Logic and Mathematics

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This article is an overview of logic and the philosophy of mathematics. It is intended for the general reader. It has appeared in the volume *The Examined Life: Readings from Western Philosophy from Plato to Kant*, edited by Stanley Rosen, published in 2000 by Random House.

Contents

1 Logic

Logic is the science of formal principles of reasoning or correct inference. Historically, logic originated with the ancient Greek philosopher Aristotle. Logic was further developed and systematized by the Stoics and by the medieval scholastic philosophers. In the late 19th and 20th centuries, logic saw explosive growth, which has continued up to the present.

One may ask whether logic is part of philosophy or independent of it. According to Bocheński [2, §10B], this issue is nowhere explicitly raised in the writings of Aristotle. However, Aristotle did go to great pains to formulate the basic concepts of logic (terms, premises, syllogisms, etc.) in a neutral way, independent of any particular philosophical orientation. Thus Aristotle seems to have viewed logic not as part of philosophy but rather as a tool or instrument[1] to be used by philosophers and scientists alike. This attitude about logic is in agreement with the modern view, according to which the predicate calculus (see 1.2 below) is a general method or framework not only for philosophical reasoning but also for reasoning about any subject matter whatsoever.

Logic is the science of correct reasoning. What then is reasoning? According to Aristotle [13, Topics, 100a25], reasoning is any argument in which certain assumptions or *premises* are laid down and then something other than these necessarily follows. Thus logic is the science of necessary inference. However, when logic is applied to specific subject matter, it is important to note that not all logical inference constitutes a scientifically valid demonstration. This is because a piece of formally correct reasoning is not scientifically valid unless it is based on a true and primary starting point. Furthermore, any decisions about what is true and primary do not pertain to logic but rather to the specific subject matter under consideration. In this way we limit the scope of logic, maintaining a sharp distinction between logic and the other sciences. All reasoning, both scientific and non-scientific, must take place within the logical framework, but it is only a framework, nothing more. This is what is meant by saying that logic is a *formal* science.

For example, consider the following inference:

> Some real estate will increase in value.
> Anything that will increase in value is a good investment.
> Therefore, some real estate is a good investment.

This inference is logically correct, because the conclusion "some real estate is a good investment" necessarily follows once we accept the premises "some real estate will increase in value" and "anything that will increase in value is a good investment". Yet this same inference may not be a demonstration of its conclusion, because one or both of the premises may be faulty. Thus logic can help us to clarify our reasoning, but it can only go so far. The real issue in this particular inference is ultimately one of finance and economics, not logic.

[1] The Greek word for instrument is *organon*. The collection of Aristotle's logical writings is known as the Organon.

We shall now briefly indicate the basics of Aristotelean logic.

1.1 Aristotelean logic

Aristotle's collection of logical treatises is known as the Organon. Of these treatises, the Prior Analytics contains the most systematic discussion of formal logic. In addition to the Organon, the Metaphysics[2] also contains relevant material. See Aristotle [13] and Ross [19].

1.1.1 Subjects and predicates

Aristotelean logic begins with the familiar grammatical distinction between subject and predicate. A *subject* is typically an individual entity, for instance a man[3] or a house or a city. It may also be a class of entities, for instance all men. A *predicate* is a property or attribute or mode of existence which a given subject may or may not possess. For example, an individual man (the subject) may or may not be skillful (the predicate), and all men (the subject) may or may not be brothers (the predicate).

The fundamental principles of predication are:

1. Identity. Everything is what it is and acts accordingly. In symbols:

   $$A \text{ is } A.$$

   For example, an acorn will grow into an oak tree and nothing else.

2. Non-contradiction. It is impossible for a thing both to be and not to be. A given predicate cannot both belong and not belong to a given subject in a given respect at a given time. Contradictions do not exist. Symbolically:

   $$A \text{ and non-}A \text{ cannot both be the case.}$$

   For example, an honest man cannot also be a thief.

3. Either-or. Everything must either be or not be. A given predicate either belongs or does not belong to a given subject in a given respect at a given time. Symbolically:

   $$\text{Either } A \text{ or non-}A.$$

   For example, a society must be either free or not free.

[2]The Metaphysics is Aristotle's treatise on the science of existence, i.e., being as such. It includes a detailed analysis of the various ways in which a thing can be said to be.

[3]We use *man* in the traditional sense, equivalent to "human being". There is no intention to exclude persons of the female gender.

These principles have exercised a powerful influence on subsequent thinkers. For example, the 20th century intellectual Ayn Rand titled the three main divisions of her best-selling philosophical novel *Atlas Shrugged*[4] [18] after the three principles above, in tribute to Aristotle.

1.1.2 Syllogisms

According to Aristotelean logic, the basic unit of reasoning is the *syllogism.* For example, the real estate inference which was presented above is a syllogism. It is of the form

Some A is B.
All B is C.
Therefore, some A is C.

Here A denotes real estate, B denotes increase in value, and C denotes a good investment. Just as in the case of this example, every syllogism consists of two premises and one conclusion. Each of the premises and the conclusion is of one of four types:

universal affirmative:	All A is B.
universal negative:	No A is B.
particular affirmative:	Some A is B.
particular negative:	Some A is not B.

The letters A, B, C are known as *terms.* Every syllogism contains three terms. The two premises always share a common term which does not appear in the conclusion. This is known as the *middle term.* In our real estate example, the middle term is B, i.e., that which increases in value.

