{ of length } n)\, (\text{if } (\forall m \leq n)\, U_1(e, \sigma[m]) \text{ then } U_1(e, X[n])).$$

Note that $U^+(e, X)$ is again Π^0_1. Now for all e such that P_e is nonempty, we have $P_e = P^+_e = \{X : U^+(e, X)\}$. On the other hand, for all e, P^+_e is nonempty, by compactness of 2^ω. Put

$$Q = \textstyle\prod_e P^+_e = \{Y : \forall e\, U^+(e, (Y)_e)\}.$$

Obviously Q is a nonempty Π^0_1 subset of 2^ω. For all e such that P_e is nonempty, we have $P_e \leq_M Q$ via the recursive functional $Y \mapsto (Y)_e$. Thus Q is Medvedev complete. □

Remark 3.4. Another construction of a Medvedev complete set is as follows. Let Q be the Π^0_1 set of complete extensions of Peano arithmetic. It can be shown that Q is Medvedev complete; see Scott/Tennenbaum [27] and Jockusch/Soare [17]. Instead of Peano arithmetic, we may use any effectively inseparable theory; see Pour-El/Kripke [23]. Or, we may use any effectively essentially incomplete theory; see Remark 3.18 below. Yet another construction of a Medvedev complete set may be obtained from Lemmas 3.14 and 3.16 below.

We are going to show that any two Medvedev complete Π^0_1 subsets of 2^ω are recursively homeomorphic (Theorem 3.21). In order to prove this, we shall first consider the nature of Medvedev reducibility in more detail.

Lemma 3.5. *Let $R \subseteq \omega \times 2^\omega \times 2^\omega$. If the predicate $R(k, X, Y)$ is Π^0_1, then the predicate $S(k, X) \equiv \exists Y\, R(k, X, Y)$ is also Π^0_1.*

Proof. By the Normal Form Theorem for Π^0_1 predicates, we have

$$R(k, X, Y) \equiv \forall n\, R_1(k, X[n], Y[n])$$

where $R_1(k, \sigma, \tau)$ is primitive recursive. Thus $S(k, X)$ holds if and only if $\exists Y\, \forall n\, R_1(k, X[n], Y[n])$. By compactness of 2^ω, this is equivalent to $\forall n\, (\exists \tau$ of length $n)\, (\forall m \leq n)\, R_1(k, X[m], \tau[m])$, which is explicitly Π^0_1. □

Lemma 3.6. *Let Q be a Π^0_1 subset of 2^ω, and let $\Phi : Q \to 2^\omega$ be a recursive functional.*

1. *The image $\Phi(Q)$ is a Π^0_1 subset of 2^ω.*
2. *For any Π^0_1 subset P of 2^ω, the inverse image $\Phi^{-1}(P)$ is a Π^0_1 subset of 2^ω.*

Proof. To prove part 1, note that for all $X \in 2^\omega$ we have $X \in \Phi(Q)$ if and only if $\exists Y\, (Y \in Q$ and $\Phi(X) = Y)$. By Lemma 3.5, this is Π^0_1. For part 2 we have $\Phi^{-1}(P) = \{Y : Y \in Q$ and $\Phi(Y) \in P\}$ and this is obviously Π^0_1. □

Definition 3.7. We use $\mathcal{B}$ to denote the free Boolean algebra on a countable set of generators $\{a_n : n \in \omega\}$. There is a well known isomorphism $b \mapsto [b]$ of $\mathcal{B}$ onto the Boolean algebra of clopen subsets of 2^ω, given by $[a_n] = \{X : X(n) = 1\}$, $[b \cdot c] = [b] \cap [c]$, $[b + c] = [b] \cup [c]$, $[-b] = 2^\omega \setminus [b]$, $[0] = \emptyset$, $[1] = 2^\omega$. For $T \subseteq \mathcal{B}$ we write $[T] = \bigcap\{[b] : b \in T\}$.

Remark 3.8. The mapping $b \mapsto [b]$ is essentially just the usual semantics for propositional calculus. The Compactness Theorem for propositional calculus says: For all $T \subseteq \mathcal{B}$, $[T] = \emptyset$ if and only if $[T_0] = \emptyset$ for some finite $T_0 \subseteq T$. This is a consequence of the fact that 2^ω is compact as a topological space.

Remark 3.9. If $T \subseteq \mathcal{B}$ is recursively enumerable, then $[T] \subseteq 2^\omega$ is Π^0_1. Conversely, if $P \subseteq 2^\omega$ is Π^0_1, then $P = [T_P]$ where $T_P = \{b \in \mathcal{B} : P \subseteq [b]\}$. Note that T_P is recursively enumerable. A standard, recursive enumeration $\{P_e : e \in \omega\}$ of the Π^0_1 subsets of 2^ω may be obtained by setting $P_e = [T_e]$, where $\{T_e : e \in \omega\}$ is a standard, recursive enumeration of the recursively enumerable subsets of $\mathcal{B}$.

Remark 3.10. The mapping $b \mapsto [b]$ gives a one-to-one correspondence between nonempty Π^0_1 subsets of 2^ω and Stone spaces of Boolean algebras of the form $\mathcal{B}/I$ where I is a recursively enumerable ideal. These are the so-called "recursively enumerable Boolean algebras" of Cenzer/Remmel [3, §5]. Recursively presented homomorphisms on the Boolean algebras correspond to recursive functionals on the Stone spaces.

Lemma 3.11. *Let Q be a Π^0_1 subset of 2^ω, and let $\Phi : Q \to 2^\omega$ be a recursive functional. Then there is a recursive function $f : \mathcal{B} \to \mathcal{B}$ such that $\Phi^{-1}[b] = [f(b)] \cap Q$ for all $b \in \mathcal{B}$.*

Proof. The predicate ($Y \in Q$ and $\Phi(Y) \in [b]$) is Π^0_1, so by the Normal Form Theorem, let $R(\tau, b)$ be a primitive recursive predicate such that

$$(Y \in Q \text{ and } \Phi(Y) \in [b]) \equiv \forall n\, R(Y[n], b).$$

Trivially we have

$$(\forall b \in \mathcal{B})\,(\forall Y \in 2^\omega)\,(\Phi(Y) \notin [b] \text{ or } \Phi(Y) \notin [-b] \text{ or } Y \notin Q),$$

or in other words,

$$(\forall b \in \mathcal{B})\,(\forall Y \in 2^\omega)\,\exists n\,(\text{not } R(Y[n], b) \text{ or not } R(Y[n], -b) \text{ or not } R(Y[n], 1)).$$

By compactness of 2^ω, it follows that $(\forall b \in \mathcal{B})\,\exists n\,(\forall \tau \text{ of length } n)\,(\exists m \leq n)\,(\text{not } R(\tau[m], b) \text{ or not } R(\tau[m], -b) \text{ or not } R(\tau[m], 1))$. For $b \in \mathcal{B}$ let $n(b)$ be the least such n, and form $f(b) \in \mathcal{B}$ such that

$$[f(b)] = \{Y \in 2^\omega : (\forall m \leq n(b))\, R(Y[m], b)\}.$$

Clearly $n : \mathcal{B} \to \omega$ and $f : \mathcal{B} \to \mathcal{B}$ are recursive, and $\Phi^{-1}[b] = [f(b)] \cap Q$. □

Remark 3.12. In Lemma 3.11, we may replace f by the unique recursive homomorphism $\overline{f} : \mathcal{B} \to \mathcal{B}$ given by $\overline{f}(a_n) = f(a_n)$ for all n. For $X \in Q$ and $b \in \mathcal{B}$, we have $\Phi(X) \in [b]$ if and only if $X \in [f(b)]$. Thus Φ is a truth-table reducibility operator. This is closely related to results of Nerode as presented in Rogers [25, pages 143 and 154].

We now introduce a property of nonempty Π^0_1 subsets of 2^ω, called productiveness, which will turn out to be equivalent to Medvedev completeness (Lemma 3.20).

Definition 3.13. Let P be a nonempty Π^0_1 subset of 2^ω. A *splitting function* for P is a recursive function $g : \omega \to \mathcal{B}$ such that for all e, if $P_e \subseteq P$ and P_e is nonempty, then $P_e \cap [g(e)]$ and $P_e \cap [-g(e)]$ are nonempty. P is said to be *productive* if there exists a splitting function for P.

Lemma 3.14. *There exists a nonempty Π^0_1 set $P \subseteq 2^\omega$ such that P is productive.*

Proof. Put $P = \{X \in 2^\omega : \forall n\,(X(n) \neq \{n\}(n))\}$. Clearly P is nonempty and Π^0_1. By Lemma 3.5, the predicate

$$S(e, n, k) \equiv \forall X\,(\text{if } X \in P_e \text{ then } X(n) = k)$$

is Σ^0_1. By the Σ^0_1 Uniformization Principle and the S-m-n Theorem, let h be a primitive recursive function such that, for all e and n, $\{h(e)\}(n) =$ some k such that $S(e, n, k)$ holds, if such a k exists. Define $g : \omega \to \mathcal{B}$ by $g(e) = a_{h(e)}$. We claim that g is a splitting function for P. To see this, suppose $P_e \subseteq P$ and $P_e \neq \emptyset$. If $P_e \cap [a_{h(e)}] = \emptyset$, then $\forall X\,(\text{if } X \in P_e \text{ then } X(h(e)) = 0)$, hence $\{h(e)\}(h(e)) = 0$, a contradiction. If $P_e \cap [-a_{h(e)}] = \emptyset$, then $\forall X\,(\text{if } X \in P_e \text{ then } X(h(e)) = 1)$, hence $\{h(e)\}(h(e)) = 1$, a contradiction. □

Lemma 3.15. *Let P and Q be nonempty Π^0_1 subsets of 2^ω. Given $a, b, a' \in \mathcal{B}$ such that $a \neq a \cdot b = b \neq 0$ and $a' \neq 0$, and given a splitting function for P, we can effectively find $b' \in \mathcal{B}$ with the following properties: $a' \neq a' \cdot b' = b' \neq 0$, and if $Q \cap [a] \neq \emptyset$ and $P \cap [a'] \neq \emptyset$ then*

1. *$Q \cap [a] \cap [b] = \emptyset$ if and only if $P \cap [a'] \cap [b'] = \emptyset$,*

2. *$Q \cap [a] \cap [-b] = \emptyset$ if and only if $P \cap [a'] \cap [-b'] = \emptyset$.*