In order to classify the various types of syllogisms, one must take account of certain symmetries. In particular, "no A is B" and "no B is A" are equivalent, as are "some A is B" and "some B is A". Furthermore, the order of the two premises in a syllogism does not matter. Allowing for these symmetries, we can enumerate a total of 126 possible syllogistic forms. Of these 126, only 11 represent correct inferences. For example, the form

all A is B, all B is C, therefore all A is C

represents a correct inference, while

all A is B, all C is B, therefore some A is C

does not.

The classification of syllogisms leads to a rather complex theory. Medieval thinkers perfected it and developed ingenious mnemonics to aid in distinguishing the correct forms from the incorrect ones. This culminated in the famous *pons asinorum* ("bridge of asses"), an intricate diagram which illustrates all of the syllogistic forms by means of a contrast between the good and the pleasurable. See Bocheński [2, §24H, §32F].

[4] A survey conducted for the Book-of-the-Month Club and the Library of Congress in 1991 found that *Atlas Shrugged* is the most influential book in the United States of America, second only to the Bible. See `http://www.lcweb.loc.gov/loc/cfbook/bklists.html`.

1.2 The predicate calculus

In 1879 the German philosopher Gottlob Frege published a remarkable treatise, the Begriffsschrift ("concept script") [22]. This brilliant monograph is the origin of modern logical theory. However, Frege's account was defective in several respects, and notationally awkward to boot. Instead of Frege's system, we shall present a streamlined system known as *first-order logic* or *the predicate calculus.*

The predicate calculus dates from the 1910's and 1920's. It is basic for all subsequent logical research. It is a very general system of logic which accurately expresses a huge variety of assertions and modes of reasoning. We shall see that it is much more flexible than the Aristotelean syllogistic.

1.2.1 Predicates and individuals

In the predicate calculus, the subject/predicate distinction is drawn somewhat differently from the way it is drawn in Aristotelean logic. The main point here is that, in the predicate calculus, a subject is always an individual entity, never a class of entities. For example, an individual man can be treated as a subject, but the class of all men must be treated as a predicate. Since a subject in the predicate calculus is always an individual entity, it is usual to speak of *individuals* rather than subjects. We shall follow this customary practice.

The predicate calculus makes heavy use of symbolic notation. Lower-case letters a, b, c, $\ldots$, x, y, z, $\ldots$ are used to denote individuals. Upper-case letters M, N, P, Q, R, $\ldots$ are used to denote predicates. Simple assertions may be formed by juxtaposing a predicate with an individual.

For example, if M is the predicate "to be a man" and a is the individual "Socrates", then Ma denotes the assertion "Socrates is a man". The symbol a is called an *argument* of M. The predicate M may be applied to any individual, and that individual is then an argument of M. If b is the individual "New York", then Mb asserts, falsely, that New York is a man. In general, if x is any individual whatsoever, then Mx is the assertion that x is a man. This assertion may or may not be true, depending on what x is. The expression Mx is called an *atomic formula* of the predicate calculus.

Some predicates require more than one argument. For example, if B is the predicate "bigger than", then Bxy denotes the assertion "x is bigger than y". Thus B requires two arguments, and Bxy is an atomic formula. If we try to use B with only one argument, we obtain something like Bx, i.e., "x is bigger than". This is not an atomic formula or any other kind of assertion. It is only a meaningless combination of symbols. In analogy with English grammar, we could say that Bxy is like a grammatically correct sentence, while Bx is merely a sentence fragment. Such fragments play no role in the predicate calculus.

Let us now go into more detail about the role of individuals in the predicate calculus. We have already said that lower-case letters denote individuals. We now divide the lower-case letters into two groups: a, b, c, $\ldots$ near the beginning of the alphabet, and x, y, z, $\ldots$ near the end of the alphabet. We insist on an important grammatical or logical distinction between these two groups. Letters of the first group are known as *individual constants* or simply *constants*. As in the above examples, we think of them as denoting specific individuals, such as Socrates or New York. Letters of the second group are known as *individual variables* or simply *variables*. For example, x is a variable. We think of x as denoting not a specific individual but rather an arbitrary or unspecified individual.[5]

1.2.2 Formulas and logical operators

We have already mentioned two kinds of symbols: lower-case letters for individuals (constants and variables), and upper-case letters for predicates. In addition to these, the predicate calculus employs seven special symbols known as *logical operators*[6]:

$$\& \qquad \lor \qquad \sim \qquad \supset \qquad \equiv \qquad \forall \qquad \exists$$

The names and meanings of the logical operators are given by

symbol	name	usage	meaning
$\&$	conjunction	$\ldots \& \ldots$	"both ... and ..."
$\lor$	disjunction	$\ldots \lor \ldots$	"either ... or ... (or both)"
$\sim$	negation	$\sim \ldots$	"it is not the case that ..."
$\supset$	implication[7]	$\ldots \supset \ldots$	"if ... then ..."
$\equiv$	bi-implication[8]	$\ldots \equiv \ldots$	"... if and only if ..."
$\forall$	universal quantifier	$\forall x \ldots$	"for all x, ..."
$\exists$	existential quantifier	$\exists x \ldots$	"there exists x such that ..."

Here x is any variable.

A *formula* is a meaningful expression built up from atomic formulas by repeated application of the logical operators. In the above table, an ellipsis mark ... stands for a formula within a larger formula.

For example, suppose we have a predicate M meaning "is a man", another predicate T meaning "is a truck", and another predicate D meaning "drives". Here M and T are predicates which require only one argument apiece. The predicate D requires two arguments: the driver, and the vehicle being driven.