Proof. Because P is productive, P is nowhere dense in 2^ω, so given $a' \neq 0$ we can effectively find $a'_0, a'_1, a'_2 \in \mathcal{B}$ such that $a' = a'_0 + a'_1 + a'_2$ and $a'_0 \cdot a'_1 = a'_0 \cdot a'_2 = a'_1 \cdot a'_2 = 0$ and $a'_0 \neq 0$ and $a'_1 \neq 0$ and $a'_2 \neq 0$ and $P \cap [a'_0] = P \cap [a'_1] = \emptyset$. Thus $P \cap [a'] = P \cap [a'_2]$. Now let g be a splitting function for P. By the Recursion Theorem, we can effectively find $e \in \omega$ such that

$$P_e = \begin{cases} P \cap [a'_2] & \text{if } Q \cap [a] \cap [b] \neq \emptyset \text{ and } Q \cap [a] \cap [-b] \neq \emptyset, \\ P \cap [a'_2] \cap [g(e)] & \text{if } Q \cap [a] \cap [b] = \emptyset \text{ and } Q \cap [a] \cap [-b] \neq \emptyset, \\ P \cap [a'_2] \cap [-g(e)] & \text{if } Q \cap [a] \cap [b] \neq \emptyset \text{ and } Q \cap [a] \cap [-b] = \emptyset, \\ \emptyset & \text{if } Q \cap [a] \cap [b] = \emptyset \text{ and } Q \cap [a] \cap [-b] = \emptyset. \end{cases}$$

Put $b' = a'_1 + a'_2 \cdot g(e)$. Clearly $a' \neq a' \cdot b' = b' \neq 0$. Now assume $Q \cap [a] \neq \emptyset$ and $P \cap [a'] \neq \emptyset$. If $Q \cap [a] \cap [b] = \emptyset$, then we have $P_e = P \cap [a'_2] \cap [g(e)]$, hence $P_e \cap [-g(e)] = \emptyset$, hence $P_e = \emptyset$ (because g is a splitting function for P), hence $P \cap [a'] \cap [b'] = P \cap [a'_2] \cap [g(e)] = P_e = \emptyset$. Similarly, if $Q \cap [a] \cap [-b] = \emptyset$, then $P \cap [a'] \cap [-b'] = \emptyset$. On the other hand, if $Q \cap [a] \cap [b] \neq \emptyset$ and $Q \cap [a] \cap [-b] \neq \emptyset$, then we have $P_e = P \cap [a'_2] = P \cap [a'] \neq \emptyset$, hence $P \cap [a'] \cap [b'] = P_e \cap [g(e)] \neq \emptyset$ and $P \cap [a'] \cap [-b'] = P_e \cap [-g(e)] \neq \emptyset$. This completes the proof. □

Lemma 3.16. *Let P and Q be nonempty Π^0_1 subsets of 2^ω.*

1. *If P is productive, then there exists a recursive functional from P onto Q.*

2. *If P and Q are productive, then P and Q are recursively homeomorphic, i.e., there exists a recursive functional from P one-to-one onto Q.*

Proof. Let P and Q be as in the hypothesis of our lemma. If $\mathcal{B}^*$ is a subalgebra of $\mathcal{B}$ and if $f^* : \mathcal{B}^* \to \mathcal{B}$ is a monomorphism, let us call f^* *good* if, for all $b \in \mathcal{B}^*$, $Q \cap [b] = \emptyset$ if and only if $P \cap [f^*(b)] = \emptyset$.

For part 1, to find a recursive functional Φ from P onto Q, it suffices to find a good recursive monomorphism $f : \mathcal{B} \to \mathcal{B}$. Assume inductively that we have already found a good finite monomorphism $f_n : \mathcal{B}_n \to \mathcal{B}$, where $\mathcal{B}_n$ is a finite subalgebra of $\mathcal{B}$. (We start with $\mathcal{B}_0 = \{0, 1\}$.) Let b be the nth element of $\mathcal{B}$ with respect to some fixed recursive enumeration of $\mathcal{B}$. Let $\mathcal{B}_{n+1}$ be the finite subalgebra of $\mathcal{B}$ generated by $\mathcal{B}_n \cup \{b\}$. We effectively extend f_n to a good finite monomorphism $f_{n+1} : \mathcal{B}_{n+1} \to \mathcal{B}$, as follows. For each atom a of $\mathcal{B}_n$, if $a \cdot b = a$ or $a \cdot b = 0$ put $f_{n+1}(a \cdot b) = f_n(a \cdot b)$, otherwise use Lemma 3.15 and a splitting function for P to effectively find $f_{n+1}(a \cdot b) = b' \in \mathcal{B}$ such that $f_n(a) \neq f_n(a) \cdot b' = b' \neq \emptyset$, and $Q \cap [a] \cap [b] = \emptyset$ if and only if $P \cap [f_n(a)] \cap [b'] = \emptyset$, and $Q \cap [a] \cap [-b] = \emptyset$ if and only if $P \cap [f_n(a)] \cap [-b'] = \emptyset$. Finally we obtain a good recursive monomorphism $f = \bigcup_n f_n : \mathcal{B} \to \mathcal{B}$, and part 1 is proved.

For part 2 we proceed as above, except that we use a back-and-forth argument involving splitting functions for both P and Q. The inductive hypothesis is that we have a good finite isomorphism $f_{2n} : \mathcal{B}_{2n} \cong \mathcal{B}'_{2n}$, where $\mathcal{B}_{2n}$ and

$\mathcal{B}'_{2n}$ are finite subalgebras of $\mathcal{B}$. Let b be the nth element of $\mathcal{B}$ with respect to some fixed recursive enumeration of $\mathcal{B}$. Let $\mathcal{B}_{2n+1}$ be the finite subalgebra of $\mathcal{B}$ generated by $\mathcal{B}_{2n} \cup \{b\}$. Use Lemma 3.15 and a splitting function for P to effectively extend f_{2n} to a good finite isomorphism $f_{2n+1} : \mathcal{B}_{2n+1} \cong \mathcal{B}'_{2n+1}$. Then let $\mathcal{B}'_{2n+2}$ be the finite subalgebra of $\mathcal{B}$ generated by $\mathcal{B}'_{2n+1} \cup \{b\}$. Use Lemma 3.15 and a splitting function for Q to effectively extend f_{2n+1} to a good finite isomorphism $f_{2n+2} : \mathcal{B}_{2n+2} \cong \mathcal{B}'_{2n+2}$. Finally we obtain a good recursive automorphism $f = \bigcup_n f_n : \mathcal{B} \to \mathcal{B}$, and part 2 is proved. □

Remark 3.17. The ideas used in the proofs of Lemmas 3.15 and 3.16 can be traced to Myhill [21], Smullyan [33], and Pour-El/Kripke [23].

Remark 3.18. Pour-El and Kripke [23] have obtained some interesting results concerning deduction-preserving isomorphisms of theories. In the Pour-El/Kripke terminology, some of our results in this section can be reformulated as follows. Let T, T', T_1, T_2 denote consistent, recursively axiomatized theories in the predicate calculus, or in the propositional calculus. Note that the Lindenbaum sentence algebras of such theories correspond precisely to nonempty Π^0_1 subsets of 2^ω, via Stone duality. Let us say that T is *effectively essentially incomplete* if, given T' extending T, we can effectively find a sentence σ in the language of T such that both $T' \cup \{\sigma\}$ and $T' \cup \{\neg\sigma\}$ are consistent. Note that T is effectively essentially incomplete if and only if the nonempty Π^0_1 subset of 2^ω corresponding to T is productive, in the sense of Definition 3.13. Thus by Lemma 3.16 we have: (1) If T_2 is effectively essentially incomplete, then for all T_1 there exists a deduction-preserving recursive monomorphism of T_1 into T_2. (2) If T_1 and T_2 are effectively essentially incomplete, then there exists a deduction-preserving recursive isomorphism of T_1 onto T_2. Pour-El/Kripke [23] obtain similar results under the stronger hypothesis that T_2 is effectively inseparable. Our results (1) and (2) are best possible, in the sense that effective essential incompleteness is a necessary as well as sufficient condition for them to hold.

Lemma 3.19. *Let P and Q be nonempty Π^0_1 subsets of 2^ω. If $P \leq_M Q$ and P is productive, then Q is productive.*

Proof. Since $P \leq_M Q$, there is a recursive functional $\Phi : Q \to P$. By Lemma 3.11, let $f : \mathcal{B} \to \mathcal{B}$ be recursive such that $\Phi^{-1}[b] = [f(b)] \cap Q$ for all $b \in \mathcal{B}$. Let $g : \omega \to \mathcal{B}$ be a splitting function for P. The predicate $S(e, X) \equiv (X \in \Phi(P_e \cap Q))$ is Π^0_1 (see the proof of part 1 of Lemma 3.6), so by the S-m-n Theorem, let $h : \omega \to \omega$ be primitive recursive such that $P_{h(e)} = \Phi(P_e \cap Q)$ for all e. Now if $P_e \subseteq Q$ and $P_e \neq \emptyset$, we have $P_{h(e)} = \Phi(P_e) \subseteq P$ and $P_{h(e)} \neq \emptyset$, hence $P_{h(e)} \cap [gh(e)] \neq \emptyset$ and $P_{h(e)} \cap [-gh(e)] \neq \emptyset$, hence $P_e \cap [fgh(e)] \neq \emptyset$ and $P_e \cap [-fgh(e)] \neq \emptyset$. Thus $fgh : \omega \to \mathcal{B}$ is a splitting function for Q. □

Lemma 3.20. *Let P be a nonempty Π^0_1 subset of 2^ω. P is productive if and only if P is Medvedev complete.*

Proof. By Lemma 3.19, Medvedev completeness implies productiveness. By part 1 of Lemma 3.16, productiveness implies Medvedev completeness. □

Theorem 3.21. *Let P and Q be nonempty Π^0_1 subsets of 2^ω.*

1. *If P is Medvedev complete, then there exists a recursive functional from P onto Q.*
2. *If P and Q are Medvedev complete, then P and Q are recursively homeomorphic, i.e., there exists a recursive functional from P one-to-one onto Q.*

Proof. This is immediate from Lemmas 3.16 and 3.20. □

Remark 3.22. Let $\mathcal{P}_M$ be the lattice of Medvedev degrees of nonempty Π^0_1 subsets of 2^ω, as in Theorem 3.2. There are many obvious structural questions to ask about $\mathcal{P}_M$. One may ask about embeddability, initial segments, final segments, definability, automorphisms, etc. There is reason to believe that a study of structural aspects of the distributive lattice $\mathcal{P}_M$ could be more rewarding than the ongoing study of the structural aspects of $\mathcal{R}_T$, the upper semilattice of Turing degrees of recursively enumerable subsets of ω, as pursued for instance in Soare [34]. For one thing, there is a well known lack of natural examples of elements of $\mathcal{R}_T$, but there are some interesting natural examples of elements of $\mathcal{P}_M$. In particular, putting

$$\mathrm{DNR}_k = \{X \in k^\omega : \forall n\,(X(n) \neq \{n\}(n))\},$$

Jockusch [16] has shown that

$$\mathrm{DNR}_2 >_M \mathrm{DNR}_3 >_M \cdots >_M \mathrm{DNR}_k >_M \cdots,\ 2 \le k \in \omega,$$

and of course DNR_2 is Medvedev complete (see the proof of Lemma 3.14). See also Simpson [29, 30] and other FOM postings in the same thread.

4 Relativization, Iteration, ω-Models

In this section we relativize and iterate the results of §3. Our construction is inspired by the idea of iterated forcing in set theory, as exposited in Jech [15, page 458] and Kunen [19, page 273]. We show that our construction gives rise to a Π^0_1 set of countable ω-models of WKL_0 with a strong homogeneity property (Theorem 4.11).

Definition 4.1. All of the concepts and results of §3 can be *uniformly relativized* (Rogers [25, pages 128–137]) to an arbitrary $X \in 2^\omega$. Say that $P^X \subseteq 2^\omega$ is $\Pi^{0,X}_1$ if $P^X = \{Y \in 2^\omega : \forall n\, R(X,Y,n)\}$ for some recursive predicate $R \subseteq 2^\omega \times 2^\omega \times \omega$. We employ a uniform, standard, recursive enumeration $\{P^X_e : e \in \omega\}$ of the $\Pi^{0,X}_1$ subsets of 2^ω. A nonempty $\Pi^{0,X}_1$ set $P^X \subseteq 2^\omega$ is said to be *X-productive* if there exists an X-recursive *splitting function*, i.e., an X-recursive function $g : \omega \to \mathcal{B}$ such that for all e, if $\emptyset \neq P^X_e \subseteq P^X$ then $P^X_e \cap [g(e)] \neq \emptyset \neq P^X_e \cap [-g(e)]$.

Recall that for $Y \in 2^\omega$ and $e \in \omega$ we have $(Y)_e \in 2^\omega$ where $(Y)_e(n) = Y((e,n))$. (See the proofs of Theorem 3.2 and Lemma 3.3.) For $Q \subseteq 2^\omega$ put $(Q)_e = \{(Y)_e : Y \in Q\}$.

Lemma 4.2. *There is a Π^0_1 predicate $\widehat{P} \subseteq 2^\omega \times 2^\omega$ such that*

$$\forall X\, \forall e\, (\textit{if } P^X_e \neq \emptyset \textit{ then } P^X_e = (\widehat{P}^X)_e)$$

where $\widehat{P}^X = \{Y : \widehat{P}(X,Y)\}$.

Proof. This comes from a uniform relativization of the proof of Lemma 3.3. The predicate $U(e,X,Z) \equiv (Z \in P^X_e)$ is Π^0_1, so by the Normal Form Theorem for Π^0_1 predicates, let $U_1 \subseteq \omega \times 2^{<\omega} \times 2^{<\omega}$ be primitive recursive such that $U(e,X,Z) \equiv \forall n\, U_1(e,X[n],Z[n])$. Put $\widehat{P}(X,Y) \equiv \forall e\, U^+(e,X,(Y)_e)$, where $U^+(e,X,Z) \equiv$

$$\forall n\, (\forall \tau \text{ of length } n)\, (\text{if } (\forall m \leq n)\, U_1(e,X[m],\tau[m]) \text{ then } U_1(e,X[n],Z[n])).$$

It is straightforward to verify that $\widehat{P}$ has the desired property. The details are as in the proof of Lemma 3.3. □

Lemma 4.3. *With notation as in Lemma 4.2, for all $X \in 2^\omega$, $\widehat{P}^X$ is X-productive with a fixed primitive recursive splitting function $g : \omega \to \mathcal{B}$.*

Proof. This comes from a uniform relativization of Lemmas 3.14 and 3.19. Let $e_0 \in \omega$ be such that, for all $X \in 2^\omega$, $P^X_{e_0} = \{Y \in 2^\omega : \forall n\, (Y(n) \neq \{n\}^X(n))\}$. The argument for Lemma 3.14 gives a primitive recursive function $g_0 : \omega \to \mathcal{B}$ such that, for all X, $P^X_{e_0}$ is X-productive with splitting function g_0. Note also that $(Y)_{e_0} \in P^X_{e_0}$ for all $Y \in \widehat{P}^X$. Let $f_0 : \mathcal{B} \to \mathcal{B}$ be primitive recursive such that, for all $Y \in 2^\omega$ and $b \in \mathcal{B}$, $Y \in [f_0(b)]$ if and only if $(Y)_{e_0} \in [b]$. Let $h_0 : \omega \to \omega$ be primitive recursive such that, for all $X \in 2^\omega$ and all $e \in \omega$, $P^X_{h_0(e)} = \{(Y)_{e_0} : Y \in P^X_e\}$. As in the proof of Lemma 3.19 we have that, for all $X \in 2^\omega$, $\widehat{P}^X$ is X-productive with splitting function $g = f_0 g_0 h_0 : \omega \to \mathcal{B}$. □

Definition 4.4. Recall that $(Y)_i(n) = Y((i,n))$. We also put

$$(Y)^i((j,n)) = \begin{cases} Y((j,n)) & \text{if } j < i, \\ 0 & \text{otherwise.} \end{cases}$$

(Compare the dependent choice notation of Simpson [31, Definition VII.6.1 and Lemmas VIII.2.5 and VIII.2.9].) From now on, fix $\widehat{P}$ as in Lemma 4.2, and put $\widehat{Q} = \{Y \in 2^\omega : \forall i\, \widehat{P}((Y)^i,(Y)_i)\}$. Clearly $\widehat{Q}$ is a nonempty Π^0_1 subset of 2^ω.

Lemma 4.5. *For all $Y \in \widehat{Q}$ we have*

$$\{((Y)_i)_e : i,e \in \omega\} = \{X \in 2^\omega : \exists i\, (X \leq_T (Y)^i)\}$$

and this is an ω-model of WKL_0.

Proof. If $X \leq_T (Y)^i$ then clearly $\{X\} = P^{(Y)^i}_e$ for an appropriate e, hence $X = ((Y)_i)_e$. This gives $\{X : \exists i\, (X \leq_T (Y)^i)\} \subseteq \{((Y)_i)_e : i,e \in \omega\}$. The converse inclusion follows from $((Y)_i)_e \leq_T (Y)_i \leq_T (Y)^{i+1}$. To see that this is an ω-model of WKL_0, use Lemma 4.2 plus the following characterization: $S \subseteq 2^\omega$ is an ω-model of WKL_0 if and only if (i) $S \neq \emptyset$, (ii) $X \oplus Y \in S$ for all $X,Y \in S$, and (iii) for all $X \in S$ and $e \in \omega$, if $P^X_e \neq \emptyset$ then $P^X_e \cap S \neq \emptyset$. □

Definition 4.6. Let P and Q be nonempty Π^0_1 subsets of 2^ω. Let Φ be a recursive functional from Q onto P. A *splitting functional for* Φ is a recursive functional $g : P \times \omega \to \mathcal{B}$ such that for all $X \in P$, the X-recursive function $e \mapsto g^X(e)$ is a splitting function for the $\Pi^{0,X}_1$ set $\Phi^{-1}(X) = \{Y \in Q : \Phi(Y) = X\}$. We say that Φ is *productive* if there exists a splitting functional for Φ.

Lemma 4.7. *Let P_1, P_2, Q_1, Q_2 be nonempty Π^0_1 subsets of 2^ω. Suppose that Φ_i is a recursive functional from Q_i onto P_i, $i = 1, 2$. If P_1 and P_2 are recursively homeomorphic and Φ_1 and Φ_2 are productive, then Q_1 and Q_2 are recursively homeomorphic. Indeed, given a recursive homeomorphism $\Psi : P_1 \cong P_2$ and splitting functionals for Φ_1 and Φ_2, we can effectively find a recursive homeomorphism $\Psi' : Q_1 \cong Q_2$ such that $\Psi \circ \Phi_1 = \Phi_2 \circ \Psi'$.*

Proof. This is a uniform relativization of part 2 of Lemma 3.16. □

Definition 4.8. For any $i \in \omega$ and any nonempty Π^0_1 set $Q \subseteq 2^\omega$, put $(Q)_i = \{(Y)_i : Y \in Q\}$ and $(Q)^i = \{(Y)^i : Y \in Q\}$. Note that $(Q)_i$ and $(Q)^i$ are again nonempty Π^0_1 subsets of 2^ω. In particular $(Q)^0$ is a trivial Π^0_1 set, namely $(Q)^0 = \{Z_0\}$ where $Z_0 \in 2^\omega$ is defined by $Z_0(n) = 0$ for all n.