[5]The idea of using letters such as x and y as variables is of great value. Historically, the creators of the predicate calculus borrowed this idea from the mathematical discipline known as *algebra*. Recall that algebra is a kind of generalized arithmetic. In algebra there are constants, i.e., specific quantities such as 2, the square root of 10, etc., but there are also variables such as x, y, etc. The key idea of algebra is that a variable x represents an unspecified or unknown quantity. It always stands for *some* quantity, but it *may* stand for *any* quantity. The use of variables makes algebra much more powerful than arithmetic. Variables help us to express and solve equations such as $2x + 3y = 11$ involving one or more unknown quantities. Variables can also be used to express arithmetical laws such as $x + y = y + x$.

[6]The first five logical operators ($\&$, $\lor$, $\sim$, $\supset$, $\equiv$) are equivalent to so-called "Boolean logic gates" of electrical engineering. Formulas built from them may be viewed as representations of the binary switching circuits that control the operation of modern digital computers. See Mendelson [14, 15].

[7]This is the so-called "material implication": $\Phi_1 \supset \Phi_2$ is equivalent to $\sim(\Phi_1 \& \sim\Phi_2)$.

[8]This is called bi-implication because $\Phi_1 \equiv \Phi_2$ is equivalent to $(\Phi_1 \supset \Phi_2) \& (\Phi_2 \supset \Phi_1)$.

Thus Mx, Ty, and Dxy are atomic formulas meaning "x is a man", "y is a truck", and "x drives y", respectively. A typical formula built from these atomic formulas is

$$\forall x\,(Mx \supset \exists y\,(Ty \,\&\, Dxy))$$

which we can translate as "for all x, if x is a man then there exists y such that y is a truck and x drives y". In other words,

Every man drives at least one truck.

Similarly, the formula

$$\forall y\,(Ty \supset \exists x\,(Mx \,\&\, Dxy))$$

translates to

Every truck is driven by at least one man.

In writing formulas, we often use parentheses as punctuation marks to indicate grouping and thereby remove ambiguity. If parentheses were not used, one could construe the formula $\sim Ty \,\&\, Dxy$ in two logically inequivalent ways: as $(\sim Ty) \,\&\, Dxy$ ("y is not a truck, and x drives y"), or as $\sim (Ty \,\&\, Dxy)$ ("y is not a truck that x drives"). The parentheses allow us to choose the meaning that we intend.

The predicate calculus is very rich in expressive power. For example, the four Aristotelean premise types discussed in 1.1.2 can easily be rendered as formulas of the predicate calculus. Letting A and B be predicates which require one argument apiece, we have

universal affirmative	all A is B	$\forall x\,(Ax \supset Bx)$
universal negative	no A is B	$\forall x\,(Ax \supset \sim Bx)$
particular affirmative	some A is B	$\exists x\,(Ax \,\&\, Bx)$
particular negative	some A is not B	$\exists x\,(Ax \,\&\, \sim Bx)$

In the second line of this table, the universal negative "no A is B" could have been rendered equivalently as $\sim \exists x\,(Ax \,\&\, Bx)$, or as $\forall x\,(Bx \supset \sim Ax)$.

The above table may tend to gloss over a subtle but philosophically significant difference between Aristotelean logic and the predicate calculus. Namely, where Aristotelean logic views A as a subject and B as a predicate, the predicate calculus views both A and B as predicates. This is typical of the different perspectives involved. Aristotelean logic emphasizes the universal essences of subjects or entities, while the predicate calculus elevates predicates to a position of supreme importance.

1.2.3 Logical validity and logical consequence

A formula of the predicate calculus is said to be *logically valid* if it is necessarily always true, regardless of the specific predicates and individuals involved. For example, the three fundamental principles of Aristotelean logic (see 1.1.1 above) correspond to formulas as follows:

Identity:	$\forall x\,(Ax \equiv Ax)$.
Non-contradiction:	$\sim\exists x\,(Ax \mathbin{\&} \sim Ax)$.
Either-or:	$\forall x\,(Ax \vee \sim Ax)$.

These formulas are logically valid, because they are "necessarily" or "automatically" or "formally" true, no matter what predicate may be denoted by the symbol A.

The predicate calculus concept of logical validity subsumes the Aristotelean syllogism. Each syllogism corresponds to a logically valid implication

$$(\Phi_1 \mathbin{\&} \Phi_2) \supset \Psi$$

where Φ_1 and Φ_2 are formulas expressing the two premises and Ψ expresses the conclusion. For example, the syllogism

some A is B, all B is C, therefore some A is C

has a predicate calculus rendition

$$((\exists x\,(Ax \mathbin{\&} Bx)) \mathbin{\&} (\forall x\,(Bx \supset Cx))) \;\supset\; (\exists x\,(Ax \mathbin{\&} Cx))$$

and this formula is logically valid.

More generally, a formula Ψ is said to be a *logical consequence* of a set of formulas $\Phi_1, \ldots, \Phi_n$ just in case

$$(\Phi_1 \mathbin{\&} \cdots \mathbin{\&} \Phi_n) \supset \Psi$$

is logically valid. Here $\Phi_1, \ldots, \Phi_n$ are premises and Ψ is a conclusion. This is similar to the Aristotelean syllogism, but it is of wider applicability, because the premises and the conclusion can be more complex. As an example, the 19th century logician Augustus DeMorgan noted[9] that the inference

all horses are animals,
therefore, the head of a horse is the head of an animal

is beyond the reach of Aristotelean logic. Yet this same inference may be paraphrased as "if all horses are animals, then for all x, if x is the head of some horse then x is the head of some animal", and this corresponds to a logically valid formula

$$(\forall y\,(Hy \supset Ay)) \;\supset\; (\forall x\,((\exists y\,(Rxy \mathbin{\&} Hy)) \supset (\exists y\,(Rxy \mathbin{\&} Ay))))$$

of the predicate calculus. Here H, A, R denote "is a horse", "is an animal", "is the head of", respectively. Thus DeMorgan's conclusion is indeed a logical consequence of his premise.

[9] See however Bocheński [2, §16E].

1.2.4 The completeness theorem

Formulas of the predicate calculus can be exceedingly complicated. How then can we distinguish the formulas that are logically valid from the formulas that are not logically valid? It turns out that there is an algorithm[10] for recognizing logically valid formulas. We shall now sketch this algorithm.

In order to recognize that a formula Φ is logically valid, it suffices to construct what is known as a *proof tree* for Φ, or equivalently a *refutation tree* for $\sim\Phi$. This is a tree which carries $\sim\Phi$ at the root. Each node of the tree carries a formula. The growth of the tree is guided by the meaning of the logical operators appearing in Φ. New nodes are added to the tree depending on what nodes have already appeared. For example, if a node carrying $\sim(\Phi_1 \,\&\, \Phi_2)$ has appeared, we create two new nodes carrying $\sim\Phi_1$ and $\sim\Phi_2$ respectively. The thought behind these new nodes is that the only way for $\sim(\Phi_1 \,\&\, \Phi_2)$ to be the case is if at least one of $\sim\Phi_1$ or $\sim\Phi_2$ is the case. Similarly, if a node carrying $\sim\forall x\,\Psi$ has already appeared, we create a new node carrying $\sim\Psi'$, where Ψ' is the result of substituting a new constant a for the variable x throughout the formula Ψ. The idea here is that the only way for the universal statement $\forall x\,\Psi$ to be false is if Ψ is false for some particular x. Since a is a new constant, Ψ' is a formula which may be considered as the most general false instance of Ψ. Corresponding to each of the seven logical operators, there are prescribed procedures for adding new nodes to the tree. We apply these procedures repeatedly until they cannot be applied any more. If explicit contradictions[11] are discovered along each and every branch of the tree, then we have a refutation tree for $\sim\Phi$. Thus $\sim\Phi$ is seen to be logically impossible. In other words, Φ is logically valid.

The adequacy of proof trees for recognizing logically valid formulas is a major insight of 20th century logic. It is a variant of the famous *completeness theorem*, first proved in 1930 by the great logician Kurt Gödel [5, 22].

On the other hand, the class of logically valid formulas is known to be extremely complicated. Indeed, this class is *undecidable*: there is no algorithm[12] which accepts as input an arbitrary formula Φ and outputs "yes" if Φ is logically valid and "no" if Φ is not logically valid. In this sense, the concept of logical validity is too general and too intractable to be analyzed thoroughly. There will never be a predicate calculus analog of the pons asinorum.

1.2.5 Formal theories

The predicate calculus is a very general and flexible framework for reasoning. By choosing appropriate predicates, one can reason about any subject whatsoever. These considerations lead to the notion of a *formal theory*.

[10] The details of this algorithm are explained in modern logic textbooks. Variants of it have been programmed to run on digital computers. They form the basis of a system of computer logic. See Fitting [4].

[11] An *explicit contradiction* is a pair of formulas of the form Ψ, $\sim\Psi$.

[12] The algorithms in question may be implemented as Turing machine programs. This undecidability result is known as *Church's theorem*. See Mendelson [15].

In order to specify a formal theory, one first chooses a small collection of predicates which are regarded as basic for a given field of study. These predicates are the *primitives* of the theory. They delimit the scope of the theory. Other predicates must be defined in terms of the primitives. Using them, one writes down certain formulas which are regarded as basic or self-evident within the given field of study. These formulas are the *axioms* of the theory. It is crucial to make all of our underlying assumptions explicit as axioms. Once this has been done, a *theorem* is any formula which is a logical consequence of the axioms. A *formal theory* is this structure of primitives, axioms, and theorems.

As a frivolous example, we could envision a theory of cars, trucks, and drivers. We would begin with some primitives such as C ("is a car"), T ("is a truck"), D ("drives"), M ("is a man"), etc. We could then write down certain obvious or self-evident axioms such as $\forall x\,(Mx \supset \sim Cx)$ ("no man is a car"), $\forall x\,((\exists y\, Dxy) \supset Mx)$ ("every driver is a man"), etc. Then, within the constraints imposed by the axioms, we could investigate the logical consequence relationships among various non-obvious assertions, such as

$$\sim \exists x\,(Mx \,\&\, \exists y\,(Dxy \,\&\, Cy) \,\&\, \exists z\,(Dxz \,\&\, Tz))$$

("nobody drives both a car and a truck"). Additional predicates V ("is a vehicle") and P ("is a driver") can be defined in terms of the primitives. The defining axioms for V and P would be $\forall y\,(Vy \equiv (Cy \vee Ty))$ and $\forall x\,(Px \equiv \exists y\, Dxy)$, respectively. In this fashion, we could attempt to codify all available knowledge about vehicles and drivers.

More seriously, one could try to write down formal theories corresponding to various scientific disciplines, such as mechanics or statistics or law. In this way one could hope to analyze the logical structure of the respective disciplines.

The process of codifying a scientific discipline by means of primitives and axioms in the predicate calculus is known as *formalization*. The key issue here is the choice of primitives and axioms. They cannot be chosen arbitrarily. The scientist who chooses them must exercise a certain aesthetic touch. They must be small in number; they must be basic and self-evident; and they must account for the largest possible number of other concepts and facts.