Lemma 4.9. *Let P and Q be nonempty Π^0_1 subsets of $\widehat{Q}$. Then there are a recursive homeomorphism $\Psi : P \cong Q$ and a recursive sequence of recursive homeomorphisms $\Psi^i : (P)^i \cong (Q)^i$, $i \in \omega$, such that $\Psi^i((Y)^i) = (\Psi(Y))^i$ for all $Y \in P$.*

Proof. By Lemma 4.3, the recursive functionals from $(\widehat{Q})^{i+1}$ onto $(\widehat{Q})^i$ given by $(Y)^{i+1} \mapsto (Y)^i$ are uniformly productive. Hence their restrictions, from $(P)^{i+1}$ onto $(P)^i$ and from $(Q)^{i+1}$ onto $(Q)^i$, are uniformly productive. Begin with the trivial recursive homeomorphism $\Psi^0 : (P)^0 \cong (Q)^0$. Repeatedly apply Lemma 4.7 to obtain a recursive sequence of recursive homeomorphisms $\Psi^1 : (P)^1 \cong (Q)^1$, $\Psi^2 : (P)^2 \cong (Q)^2$, …, $\Psi^i : (P)^i \cong (Q)^i$, …, such that $\Psi^i((Y)^i) = (\Psi^{i+1}((Y)^{i+1}))^i$ for all $Y \in P$. Finally $\Psi : P \cong Q$ is given by $\Psi = \lim_i \Psi^i$. □

Lemma 4.10. *Let P and Q be nonempty Π^0_1 subsets of $\widehat{Q}$. Then there exists a recursive homeomorphism $\Psi : P \cong Q$ such that for all $Y \in P$ and $Z \in Q$, if $\Psi(Y) = Z$ then $\{((Y)_i)_e : i, e \in \omega\} = \{((Z)_i)_e : i, e \in \omega\}$.*

Proof. Let $\Psi : P \cong Q$ be as in Lemma 4.9. Let $Y \in P$ and $Z \in Q$ be such that $\Psi(Y) = Z$. By Lemma 4.9 we have $\Psi^i((Y)^i) = (Z)^i$, hence $(Y)^i \equiv_T (Z)^i$ for all i, hence $\{X : \exists i\,(X \leq_T (Y)^i)\} = \{X : \exists i\,(X \leq_T (Z)^i)\}$. By Lemma 4.5 it follows that $\{((Y)_i)_e : i, e \in \omega\} = \{((Z)_i)_e : i, e \in \omega\}$. □

Theorem 4.11. *There is a nonempty Π^0_1 subset of 2^ω, $\widehat{Q}'$, with the following properties:*

1. *For all $Y \in \widehat{Q}'$, $\{(Y)_m : m \in \omega\}$ is an ω-model of WKL_0.*
2. *Any two nonempty Π^0_1 subsets of $\widehat{Q}'$ are recursively homeomorphic.*

3. *For all nonempty* Π^0_1 *sets* $P, Q \subseteq \widehat{Q}'$*, there is a recursive homeomorphism* $\Psi : P \cong Q$ *such that for all* $Y \in P$ *and* $Z \in Q$*, if* $\Psi(Y) = Z$ *then* $\{(Y)_m : m \in \omega\} = \{(Z)_m : m \in \omega\}$.

Proof. This follows from Lemmas 4.5 and 4.10 if we let $\widehat{Q}' = \{Y' : Y \in \widehat{Q}\}$, where $Y'(((i,e),n)) = Y((i,(e,n)))$ for all $i, e, n \in \omega$. (Note: Y' is not the Turing jump of Y.) □

5 Jockusch/Soare Genericity

In this section we combine the previous theorem with so-called Jockusch/Soare forcing, to obtain an ω-model of WKL_0 in which all definable elements are recursive (Theorem 5.11).

Definition 5.1. A relation $R \subseteq \omega^k$ is said to be *arithmetical* if it is first order definable over the standard model of arithmetic $(\omega, +, \cdot, 0, 1, <, =)$. We write $\mathrm{REC} = \{A \in 2^\omega : A \text{ is recursive}\}$, and $\mathrm{ARITH} = \{A \in 2^\omega : A \text{ is arithmetical}\}$.

Definition 5.2. Let $\mathcal{P}$ be the set of all nonempty Π^0_1 subsets of 2^ω. $\mathcal{D} \subseteq \mathcal{P}$ is said to be *arithmetical* if $\{e \in \omega : P_e \in \mathcal{D}\}$ is arithmetical. $\mathcal{D}$ is said to be *dense* if for all $P \in \mathcal{P}$ there exists $Q \in \mathcal{D}$ such that $Q \subseteq P$. $G \in 2^\omega$ is said to *meet* $\mathcal{D}$ if there exists $Q \in \mathcal{D}$ such that $G \in Q$. G is said to be *Jockusch/Soare generic* (cf. Jockusch/Soare [17, proof of Theorem 2.4]), or simply, *generic*, if G meets every dense arithmetical $\mathcal{D} \subseteq \mathcal{P}$.

Lemma 5.3. *Given* $P \in \mathcal{P}$*, there exists* $G \in P$ *such that* G *is generic.*

Proof. Let $\mathcal{D}_n$, $n \in \omega$ be an enumeration of the dense arithmetical subsets of $\mathcal{P}$. Construct a sequence $P_0 \supseteq P_1 \supseteq \ldots P_n \supseteq \ldots$ in $\mathcal{P}$ as follows. Begin with $P_0 = P$. Given P_n, let $P_{n+1} \subseteq P_n$ be such that $P_{n+1} \in \mathcal{D}_n$. Finally let G be the unique element of $\bigcap_n P_n$. Clearly $G \in P$ and G meets each $\mathcal{D}_n$, hence G is generic. □

Lemma 5.4. *Let* A_i*,* $i \in \omega$ *be a sequence of nonrecursive elements of* 2^ω*. Given* $P \in \mathcal{P}$*, there exists* $G \in P$ *such that* G *is generic and* $\forall i\,(A_i \not\leq_T G)$.

Proof. For all $Y \in 2^\omega$ we have $A_i \leq_T Y$ if and only if $\exists e\,\forall n\,(\{e\}^Y(n) = A_i(n))$. For $e, i \in \omega$, put $\mathcal{D}_{e,i} = \{Q \in \mathcal{P} : \exists n\,(\forall Y \in Q)\,(\{e\}^Y(n) \neq A_i(n))\}$. We claim that $\mathcal{D}_{e,i}$ is dense in $\mathcal{P}$. To see this, let $P \in \mathcal{P}$ be given. If $\forall n\,(\forall Y \in P)\,(\{e\}^Y(n) = A_i(n))$, then by Lemma 3.5 A_i is recursive, contrary to assumption. So we have $\exists n\,(\exists Y \in P)\,(\{e\}^Y(n) \neq A_i(n))$. Fix such an n and put $Q = \{Y \in P : \{e\}^Y(n) \neq A_i(n)\}$. Clearly $Q \in \mathcal{P}$ and $Q \subseteq P$ and $Q \in \mathcal{D}_{e,i}$. This proves our claim. Now let $\mathcal{D}_n$, $n \in \omega$ be an enumeration of the dense arithmetical subsets of $\mathcal{P}$. As in the proof of Lemma 5.3, given $P \in \mathcal{P}$ there exists $G \in P$ such that G meets $\mathcal{D}_n$ for all n, and G meets $\mathcal{D}_{e,i}$ for all e, i. This proves our lemma. □

Lemma 5.5. *Let $G, H \in 2^\omega$. Suppose $H \leq_T G$ and G is generic. Then H is generic, and H is truth-table reducible to G.*

Proof. We are assuming $H \leq_T G$, so let $e \in \omega$ be such that $\forall n\,(H(n) = \{e\}^G(n))$. Put $\mathcal{D}'_e = \{Q \in \mathcal{P} :$ either $\exists n\,(\forall Y \in Q)\,(\{e\}^Y(n)$ is undefined) or $\forall n\,(\forall Y \in Q)\,(\{e\}^Y(n)$ is defined)$\}$. We claim that $\mathcal{D}'_e$ is dense in $\mathcal{P}$. To see this, given $P \in \mathcal{P} \setminus \mathcal{D}'_e$, we have $\exists n\,(\exists Y \in P)\,(\{e\}^Y(n)$ is undefined), so fix such an n and put $Q = \{Y \in P : \{e\}^Y(n)$ is undefined$\}$. Then clearly $Q \subseteq P$ and $Q \in \mathcal{D}'_e$. This proves our claim. Since $\mathcal{D}'_e$ is dense arithmetical, let $Q \in \mathcal{D}'_e$ be such that $G \in Q$. It follows that $\forall n\,(\forall Y \in Q)\,(\{e\}^Y(n)$ is defined), so we have a recursive functional $\Phi : Q \to 2^\omega$ given by $\Phi(Y)(n) = \{e\}^Y(n)$, and $H = \Phi(G)$. Hence by Lemma 3.11 and Remark 3.12, H is truth-table reducible to G. To show that H is generic, let $\mathcal{D} \subseteq \mathcal{P}$ be dense arithmetical. Put $\mathcal{D}^* = \{Q^* \in \mathcal{P} : Q^* \cap Q = \emptyset$ or $\Phi(Q^* \cap Q) \in \mathcal{D}\}$. By Lemma 3.6, $\mathcal{D}^*$ is dense arithmetical. Let $Q^* \in \mathcal{D}^*$ be such that $G \in Q^*$. Then $G \in Q^* \cap Q$, so $H = \Phi(G) \in \Phi(Q^* \cap Q) \in \mathcal{D}$. This completes the proof. □

Definition 5.6. Let $L_1(Y)$ be the language of first order arithmetic with an extra function symbol Y denoting an element of 2^ω, i.e., a function $Y : \omega \to \{0, 1\}$. Let $\varphi(Y)$ be an $L_1(Y)$-sentence. Let $P \in \mathcal{P}$. We say that P *forces* $\varphi(Y)$ if $\varphi(G)$ holds for all generic G such that $G \in P$.

Lemma 5.7.

1. *Let $\varphi(Y)$ be an $L_1(Y)$-sentence. If G is generic, then $\varphi(G)$ holds if and only if there exists $P \in \mathcal{P}$ such that $G \in P$ and P forces $\varphi(Y)$.*

2. *Let $\varphi(Y, n_1, \ldots, n_k)$ be an $L_1(Y)$-formula with free variables $n_1, \ldots, n_k$. Then*

$$\{(e, n_1, \ldots, n_k) : P_e \in \mathcal{P} \text{ and } P_e \text{ forces } \varphi(Y, n_1, \ldots, n_k)\}$$

is arithmetical.