To date, this kind of formal theory-building has been convincingly carried out in only a few cases. A survey is in Tarski [21]. The most notable successes have been in mathematics.

2 Foundations of mathematics

Mathematics is the science of quantity. Traditionally there were two branches of mathematics, arithmetic and geometry, dealing with two kinds of quantities: numbers and shapes. Modern mathematics is richer and deals with a wider variety of objects, but arithmetic and geometry are still of central importance.

Foundations of mathematics is the study of the most basic concepts and logical structure of mathematics, with an eye to the unity of human knowledge.

Among the most basic mathematical concepts are: number, shape, set, function, algorithm, mathematical axiom, mathematical definition, mathematical proof.

The reader may reasonably ask why mathematics appears at all in this volume. Isn't mathematics too narrow a subject? Isn't the philosophy of mathematics of rather specialized interest, all the more so in comparison to the broad humanistic issues of philosophy proper, issues such as the good, the true, and the beautiful?

There are three reasons for discussing mathematics in a volume on general philosophy:

1. Mathematics has always played a special role in scientific thought. The abstract nature of mathematical objects presents philosophical challenges that are unusual and unique.

2. Foundations of mathematics is a subject that has always exhibited an unusually high level of technical sophistication. For this reason, many thinkers have conjectured that foundations of mathematics can serve as a model or pattern for foundations of other sciences.

3. The philosophy of mathematics has served as a highly articulated testbed where mathematicians and philosophers alike can explore how various general philosophical doctrines play out[13] in a specific scientific context.

The purpose of this section is to indicate the role of logic in the foundations of mathematics. We begin with a few remarks on the geometry of Euclid. We then describe some modern formal theories for mathematics.

2.1 The geometry of Euclid

Above the gateway to Plato's academy appeared a famous inscription:

Let no one who is ignorant of geometry enter here.

In this way Plato indicated his high opinion of geometry. According to Heath [9, page 284], Plato regarded geometry as "the first essential in the training of philosophers", because of its abstract character. See also Plato [17, Republic, 527B].

In the Posterior Analytics [13], Aristotle laid down the basics of the scientific method.[14] The essence of the method is to organize a field of knowledge logically by means of primitive concepts, axioms, postulates, definitions, and theorems. The majority of Aristotle's examples of this method are drawn from arithmetic and geometry [1, 7, 9].

[13] For example, philosophical intrinsicism may play out as mathematical Platonism. Philosophical subjectivism may play out as mathematical constructivism. Nominalism may play out as formalism.

[14] Our modern notion of a formal theory (see 1.2.5 above) is a variant of Aristotle's concept of scientific method.

The methodological ideas of Aristotle decisively influenced the structure and organization of Euclid's monumental treatise on geometry, the Elements [8]. Euclid begins with 21 definitions, five postulates, and five common notions. After that, the rest of the Elements are an elaborate deductive structure consisting of hundreds of propositions. Each proposition is justified by its own demonstration. The demonstrations are in the form of chains of syllogisms. In each syllogism, the premises are identified as coming from among the definitions, postulates, common notions, and previously demonstrated propositions. For example, in Book I of the Elements, the demonstration of Proposition 16 ("in any triangle, if one of the sides be produced, the exterior angle is greater than either of the interior and opposite angles") is a chain of syllogisms with Postulate 2, Common Notion 5, and Propositions 3, 4 and 15 ("if two straight lines cut one another, they make the vertical angles equal to one another") occurring as premises. It is true that the syllogisms of Euclid do not always conform strictly to Aristotelean templates. However, the standards of rigor are very high, and Aristotle's influence is readily apparent.

The logic of Aristotle and the geometry of Euclid are universally recognized as towering scientific achievements of ancient Greece.

2.2 Formal theories for mathematics

2.2.1 A formal theory for geometry

With the advent of calculus in the 17th and 18th centuries, mathematics developed very rapidly and with little attention to logical foundations. Euclid's geometry was still regarded as a model of logical rigor, a shining example of what a well-organized scientific discipline ideally ought to look like. But the prolific Enlightenment mathematicians such as Leonhard Euler showed almost no interest in trying to place calculus on a similarly firm foundation. Only in the last half of the 19th century did scientists begin to deal with this foundational problem in earnest. The resulting crisis had far-reaching consequences. Even Euclid's geometry itself came under critical scrutiny. Geometers such as Moritz Pasch discovered what they regarded as gaps or inaccuracies in the Elements. Great mathematicians such as David Hilbert entered the fray.

An outcome of all this foundational activity was a thorough reworking of geometry, this time as a collection of formal theories within the predicate calculus. Decisive insights were obtained by Alfred Tarski. We shall sketch Tarski's formal theory for Euclidean[15] plane geometry.[16]

As his primitive predicates, Tarski takes P ("point"), B ("between"), D ("distance"), I ("identity"). The atomic formulas Px, $Bxyz$, $Dxyuv$, and Ixy mean "x is a point", "y lies between x and z", "the distance from x to y is equal to the distance from u to v", and "x is identical to y", respectively. Geometrical

[15] Here "Euclidean geometry" refers to the familiar geometry in which the angles of a triangle sum to 180 degrees, as distinct from the "non-Euclidean" (i.e., hyperbolic) geometry developed by Bolyai and Lobachevsky in the 19th century.

[16] Tarski also showed how to handle non-Euclidean plane geometry, as well as Euclidean and non-Euclidean geometries of higher dimension, in a similar fashion.

objects other than points, such as line segments, angles, triangles, circles, etc., are handled by means of the primitives. For example, the circle with center x and radius uv consists of all points y such that $Dxyuv$ holds.

In geometry, two points x and y are considered identical if the distance between them is zero. Tarski expresses this by means of an axiom

$$\forall x\,\forall y\,\forall z\,(Dxyzz \supset Ixy).$$

Another axiom

$$\forall w\,\forall x\,\forall y\,\forall z\,((Bwxy\,\&\,Bwyz) \supset Bxyz)$$

expresses the fact that, given any four points, if the second is between the first and the third, and if the third is between the first and the fourth, then the third is between the second and the fourth. A noteworthy axiom is

$$\forall x\,\forall y\,\forall z\,\forall u\,\forall v\,((Dxuxv\,\&\,Dyuyv\,\&\,Dzuzv\,\&\,\sim Iuv) \supset (Bxyz \vee Bxzy \vee Byxz))$$

which says: any three points x, y, z equidistant from two distinct points u, v must be collinear. This axiom is typical of two-dimensional (i.e., plane) geometry and does not apply to geometries of dimension greater than two.

Altogether Tarski presents twelve axioms, plus an additional collection of axioms expressing the idea that a line is continuous. The full statement of Tarski's axioms for Euclidean plane geometry is given at [10, pages 19–20]. Let T_g be the formal theory based on Tarski's axioms.

Remarkably, Tarski has demonstrated that T_g is *complete*. This means that, for any purely geometrical[17] statement Ψ, either Ψ or $\sim\Psi$ is a theorem of T_g. Thus we see that the axioms of T_g suffice to answer all yes/no questions of Euclidean plane geometry. Combining this with the completeness theorem of Gödel, we find that T_g is *decidable*: there is an algorithm[18] which accepts as input an arbitrary statement of plane Euclidean geometry, and outputs "true" if the statement is true, and "false" if it is false. This is a triumph of modern foundational research.

2.2.2 A formal theory for arithmetic

By *arithmetic* we mean elementary school arithmetic, i.e., the study of the positive whole numbers 1, 2, 3, ... along with the familiar operations of addition (+) and multiplication (×). This part of mathematics is obviously fundamental, yet it turns out to be surprisingly complicated. Below we write down some of the axioms which go into a formal theory of arithmetic.[19]

Our primitive predicates for arithmetic are N ("number"), A ("addition"), M ("multiplication"), I ("identity"). The atomic formulas Nx, $Axyz$, $Mxyz$,

[17] This means that all occurrences of variables x within the formula Ψ are within subformulas of the form $\forall x\,(Px \supset \ldots)$ or $\exists x\,(Px\,\&\,\ldots)$. Thus we are restricting attention to the realm of geometry and excluding everything else.

[18] Such algorithms have been implemented as computer programs. They are useful in robotics and other artificial intelligence applications.

[19] Two recent studies of formal arithmetic are Hájek/Pudlák [6] and Simpson [20].

Ixy mean "x is a number", "$x+y=z$", "$x\times y=z$", "$x=y$", respectively. Our axioms will use the predicates N, A, M, I to assert that for any given numbers x and y, the numbers $x+y$ and $x\times y$ always exist and are unique. We shall also have axioms expressing some well known arithmetical laws:

substitution laws:	if $x=y$ and x is a number then y is a number, etc.
commutative laws:	$x+y=y+x$ and $x\times y=y\times x$.
associative laws:	$(x+y)+z=x+(y+z)$ and $(x\times y)\times z=x\times(y\times z)$.
distributive law:	$x\times(y+z)=(x\times y)+(x\times z)$.
comparison law:	$x\neq y$ if and only if, for some z, $x+z=y$ or $x=y+z$.
unit law:	$x\times 1=x$.

Our formal axioms for arithmetic are as follows.

substitution laws:

$\forall x\, Ixx$
$\forall x\,\forall y\,(Ixy\equiv Iyx)$
$\forall x\,\forall y\,\forall z\,((Ixy\,\&\,Iyz)\supset Ixz)$
$\forall x\,\forall y\,(Ixy\supset(Nx\equiv Ny))$

existence and uniqueness of $x+y$:

$\forall x\,\forall y\,\forall z\,\forall u\,\forall v\,\forall w\,((Ixu\,\&\,Iyv\,\&\,Izw)\supset(Axyz\equiv Auvw))$
$\forall x\,\forall y\,\forall z\,(Axyz\supset(Nx\,\&\,Ny\,\&\,Nz))$
$\forall x\,\forall y\,((Nx\,\&\,Ny)\supset\exists w\,\forall z\,(Iwz\equiv Axyz))$

existence and uniqueness of $x\times y$:

$\forall x\,\forall y\,\forall z\,\forall u\,\forall v\,\forall w\,((Ixu\,\&\,Iyv\,\&\,Izw)\supset(Mxyz\equiv Muvw))$
$\forall x\,\forall y\,\forall z\,(Mxyz\supset(Nx\,\&\,Ny\,\&\,Nz))$
$\forall x\,\forall y\,((Nx\,\&\,Ny)\supset\exists w\,\forall z\,(Iwz\equiv Mxyz))$

commutative laws:

$\forall x\,\forall y\,\forall z\,(Axyz\equiv Ayxz)$
$\forall x\,\forall y\,\forall z\,(Mxyz\equiv Myxz)$

associative laws:

$\forall x\,\forall y\,\forall z\,\forall u\,\forall v\,\forall w\,((Axyu\,\&\,Ayzv)\supset(Auzw\equiv Axvw))$
$\forall x\,\forall y\,\forall z\,\forall u\,\forall v\,\forall w\,((Mxyu\,\&\,Myzv)\supset(Muzw\equiv Mxvw))$

distributive law:

$\forall x\,\forall y\,\forall z\,\forall u\,\forall v\,\forall w\,\forall t\,((Mxyu\,\&\,Mxzv\,\&\,Ayzw)\supset(Mxwt\equiv Auvt))$

comparison law:

$\forall x\,\forall y\,((Nx\,\&\,Ny)\supset(Ixy\equiv\sim\exists z\,(Axzy\vee Ayzx)))$

unit law:

$\exists z\,(Nz\,\&\,(\sim\exists x\,\exists y\,Axyz)\,\&\,\forall w\,(Nw\supset Mwzw))$

Let T_a be the formal theory specified by the above primitives and axioms.