Proof. Parts 1 and 2 are proved together by a straightforward induction on the complexity of $\varphi(Y)$. If $\varphi(Y)$ is atomic, then for all $P \in \mathcal{P}$ we have that P forces $\varphi(Y)$ if and only if $\varphi(Y)$ holds for all $Y \in P$, because $\{Y \in P : \varphi(Y)\}$ and $\{Y \in P : \neg\,\varphi(Y)\}$ are elements of $\mathcal{P}$. For arbitrary $\varphi(Y)$ and $\psi(Y)$ of $L_1(Y)$, we have that $P \in \mathcal{P}$ forces $\varphi(Y) \vee \psi(Y)$ if and only if $(\forall P' \in \mathcal{P})$ (if $P' \subseteq P$ then $(\exists P'' \in \mathcal{P})\,(P'' \subseteq P'$ and either P'' forces $\varphi(Y)$ or P'' forces $\psi(Y)$)). For arbitrary $\varphi(Y, n)$ of $L_1(Y)$, we have that $P \in \mathcal{P}$ forces $\exists n\,\varphi(Y, n)$ if and only if $(\forall P' \in \mathcal{P})$ (if $P' \subseteq P$ then $(\exists P'' \in \mathcal{P})\,(\exists n \in \omega)\,(P'' \subseteq P'$ and P'' forces $\varphi(Y, n)$)). For arbitrary $\varphi(Y)$ of $L_1(Y)$, we have that $P \in \mathcal{P}$ forces $\varphi(Y)$ if and only if $(\forall P' \in \mathcal{P})$ (if $P' \subseteq P$ then P' does not force $\varphi(Y)$). □

Lemma 5.8. *Let L_2 be the language of second order arithmetic. Given an L_2-sentence σ, we can find an $L_1(Y)$-sentence $\widetilde{\sigma}(Y)$ such that for all $Y \in 2^\omega$, $\widetilde{\sigma}(Y)$ holds if and only if the ω-model $\{(Y)_m : m \in \omega\}$ satisfies σ.*

Proof. The proof is straightforward. Set quantifiers in σ are translated as number quantifiers in $\widetilde{\sigma}(Y)$. $\square$

Lemma 5.9. *With $\widehat{Q}'$ as in Theorem 4.11, let $G, H \in \widehat{Q}'$ be generic. Then the ω-models $\{(G)_m : m \in \omega\}$ and $\{(H)_m : m \in \omega\}$ satisfy the same L_2-sentences.*

Proof. Suppose $\{(G)_m : m \in \omega\}$ and $\{(H)_m : m \in \omega\}$ satisfy σ and $\neg\sigma$ respectively. Then $\widetilde{\sigma}(G)$ and $\neg\widetilde{\sigma}(H)$ hold. By part 1 of Lemma 5.7, there exist $P, Q \in \mathcal{P}$ such that $G \in P$ and $H \in Q$ and P forces $\widetilde{\sigma}(Y)$ and Q forces $\neg\widetilde{\sigma}(Y)$. Since $G, H \in \widehat{Q}'$, we may safely assume that $P, Q \subseteq \widehat{Q}'$. Let $\Psi : P \cong Q$ be a recursive homeomorphism as in part 3 of Theorem 4.11. By Lemma 5.5 we have that $G^* = \Psi(G) \in Q$ is generic. Since Q forces $\neg\widetilde{\sigma}(Y)$, it follows that $\neg\widetilde{\sigma}(G^*)$ holds, i.e., $\{(G^*)_m : m \in \omega\} \models \neg\sigma$. But, by part 3 of Theorem 4.11, we have that $\{(G)_m : m \in \omega\} = \{(G^*)_m : m \in \omega\}$. This contradiction proves our lemma. $\square$

Lemma 5.10. *Consider an ω-model $S = \{(G)_m : m \in \omega\}$ where $G \in \widehat{Q}'$ and G is generic.*

1. *For L_2-sentences σ, we have that $S \models \sigma$ if and only if $\widehat{Q}'$ forces $\widetilde{\sigma}(Y)$.*
2. *For relations $R \subseteq \omega^k$, we have that R is definable over S without parameters if and only if R is arithmetical.*
3. *For $A \in S$, we have that A is definable over S (without parameters) if and only if A is recursive.*

Proof. Parts 1 and 2 are immediate from Lemmas 5.7 and 5.9. For part 3, suppose that $A \in S$ and A is definable without parameters over S. Let $\varphi(Z)$ be an L_2-formula with one free set variable, Z, such that A is the unique $Z \in S$ such that $S \models \varphi(Z)$. Letting σ be the L_2-sentence $(\exists \text{ unique } Z)\,\varphi(Z)$, we have that $S \models \sigma$. By part 1 we have that $\widehat{Q}'$ forces $\widetilde{\sigma}(Y)$. By Lemma 5.4, let $H \in \widehat{Q}'$ be generic such that $(\forall Z \in S)\,(\text{if } Z \notin \text{REC then } Z \not\leq_T H)$. Consider the ω-model $T = \{(H)_m : m \in \omega\}$. Then $S \cap T = \text{REC}$. Since $H \in \widehat{Q}'$ and H is generic and $\widehat{Q}'$ forces $\widetilde{\sigma}(Y)$, we have that $\widetilde{\sigma}(H)$ holds, i.e., $T \models \sigma$. Let B be the unique $Z \in T$ such that $T \models \varphi(Z)$. We claim that $A = B$. This is clear from Lemma 5.9, because for all $n \in \omega$ and $k = 0, 1$, we have $A(n) = k$ if and only if $S \models \exists Z\,(\varphi(Z) \text{ and } Z(n) = k)$, if and only if $T \models \exists Z\,(\varphi(Z) \text{ and } Z(n) = k)$, if and only if $B(n) = k$. Since $A = B$, it follows that $A \in \text{REC}$. $\square$

Theorem 5.11. *There is a countable ω-model S of WKL_0 such that every definable element of S is recursive.*

Proof. This follows immediately from Theorem 4.11 and Lemma 5.3 and part 3 of Lemma 5.10. $\square$

Remark 5.12. Theorem 5.11 is due to Friedman [11, unpublished, Theorem 1.10], announced in [12, Theorem 1.6]. Our proof here is different from Friedman's proof in [11].

6 Relative Genericity and Definability

In this section we prove a key lemma concerning relativized Jockusch/Soare genericity (Lemma 6.2). We then use our lemma to obtain an improvement of Theorem 5.11, involving relative definability and relative recursiveness, i.e., Turing reducibility (Theorem 6.9).

Definition 6.1. All of the concepts and results of §5 can be straightforwardly relativized to an arbitrary $X \in 2^\omega$. We use $\mathcal{P}^X$ to denote the set of nonempty $\Pi^{0,X}_1$ subsets of 2^ω. (See Definition 4.1.) $\mathcal{D}^X \subseteq \mathcal{P}^X$ is said to be *arithmetical in* X if $\{e : P^X_e \in \mathcal{D}^X\}$ is arithmetical in X, i.e., definable over $(\omega, +, \cdot, 0, 1, <, =, X)$ by a formula of $L_1(X)$. $G \in 2^\omega$ is said to be *Jockusch/Soare generic over* X, or simply, *generic over* X, if G meets every dense subset of $\mathcal{P}^X$ which is arithmetical in X.

Lemma 6.2. *Let $G, X \in 2^\omega$. Suppose $X \leq_T G$ and G is generic. Then G is generic over X.*

Proof. Let $\mathcal{D}^X \subseteq \mathcal{P}^X$ be given such that $\mathcal{D}^X$ is dense in $\mathcal{P}^X$ and arithmetical in X. We need to show that G meets $\mathcal{D}^X$. By Lemma 5.5, there are a Π^0_1 set $Q \subseteq 2^\omega$ and a recursive functional $\Phi : Q \to 2^\omega$ such that $\Phi(G) = X$. It suffices to show that Q forces (if $\mathcal{D}^{\Phi(Y)}$ is dense in $\mathcal{P}^{\Phi(Y)}$ then Y meets $\mathcal{D}^{\Phi(Y)}$). Equivalently, we shall show $(\forall Q' \in \mathcal{P})$ (if $Q' \subseteq Q$ and Q' forces ($\mathcal{D}^{\Phi(Y)}$ is dense in $\mathcal{P}^{\Phi(Y)}$), then $(\exists Q'' \in \mathcal{P})\,(Q'' \subseteq Q'$ and Q'' forces (Y meets $\mathcal{D}^{\Phi(Y)}$))). To see this, let $Q' \in \mathcal{P}$ be given such that $Q' \subseteq Q$ and Q' forces ($\mathcal{D}^{\Phi(Y)}$ is dense in $\mathcal{P}^{\Phi(Y)}$). Put $P' = \Phi(Q')$. Using $L_1(X)$ as our forcing language, we have that P' forces ($\mathcal{D}^X$ is dense in $\mathcal{P}^X$). In particular, since P' forces $Q' \cap \Phi^{-1}(X) \in \mathcal{P}^X$, it follows that P' forces $\exists e\,(P^X_e \in \mathcal{D}^X$ and $P^X_e \subseteq Q' \cap \Phi^{-1}(X))$. Let $e \in \omega$ and $P'' \in \mathcal{P}$ be such that $P'' \subseteq P'$ and P'' forces ($P^X_e \in \mathcal{D}^X$ and $P^X_e \subseteq Q' \cap \Phi^{-1}(X)$). Put $Q'' = \{Y \in Q' : \Phi(Y) \in P''$ and $Y \in P^{\Phi(Y)}_e\}$. Then $Q'' \in \mathcal{P}$, $Q'' \subseteq Q'$, and Q'' forces ($Y \in P^{\Phi(Y)}_e$ and $P^{\Phi(Y)}_e \in \mathcal{D}^{\Phi(Y)}$), i.e., Q'' forces (Y meets $\mathcal{D}^{\Phi(Y)}$). This proves our lemma. □

Lemma 6.3. *Let $X \in 2^\omega$ be given. Suppose $P^X, Q^X \in \mathcal{P}^X$, and suppose $\Phi^X : P^X \cong Q^X$ is an X-recursive homeomorphism of P^X onto Q^X. If $G \in P^X$ is generic over X, then $\Phi^X(G) \in Q^X$ is generic over X.*

Proof. This follows from a straightforward relativization to X of Lemma 5.5. □

Definition 6.4. Let $\widehat{P}$ be as in Lemma 4.2. Relativizing Definition 4.4, put

$$\widehat{Q}^X = \{Y \in 2^\omega : \forall i\, \widehat{P}(X \oplus (Y)^i, (Y)_i)\}$$

for all $X \in 2^\omega$. Clearly $\widehat{Q}^X \in \mathcal{P}^X$.

Lemma 6.5. *Let $X \in 2^\omega$ be given. For all $Y \in \widehat{Q}^X$ we have*

$$\{((Y)_i)_e : i, e \in \omega\} = \{W \in 2^\omega : \exists i\,(W \leq_T (Y)^i)\}$$

and this is an ω-model of WKL_0 containing X.