It is known that T_a suffices to derive many familiar arithmetical facts. For example, $2+2=4$ may be expressed, awkwardly[20] to be sure, as $(1+1)+(1+1)=$

[20]This kind of awkwardness can be alleviated by means of various devices. In particular,

$((1+1)+1)+1$ or

$$\exists x\, \exists y\, \exists z\, \exists w\, (Mxxx \,\&\, Axxy \,\&\, Axyz \,\&\, Axzw \,\&\, Ayyw)^{21}$$

and this formula is indeed a theorem of T_a, i.e., a logical consequence of the axioms of T_a. Another theorem of T_a is

$$\forall x\, \forall y\, \forall z\, \forall w\, (((Axzw \,\&\, Ayzw) \vee (Mxzw \,\&\, Myzw)) \supset Ixy)$$

expressing a familiar cancellation law: if either $x + z = y + z$ or $x \times z = y \times z$, then $x = y$.

On the other hand, the axioms of T_a are by no means exhaustive. They can be supplemented with other axioms expressing the so-called *mathematical induction* or *least number* principle: if there exists a number having some well-defined property, then among all numbers having the property there is a smallest one. The resulting formal theory is remarkably powerful, in the sense that its theorems include virtually all known arithmetical facts. But it is not so powerful as one might wish. Indeed, any formal theory which includes T_a is necessarily either inconsistent[22] or incomplete. Thus there is no hope of writing down enough axioms or developing an algorithm to decide all arithmetical facts. This is a variant of the famous 1931 *incompleteness theorem* of Gödel [5, 22]. There are several methods of coping with the incompleteness phenomenon, and this constitutes a currently active area of research in foundations of mathematics.

The contrast between the completeness of formal geometry and the incompleteness of formal arithmetic is striking. Both sides of this dichotomy are of evident philosophical interest.

2.2.3 A formal theory of sets

One of the aims of modern logical research is to devise a single formal theory which will unify all of mathematics. Such a theory will necessarily be subject to the Gödel incompleteness phenomenon, because it will incorporate not only T_g but also T_a.

One approach to a unified mathematics is to straightforwardly embed arithmetic into geometry, by identifying whole numbers with evenly spaced points on a line. This idea was familiar to the ancient Greeks. Another approach is to explain geometry in terms of arithmetic and algebra, by means of coordinate systems, like latitude and longitude on a map. This idea goes back to the 17th century mathematician and philosopher René Descartes and the 19th century mathematician Karl Weierstrass. Both approaches give rise to essentially the same formal theory, known as *second-order arithmetic*.[23] This theory includes both T_a and T_g and is adequate for the bulk of modern mathematics. Thus the

standard mathematical notation such as $x+2$ can be incorporated into the predicate calculus.

[21] In this formula, the variables x, y, z, w play the role of 1, 2, 3, 4 respectively.

[22] A formal theory is said to be *inconsistent* if its axioms logically imply an explicit contradiction. Such theories are of no scientific value.

[23] A recent study of second-order arithmetic is Simpson [20].

decision about whether to make geometry more fundamental than arithmetic or vice versa seems to be mostly a matter of taste.

A very different approach to a unified mathematics is via *set theory.* This is a peculiarly 20th century approach. It is based on one very simple-looking concept: sets. Remarkably, this one concept leads directly to a vast structure which encompasses all of modern mathematics.

A *set* is a collection of objects called the *elements* of the set. We sometimes use informal notations such as $x = \{y, z, \ldots\}$ to indicate that x is a set consisting of elements y, z, ... The number of elements in a set can be arbitrarily large or even infinite. A basic principle of set theory is that a set is determined by its elements. Thus two sets are identical if and only if they have the same elements. This principle is known as *extensionality.* For example, the set $\{a, b, c\}$ is considered to be the same set as $\{c, a, b\}$ because the elements are the same, even though written in a different order.

Much of the complexity of set theory arises from the fact that sets may be elements of other sets. For instance, the set $\{a, b\}$ is an element of the set $\{\{a, b\}, c\}$ and this is distinct from the set $\{a, b, c\}$.

For a formal theory of sets, we use three primitives: S ("set"), I ("identity"), E ("element"). The atomic formulas Sx, Ixy, Exy mean "x is a set", "x is identical to y", "x is an element of y", respectively. One of the ground rules of set theory is that only sets may have elements. This is expressed as an axiom $\forall w\, \forall x\, (Ewx \supset Sx)$. In addition there is an axiom of extensionality

$$\forall x\, \forall y\, ((Sx \,\&\, Sy) \supset (Ixy \equiv \forall w\, (Ewx \equiv Ewy)))$$

and an axiom $\exists x\, (Sx \,\&\, {\sim}\, \exists w\, Ewx)$ expressing the existence of the *empty set*, i.e. a set $\{\,\}$ having no elements. A list of all the axioms of set theory may be found in textbooks [11, 15]. Let T_s be the formal theory of sets based on these axioms.