Proof. A straightforward relativization to X of Lemma 4.5 shows that, for all $Y \in \widehat{Q}^X$, $\{((Y)_i)_e : i, e \in \omega\} = \{W \in 2^\omega : W \leq_T (Y)^i\}$ and this is an ω-model of WKL_0. Clearly $\{X\}$ is $\Pi^{0,X}_1$, hence $\{X\} = P_e^{X \oplus (Y)^0}$ for an appropriate e, hence $X = ((Y)_1)_e$. □

Lemma 6.6. *Consider an ω-model $S = \{((G)_i)_e : i, e \in \omega\}$ where $G \in \widehat{Q}$ and G is generic. For all $A \in S$, if A is definable over S, then A is recursive.*

Proof. The proof of Theorem 4.11 shows that $\widehat{Q} \cong \widehat{Q}'$ via a recursive homeomorphism $Y \mapsto Y'$ such that, for all $Y \in \widehat{Q}$, $\{((Y)_i)_e : i, e \in \omega\} = \{(Y')_m : m \in \omega\}$. In particular, $S = \{(G')_m : m \in \omega\}$ where $G' \in \widehat{Q}'$. Furthermore, by Lemma 5.5, G' is generic. Part 3 of Lemma 5.10 now gives the desired conclusion. □

Lemma 6.7. *Let $X \in 2^\omega$ be given. Consider an ω-model $S = \{((G)_i)_e : i, e \in \omega\}$ where $G \in \widehat{Q}^X$ and G is generic over X. For all $A \in S$, if A is definable over S from X, then $A \leq_T X$.*

Proof. This is a straightforward relativization to X of Lemma 6.6. □

Lemma 6.8. *Let $X \in 2^\omega$ and $i^*, e^* \in \omega$ be given. Put*

$$\widehat{Q}^X_* = \{Y \in \widehat{Q} : ((Y)_{i^*})_{e^*} = X\}.$$

If $\widehat{Q}^X_*$ is nonempty, then there exists an X-recursive homeomorphism $\Psi^X_* : \widehat{Q}^X_* \cong \widehat{Q}^X$ such that for all $Y \in \widehat{Q}^X_*$, putting $Y_* = \Psi^X_*(Y)$, we have $\{((Y)_i)_e : i, e \in \omega\} = \{((Y_*)_i)_e : i, e \in \omega\}$.

Proof. As in the proof of Lemma 4.9, construct a recursive sequence of X-recursive homeomorphisms $\Psi^{X,i}_* : (\widehat{Q}^X_*)^{i^*+i+2} \cong (\widehat{Q}^X)^{i+1}$, $i \in \omega$, such that $\Psi^{X,i}_*((Y)^{i^*+i+2}) = (\Psi^{X,i+1}_*((Y)^{i^*+i+3}))^{i+1}$ for all $Y \in \widehat{Q}^X_*$. Define $\Psi^X_* = \lim_i \Psi^{X,i}_*$. Then $\Psi^X_* : \widehat{Q}^X_* \cong \widehat{Q}^X$ is an X-recursive homeomorphism. Furthermore, for $Y \in \widehat{Q}^X_*$ and $Y_* = \Psi^X_*(Y)$, we have $(Y)^{i^*+i+2} \equiv_T (Y_*)^{i+1}$ for all $i \in \omega$. Hence $\{W : \exists i\, (W \leq_T (Y)^i)\} = \{W : \exists i\, (W \leq_T (Y_*)^i)\}$. By Lemmas 4.5 and 6.5, it follows that $\{((Y)_i)_e : i, e \in \omega\} = \{((Y_*)_i)_e : i, e \in \omega\}$. □

Theorem 6.9. *There is a countable ω-model of* WKL_0 *satisfying $\forall X\, \forall Z\, ($if Z is definable from X then $Z \leq_T X)$.*

Proof. Let $G \in \widehat{Q}$ be generic, and put $S = \{((G)_i)_e : i, e \in \omega\}$. By Lemma 4.5, S is an ω-model of WKL_0. Fix $X \in S$. Fix $i^*, e^* \in \omega$ such that $X = ((G)_{i^*})_{e^*}$. As in Lemma 6.8, put $\widehat{Q}^X_* = \{Y \in \widehat{Q} : ((Y)_{i^*})_{e^*} = X\}$, and let Ψ^X_* be an X-recursive homeomorphism of $\widehat{Q}^X_*$ onto $\widehat{Q}^X$. We have $G \in \widehat{Q}^X_*$. Put $G_* = \Psi^X_*(G) \in \widehat{Q}^X$. By Lemma 6.8, $S = \{((G_*)_i)_e : i, e \in \omega\}$. By Lemma 6.2, G is generic over X. Hence, by Lemma 6.3, G_* is generic over X. It follows by Lemma 6.7 that, for all $A \in S$, if A is definable over S from X, then $A \leq_T X$. This completes the proof. □

Remark 6.10. Our Theorem 6.9 above contradicts Friedman's Theorem 1.12 of [11, unpublished].

7 Generalization to Non-ω-Models

In this section we generalize the results of §§3,4,5,6 to countable non-ω-models of WKL_0.

As in [31, Remark I.7.6], let Σ^0_1-PA be first order Peano arithmetic with the induction scheme restricted to Σ^0_1 formulas. The following theorem is well known.

Theorem 7.1. *Let $N = (|N|, +_N, \cdot_N, 0_N, 1_N, <_N, =_N)$ be a countable model of Σ^0_1-PA. Then there exists a countable $S \subseteq P(|N|)$ such that $(N, S) \models \mathsf{WKL}_0$.*

Proof. This result is originally due to Harrington (1977, unpublished). The proof is in Simpson [31, §IX.2]. □

Thus any countable model of Σ^0_1-PA is the first order part of a countable model of WKL_0. It follows by the Gödel Completeness Theorem that Σ^0_1-PA is the first order part of WKL_0. (See Simpson [31, §IX.2].) We shall strengthen these results below (Theorems 7.6 and 7.8, Corollary 7.9).

Let N be a countable model of Σ^0_1-PA. It is well known that the familiar concepts and results of classical recursion theory can be generalized to N-*recursion theory*. See for instance Mytilinaios [22]. Let Δ^0_1-Def(N) be the set of all $A \subseteq |N|$ such that A is Δ^0_1 definable over N, i.e., both Σ^0_1 and Π^0_1 definable over N allowing parameters from $|N|$. We describe sets $A \in \Delta^0_1$-Def(N) as being N-*recursive*. It is known from Simpson [31, §IX.1] that Δ^0_1-Def(N) is the smallest $S \subseteq P(|N|)$ such that $(N, S) \models \mathsf{RCA}_0$. Thus Δ^0_1-Def(N) in N-recursion theory plays the role of REC in classical recursion theory.

A set $F \subseteq |N|$ is said to be N-*finite* if it is N-recursive and bounded in N, or equivalently, if there exists an N-canonical index of F. By an N-*canonical index* of F, we mean an $n \in |N|$ such that $F = F_n = \{m \in |N| : N \models mEn\}$, where mEn is the usual Σ^0_0 formula asserting that 2^m occurs in the binary expansion of n, i.e., $(\exists x < n)\,(\exists y < 2^m)\,(n = (2x+1)2^m + y)$. Compare Rogers [25, §5.6]. The N-finite sets play the role of finite sets in classical recursion theory. A set $A \subseteq |N|$ is said to be N-*regular* if $A \cap F$ is N-finite for each N-finite $F \subseteq |N|$. By Bounded Σ^0_1 Comprehension [31, Theorem II.3.9], every N-recursively enumerable set is N-regular. In this paper we shall have no use for sets which are not N-regular. If $S \subseteq P(|N|)$ is such that $(N, S) \models \mathsf{RCA}_0$, then every $A \in S$ is necessarily N-regular.

We denote by $(2^\omega)_N$ the set of all N-regular functions $X : |N| \to \{0, 1\}$. We denote by $(2^{<\omega})_N$ the set of N-*strings*, i.e., (N-canonical indices of) N-finite sequences of 0's and 1's. Clearly $(2^{<\omega})_N$ and the length function $\mathrm{lh}_N : (2^{<\omega})_N \to |N|$ are N-recursive. For $X \in (2^\omega)_N$ and $n \in |N|$, we have an N-string $X[n] = \langle X(0), \ldots, X(n-1)\rangle \in (2^{<\omega})_N$ of length n. We denote by $\mathcal{P}_N$ the set of nonempty sets of the form $\{X \in (2^\omega)_N : (\forall n \in |N|)\,(X[n] \in T)\}$ where $T \subseteq (2^{<\omega})_N$ is N-recursive. The sets $P \in \mathcal{P}_N$ in N-recursion theory play the role of nonempty Π^0_1 subsets of 2^ω in classical recursion theory.

We say that $P \in \mathcal{P}_N$ is *complete* if for every $Q \in \mathcal{P}_N$ there exists an N-recursive functional $\Phi : P \to Q$. Generalizing Theorem 3.21, we have:

Theorem 7.2. *Let N be a countable model of Σ^0_1-PA, and let $P, Q \in \mathcal{P}_N$. If P is complete, there exists an N-recursive functional from P onto Q. If P and Q are complete, there exists an N-recursive functional $\Phi : P \cong Q$.*

Proof. This is a straightforward generalization of the arguments of §3. □

Generalizing Theorem 4.11, we have:

Theorem 7.3. *Let N be a countable model of Σ^0_1-PA. We can find $(\widehat{Q}')_N \in \mathcal{P}_N$ with the following properties:*

1. *For all $Y \in (\widehat{Q}')_N$, if $(N, \{(Y)_n : n \in |N|\}) \models \mathsf{RCA}_0$ then $(N, \{(Y)_n : n \in |N|\}) \models \mathsf{WKL}_0$.*

2. *For all $P, Q \in \mathcal{P}_N$ such that $P, Q \subseteq (\widehat{Q}')_N$, there exists an N-recursive functional $\Psi : P \cong Q$ such that for all $Y \in P$ and $Z \in Q$, if $\Psi(Y) = Z$ then $\{(Y)_n : n \in |N|\} = \{(Z)_n : n \in |N|\}$.*

Proof. This is a straightforward generalization of the arguments of §4. □

For $G \in (2^\omega)_N$ the notion of *Jockusch/Soare genericity over N* is defined in the obvious way, in terms of dense subsets of $\mathcal{P}_N$ which are definable over N allowing parameters from $|N|$. This notion is equivalent to genericity over $(N, \Delta^0_1\text{-Def}(N))$ in the sense of Simpson [31, §IX.2]. Generalizing Lemma 5.3, we have:

Lemma 7.4. *Let N be a countable model of Σ^0_1-PA. For any $P \in \mathcal{P}_N$ there exists $G \in P$ such that G is Jockusch/Soare generic over N. For any such G we have $(N, \Delta^0_1\text{-Def}(N, G)) \models \mathsf{RCA}_0$.*

Proof. See Simpson [31, §IX.2]. □

Generalizing Lemma 5.4, we have:

Lemma 7.5. *Let N be a countable model of Σ^0_1-PA, and suppose*

$$\{A_i : i \in \omega\} \cap \Delta^0_1\text{-Def}(N) = \emptyset.$$

Then for any $P \in \mathcal{P}_N$ there exists $G \in P$ such that

$$\{A_i : i \in \omega\} \cap \Delta^0_1\text{-Def}(N, G) = \emptyset,$$

and G is Jockusch/Soare generic over N.

Proof. Combine the proof of Lemma 7.4 with a straightforward generalization of the proof of Lemma 5.4. □

Generalizing Theorem 5.11, we have:

Theorem 7.6. *Let N be a countable model of Σ^0_1-PA. Then there exists a countable $S \subseteq P(|N|)$ such that $(N, S) \models \mathsf{WKL}_0$ and furthermore, every element of S which is definable over (N, S) allowing parameters from $|N|$ is N-recursive.*

Proof. This is a straightforward generalization of the arguments of §5. □

Remark 7.7. Theorems 7.2, 7.3 and 7.6 and Lemma 7.5 are originally due to Simpson/Tanaka/Yamazaki [32].

Generalizing Theorem 6.9, we have:

Theorem 7.8. *Let N be a countable model of Σ^0_1-PA. Then there exists a countable $S \subseteq P(|N|)$ such that $(N, S) \models \mathsf{WKL}_0$ and furthermore, for all $X, Z \in S$, if Z is definable over (N, S) allowing parameters from $|N| \cup \{X\}$, then $Z \in \Delta^0_1$-$\mathrm{Def}(N, X)$.*

Proof. This is a straightforward generalization of the arguments of §6. □

Corollary 7.9. *Σ^0_1-PA is the first order part of the L_2-theory consisting of WKL_0 plus the following scheme:*

$$\forall X\, (\textit{if } (\exists \textit{ exactly one } Z)\, \varphi(X, Z) \textit{ then } (\exists Z)\, (Z \leq_T X \textit{ and } \varphi(X, Z))),$$

where $\varphi(X, Z)$ is any L_2-formula with no free set variables other than X, Z.

Proof. This follows from Theorem 7.8 plus Gödel's Completeness Theorem. □

8 A Result of Kučera

In this section we present a simplified proof of a recursion-theoretic result of Kučera [20]. Our proof is based on two easy, well-known lemmas. We present the proof in detail now, because later we shall need to generalize it to the context of N-recursion theory where N is a model of Σ^0_1-PA.

Lemma 8.1. *There exists a recursively enumerable set $A \subseteq \omega$ such that the complement $\overline{A}$ is infinite and, for all x, if $\{x\}(x)\downarrow$ then $\{x\}_{a_x}(x)\downarrow$, where*

$$\overline{A} = \{a_0 < a_1 < \cdots < a_x < \cdots\}\,.$$

Proof. We use a movable marker argument, as in Rogers [25, pages 235–236]. We shall have $a_x = \lim_s a^s_x$ where a^s_x is the position of marker x at stage s. Thus

$$\overline{A^s} = \{a^s_0 < a^s_1 < \cdots < a^s_x < \cdots\}\,.$$

Stage 0. Begin by defining $a^0_x = x$ for all x. In other words, $A^0 = \emptyset$.

Stage $s+1$. Let x_s be the least x such that $\{x\}_s(x)\downarrow$ and $\{x\}_{a^s_x}(x)\uparrow$. If x_s is undefined, let $A^{s+1} = A^s$. Thus $a^{s+1}_x = a^s_x$ for all x. If x_s is defined, let $A^{s+1} = A^s \cup \{a^s_x, \ldots, a^s_{s-1}\}$. Thus

$$a^{s+1}_x = \begin{cases} a^s_x & \text{if } x < x_s, \\ a^s_{s+x-x_s} & \text{if } x \geq x_s. \end{cases}$$

and in particular $a^{s+1}_{x_s} = a^s_s \geq s$.

Finally put $A = \bigcup_s A^s$. Clearly A is a recursively enumerable set. Note that x_s takes on each possible value at most once. Hence $x_s \to \infty$ as $s \to \infty$, and it is clear that the construction works. □

The following lemma is a strengthening due to K. Ohashi of the well-known Splitting Theorem of R. Friedberg. See Rogers [25, Exercise 12.21].

Lemma 8.2. *Let A be a nonrecursive, recursively enumerable set. Then there exists a pair of disjoint, recursively inseparable, recursively enumerable sets B_1, B_2 such that $A = B_1 \cup B_2$.*

Proof. Let $f : \omega \to \omega$ be a one-to-one recursive function such that A is the range of f. Put $A^s = \{f(0), \ldots, f(s)\}$. Let W_x $(x \in \omega)$ be a standard, recursive enumeration of the recursively enumerable sets. Let W_x^s be the finite set of elements enumerated into W_x prior to stage $s+1$.

To construct B_1, B_2 we use a no-injury priority argument. For each $i = 1, 2$ and $x = 0, 1, 2, \ldots$ there is a requirement R_{2x+i} to the effect that $B_i \cap W_x \neq \emptyset$ "if possible". The ordering of the requirements is $R_1, R_2, \ldots, R_{2x+1}, R_{2x+2}, \ldots$.

Stage 0. Put $B_1^0 = B_2^0 = \emptyset$.

Stage $s+1$. Let x_s be the least x such that $f(s) \in W_x^s$ and either $B_1^s \cap W_x^s = \emptyset$ or $B_2^s \cap W_x^s = \emptyset$. If x_s does not exist, or if $B_1^s \cap W_x^s = \emptyset$, put $f(s)$ into B_1, i.e., $B_1^{s+1} = B_1^s \cup \{f(s)\}$ and $B_2^{s+1} = B_2^s$. Otherwise put $f(s)$ into B_2, i.e., $B_1^{s+1} = B_1^s$ and $B_2^{s+1} = B_2^s \cup \{f(s)\}$.

Finally put $B_1 = \bigcup_s B_1^s$ and $B_2 = \bigcup_s B_2^s$. Clearly B_1 and B_2 are recursively enumerable sets, and $A = B_1 \cup B_2$. It is also clear that x_s takes on each possible value at most twice, hence $x_s \to \infty$ as $s \to \infty$.

We claim that B_1 and B_2 are recursively inseparable. Assume for a contradiction that X is a recursive set which separates B_1 and B_2, i.e., $B_1 \subseteq X$ and $B_2 \cap X = \emptyset$. Let x and y be such that $W_x = X$ and $W_y = \overline{X}$. Then $W_x \cap B_2 = \emptyset$ and $W_y \cap B_1 = \emptyset$. For all sufficiently large s we have $x_s > x, y$, hence $f(s) \notin W_x^s \cup W_y^s$. Thus there is a finite set F such that for all $a \in A \setminus F$ there exists s such that $a \in A^s \setminus (W_x^s \cup W_y^s)$. On the other hand, we also have that for all $a \in \overline{A}$ there exists s such that $a \in (W_x^s \cup W_y^s) \setminus A^s$. It follows that A is recursive, a contradiction. □

The following theorem and its corollaries are due to Kučera [20].

Theorem 8.3. *There exists a disjoint, recursively inseparable pair of recursively enumerable sets B_1, B_2 with the following property. If X and Y are separating sets, then for the symmetric difference $X \bigtriangleup Y = (X \setminus Y) \cup (Y \setminus X)$ we have that either $X \bigtriangleup Y$ is finite or $K \leq_T X \bigtriangleup Y$. Here $K = \{x : \{x\}(x)\downarrow\}$, the complete recursively enumerable set.*

Proof. Let A be a recursively enumerable set as in Lemma 8.1. Then clearly $K \leq_T Z$ for any infinite set $Z \subseteq \overline{A}$. By Lemma 8.2 let B_1, B_2 be a disjoint, recursively inseparable pair of recursively enumerable sets such that $A = B_1 \cup B_2$. Now let X and Y be separating sets. Then $B_1 \subseteq X \cap Y$ and $B_2 \cap (X \cup Y) = \emptyset$. Hence $X \bigtriangleup Y \subseteq \overline{A}$ and we have our result. □

The previous theorem is of interest with respect to Π^0_1 subsets of 2^ω. An extensive survey of this part of recursion theory is Cenzer/Remmel [3]. It is known that a nonempty Π^0_1 subset of 2^ω with no recursive elements necessarily has elements of $2^{\aleph_0}$ distinct Turing degrees, among which are infinitely many pairwise incomparable Turing degrees $< \mathbf{0}'$. We now get:

Corollary 8.4. *There exists a nonempty Π^0_1 set $P \subseteq 2^\omega$ with no recursive elements, such that if $\mathbf{a}$ and $\mathbf{b}$ are Turing degrees of elements of P, then either $\mathbf{a} = \mathbf{b}$ or $\mathbf{a} \cup \mathbf{b} \geq \mathbf{0}'$.*

Proof. Let B_1, B_2 be as in Theorem 8.3. Let P be the Π^0_1 set of (characteristic functions of) separating sets for B_1, B_2. If $\mathbf{a}$ and $\mathbf{b}$ are the Turing degrees of $X, Y \in P$ respectively, it follows from the conclusion of Theorem 8.3 that either $\mathbf{a} = \mathbf{b}$ or $\mathbf{a} \cup \mathbf{b} \geq \mathbf{0}'$. □

If $\mathbf{a}$ is a Turing degree, we write $\mathbf{a} \gg \mathbf{0}$ to mean that every nonempty Π^0_1 subset of 2^ω contains at least one element of Turing degree $\leq \mathbf{a}$. This is equivalent to $\mathbf{a}$ being the degree of a complete extension of Peano arithmetic. See also Jockusch/Soare [17] and Simpson [28, §6]. It is known that there exist $\mathbf{a} \gg \mathbf{0}$ and $\mathbf{b} \gg \mathbf{0}$ such that $\mathbf{a} \cap \mathbf{b} = \mathbf{0}$. We now get:

Corollary 8.5. *If $\mathbf{a} \gg \mathbf{0}$ and $\mathbf{b} \gg \mathbf{0}$ and $\mathbf{a} \cap \mathbf{b} = \mathbf{0}$, then $\mathbf{a} \cup \mathbf{b} \geq \mathbf{0}'$.*

9 An Application to ω-Models

In this section we apply Kučera's result to the study of ω-models of WKL_0. For background on this subject, see Simpson [31, §VIII.2].

It is known that minimal ω-models of WKL_0 do not exist, i.e., every ω-model of WKL_0 has a proper ω-submodel of WKL_0. It is also known that the intersection of all ω-models of WKL_0 is REC, the set of recursive sets. We now have:

Theorem 9.1. *There exists a countable ω-model S of WKL_0 such that*

$$\bigcap\{S' \subseteq S : S' \text{ is an } \omega\text{-model of } \mathsf{WKL}_0\} \neq \mathrm{REC}.$$

Here REC *is the set of recursive sets.*

Proof. Let B_1, B_2 be as in Theorem 8.3. Since WKL_0 proves Σ^0_1 separation [31, Lemma IV.4.4], every ω-model of WKL_0 contains a separating set for B_1, B_2. Let S be an ω-model of WKL_0 such that $K \notin S$, where K is the complete recursively enumerable set. Let $X \in S$ be a separating set for B_1, B_2. Obviously X is not recursive. We claim that $X \in \bigcap\{S' \subseteq S : S' \models \mathsf{WKL}_0\}$. Given $S' \subseteq S$ such that $S' \models \mathsf{WKL}_0$, let $Y \in S'$ be a separating set for B_1, B_2. Since $X, Y \in S$ but $K \notin S$, the conclusion of Theorem 8.3 implies that the symmetric difference $X \triangle Y$ is finite. Since $Y \in S'$, it follows that $X \in S'$. This gives our result. □

Remark 9.2. For any ω-model S of WKL_0, it can be shown that $\bigcap\{S' \subseteq S : S'$ is an ω-model of $\mathsf{WKL}_0\} = \mathrm{REC}$ if and only if $K \in S$.

The above theorem concerning ω-models of WKL_0 is in contrast to the following theorem of Simpson [31, Corollary VIII.6.10] concerning β-models of ATR_0. This is one instance where the well-known analogy [31, pages 40 and 314] between ω-models of WKL_0 and β-models of ATR_0 breaks down.

Theorem 9.3. *If S is a β-model of ATR_0, then*

$$\bigcap\{S' \subseteq S : S' \text{ is a } \beta\text{-model of } \mathsf{ATR}_0\} = \mathrm{HYP}.$$

Here HYP *is the set of hyperarithmetical sets.*

Actually Simpson [31, Corollary VIII.6.10] obtains not only Theorem 9.3 for β-models of ATR_0, but also an appropriate generalization for arbitrary models of ATR_0.

10 Applications to Non-ω-Models

In this section we generalize Kučera's result and apply the generalization to the study of non-ω-models of WKL_0. For background on non-ω-models of WKL_0, see Simpson [31, §VIII.2].

Lemma 10.1. *There exists a Σ^0_1 formula $\varphi(n,i)$ with the following properties. Let $\psi(X)$ be the Π^0_1 formula $\forall n\,((\varphi(n,1) \rightarrow n \in X) \wedge (\varphi(n,0) \rightarrow n \notin X))$. Then*

1. WKL_0 *proves* $\exists X\, \psi(X)$.
2. RCA_0 *proves* $\forall X\, (\psi(X) \rightarrow X$ *is not recursive*).
3. RCA_0 *proves* $\forall X\, \forall Y\, ((\psi(X) \wedge \psi(Y)) \rightarrow (X \triangle Y$ *is finite or* $K \leq_T X \triangle Y))$.

Here K is the complete recursively enumerable set.

Proof. This follows from a straightforward formalization of our proof of Theorem 8.3 via Lemmas 8.1 and 8.2 above, with $\varphi(n,i) \equiv (n \in B_1 \wedge i = 1) \vee (n \in B_2 \wedge i = 0)$. The key point for the success of the priority argument for Lemma 8.2 is that $x_s \rightarrow \infty$ as $s \rightarrow \infty$. This is provable in RCA_0 because of Bounded Σ^0_1 Comprehension [31, Theorem II.3.9]. □

Remark 10.2. For applications of Bounded Σ^0_1 Comprehension in formalization of more sophisticated priority arguments, see Mytilinaios [22].

Theorem 10.3. *Let (N,S) be a model of WKL_0 + "K does not exist". Then with the Π^0_1 formula $\psi(X)$ of Lemma 10.1, we have that (N,S) satisfies*

1. $\exists X\, \psi(X)$
2. $\forall X\, (\psi(X) \rightarrow X$ *is not recursive*)
3. $\forall X\, \forall Y\, ((\psi(X) \wedge \psi(Y)) \rightarrow X \triangle Y$ *is finite*).

Furthermore, $\bigcap\{S' \subseteq S : (N,S') \models \mathsf{WKL}_0\} \neq \Delta^0_1\text{-Def}(N)$.

Proof. That (N, S) satisfies 1, 2, 3 is immediate from Lemma 10.1. Let $X \in S$ be such that $(N, S) \models \psi(X)$. Then $X \notin \Delta^0_1$-Def(N), but for every $S' \subseteq S$ such that $(N, S') \models \mathsf{WKL}_0$ we have $X \in S'$. See also Theorem 9.1 and its proof. □

Remark 10.4. Friedman [12, Theorem 1.7] states the following result: If $\varphi(X)$ is an arithmetical formula and WKL proves $\exists X\,(\varphi(X) \wedge X$ is not recursive), then WKL proves $\forall X\,\exists Y\,(\varphi(Y) \wedge Y$ is not recursive $\wedge\ \forall n\,(Y \neq (X)_n))$. But this is refuted by our Theorem 10.3, taking $\varphi(X)$ to be the Π^0_1 formula $\psi(X)$.

As in §7, let Σ^0_1-PA be first order Peano arithmetic with the induction scheme restricted to Σ^0_1 formulas. For $k \geq 1$ we define Σ^0_k-PA similarly, with the induction scheme restricted to Σ^0_k formulas.

Corollary 10.5. *Any countable model of* Σ^0_1-PA *is the first order part of a countable model of* WKL_0 *with the properties mentioned in Theorem* 10.3.

Proof. Let N be a countable model of Σ^0_1-PA. By [31, Exercise IX.2.8], there exists a countable $S \subseteq P(|N|)$ such that (N, S) is a model of WKL_0 + "K does not exist". Our result now follows from Theorem 10.3. □

Corollary 10.6. *Let N be a model of* Σ^0_1-PA *which is not a model of* Σ^0_2-PA*. Then any model of* WKL_0 *having N as its first order part has the properties mentioned in Theorem* 10.3.

Proof. Any model of WKL_0 having N as its first order part is of the form (N, S) where $S \subseteq P(|N|)$. We claim that (N, S) necessarily satisfies "K does not exist". Otherwise, let $K \in S$ be such that $(N, S) \models$ "K is the complete recursively enumerable set". Clearly any Σ^0_2 formula without set parameters is equivalent over (N, S) to a Σ^0_1 formula with parameter K. Since WKL_0 includes induction for all Σ^0_1 formulas with set parameters, (N, S) satisfies induction for all Σ^0_1 formulas with parameter K. Hence (N, S) satisfies induction for all Σ^0_2 formulas without set parameters. In other words, $N \models \Sigma^0_2$-PA, a contradiction. This proves the claim. Our result now follows from Theorem 10.3. □

Theorem 10.7. *There is a Π^0_1 formula $\widetilde{\psi}(X)$ with no free variables other than X, such that*

1. WKL_0 *proves* $\exists X\,\widetilde{\psi}(X)$.
2. RCA_0 *proves* $\forall X\,\forall Y\,((\widetilde{\psi}(X) \wedge \widetilde{\psi}(Y)) \rightarrow X \triangle Y$ *is finite*).
3. WKL_0 *does not prove* $(\exists$ *recursive* $X)\,\widetilde{\psi}(X)$.
4. RCA_0 *does not prove* $\exists X\,\widetilde{\psi}(X)$.

Proof. Let $\psi(X)$ be the Π^0_1 formula of Lemma 10.1. Let $\forall n\,\theta(n)$ be a Π^0_1 sentence which is provable in Σ^0_2-PA but not in Σ^0_1-PA. (For instance, we may take $\forall n\,\theta(n)$ to be the Π^0_1 sentence expressing consistency of Σ^0_1-PA.) We may assume that $\theta(n)$ is Σ^0_0. Let $\widetilde{\psi}(X)$ be the Π^0_1 formula

$\forall n\,(\text{if }(\forall m \le n)\,(m \notin X)\text{ then }\theta(n))$, and
$\forall n\,(\text{if }n = \text{least element of }X\text{ then }\neg\,\theta(n)\text{ and }\psi(\{k : n{+}1{+}k \in X\}))$.

Reasoning in RCA_0, suppose X is such that $\widetilde{\psi}(X)$ holds. If $X = \emptyset$ then we have $\forall n\,\theta(n)$, hence $\forall X\,(\widetilde{\psi}(X) \to X = \emptyset)$. Now suppose $X \neq \emptyset$. Then $X = \{n_0\} \cup \{n_0 + 1 + k : k \in X_0\}$ where $\psi(X_0)$ holds and n_0 is the least n such that $\neg\,\theta(n)$. Since $\forall n\,\theta(n)$ fails, Σ^0_2-PA fails. Hence by Corollary 10.6 we have $\forall X_0\,\forall Y_0\,((\psi(X_0) \wedge \psi(Y_0)) \to X_0 \triangle Y_0$ is finite). This implies $\forall X\,\forall Y\,((\widetilde{\psi}(X) \wedge \widetilde{\psi}(Y)) \to X \triangle Y$ is finite). The rest follows easily from Theorem 10.3. □

Remark 10.8. For Π^0_1 formulas $\varphi(X)$ with no free set variables other than X, it is known that WKL_0 proves

$$(\exists\text{ exactly one }X)\,\varphi(X) \to (\exists\text{ recursive }X)\,\varphi(X).$$

(This follows from [31, Lemma VIII.2.4.2] using Δ^0_1 comprehension.) One might conjecture that this result would continue to hold with "($\exists$ exactly one X)" weakened to "($\exists$ countably many X)". However, such a conjecture is refuted by Theorem 10.7, taking $\varphi(X)$ to be $\widetilde{\psi}(X)$.

Remark 10.9. Tanaka [36] conjectured that WKL_0 is conservative over RCA_0 for sentences of the form ($\exists$ countably many X) $\varphi(X)$, where $\varphi(X)$ is arithmetical with no free set variables other than X. This conjecture is refuted by Theorem 10.7, taking $\varphi(X)$ to be the Π^0_1 formula $\widetilde{\psi}(X)$.

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