The set theory approach to arithmetic is in terms of the non-negative whole numbers 0, 1, 2, 3, ... These numbers are identified with specific sets. Namely, we identify 0 with the empty set $\{\,\}$, 1 with $\{\{\,\}\}$, 2 with $\{\{\,\}, \{\{\,\}\}\}$, 3 with $\{\{\,\}, \{\{\,\}\}, \{\{\,\}, \{\{\,\}\}\}\}$, etc. In general, we identify the number n with the set of smaller numbers $\{0, 1, \ldots, n-1\}$. Among the axioms of T_s is an *axiom of infinity* asserting the existence of the infinite set $\omega = \{0, 1, 2, 3, \ldots\}$. One can use the set ω to show that T_s includes a theory equivalent to T_a. After that, one can follow the ideas of Descartes and Weierstrass to see that T_s also includes a theory equivalent to T_g. It turns out that the rest of modern mathematics can also be emulated within T_s. This includes an elaborate theory of infinite sets which are much larger than ω.

The set-theoretical approach to arithmetic and geometry is admittedly somewhat artificial. However, the idea of basing all of mathematics on one simple concept, sets, has exerted a powerful attraction.[24] The implications of this idea are not yet fully understood and are a topic of current research.

[24]The idea of set-theoretical foundations gave rise to the "new math" pedagogy of the 1960's. For a lively discussion, see Kline [12].

3 Philosophy of mathematics

In this section we indicate some issues and trends in the philosophy of mathematics.

3.1 Plato and Aristotle

The objects that are studied in mathematics tend to be somewhat abstract and remote from everyday perceptual experience. Therefore, the existence and nature of mathematical objects present special philosophical challenges. For example, is a geometrical square different from a square floor tile? If so, then where is the geometrical square? Is it on the floor, in our minds, or somewhere else? And what about sets? Is a set of 52 cards something other than the cards themselves?

The ancient Greek philosophers took such questions very seriously. Indeed, many of their general philosophical discussions were carried on with extensive reference to geometry and arithmetic. Plato seemed to insist that mathematical objects, like the Platonic forms or essences, must be perfectly abstract and have a separate, non-material kind of existence. Aristotle [1, 7, 13, 19] dissected and refuted this view in books M and N of the Metaphysics. According to Aristotle, the geometrical square is a significant aspect of the square floor tile, but it can only be understood by discarding other irrelevant aspects such as the exact measurements, the tiling material, etc. Clearly these questions provide much food for philosophical analysis and debate.

3.2 The 20th century

In the 20th century, the advent of the predicate calculus and the digital computer profoundly affected our view of mathematics. The discovery that all of mathematics can be codified in formal theories created a huge stir. One expression of this excitement was the rise of an extreme philosophical doctrine known as *formalism*.[25]

According to formalism, mathematics is only a formal game, concerned solely with algorithmic manipulation of symbols. Under this view, the symbols of the predicate calculus do not denote predicates or anything else. They are merely marks on paper, or bits and bytes in the memory of a computer. Therefore, mathematics cannot claim to be any sort of knowledge of mathematical objects. Indeed, mathematical objects do not exist at all, and the profound questions debated by Plato and Aristotle become moot. Mathematics is nothing but a kind of blind calculation.

The formalist doctrine fits well with certain modern trends in computer science, e.g., artificial intelligence. However, formalism has proved inadequate as an integrated philosophy of mathematics, because it fails to account for human mathematical understanding, not to mention the spectacular applications of mathematics in fields such as physics and engineering.

[25] See for example Curry [3].

By way of reaction against formalism, several alternative doctrines have been advocated. One of these is *constructivism*, the idea that mathematical knowledge can be obtained by means of a series of purely mental constructions. Under this view, mathematical objects exist solely in the mind of the mathematician, so mathematical knowledge is absolutely certain. However, the status of mathematics vis a vis the external world becomes doubtful. An extreme version of constructivism is so solipsistic that it does not even allow for the possibility of mathematical communication from one mind to another.

An additional disturbing feature of constructivism is that it entails rejection of the basic laws of logic. To see how this comes about, consider some specific mathematical problem or question[26] of a yes/no nature, for which the answer is currently unknown. (Mathematics abounds with such questions, and the Gödel incompleteness phenomenon suggests that such questions will always exist.) Express the "yes" answer as a formula Ψ and the "no" answer as the negated formula $\sim\Psi$. Since the answer is unknown, neither Ψ nor $\sim\Psi$ is in the mind of the mathematician. Therefore, according to the constructivists, the disjunction $\Psi\vee\sim\Psi$ is not a legitimate mathematical assumption. Thus Aristotle's either-or principle (see 1.1.1 and 1.2.3 above) must be abandoned.[27]

Constructivism has the merit of allowing human beings to possess mathematical knowledge. However, the constructivist rejection of the external world and of Aristotelean logic are highly unpalatable to most mathematicians and mathematically oriented scientists. For this reason, constructivism remains a fringe movement on the 20th century mathematical landscape.

Another 20th century philosophical doctrine has arisen from set-theoretical foundations. The reliance on infinite sets suggests many perplexing questions. What do such sets correspond to in reality? Where are they, and how can the human mind grasp them? In order to boldly answer these questions, and as a reaction against formalism, many researchers in axiomatic set theory have subscribed to what is known as *set-theoretical Platonism.* According to this variant of the Platonic doctrine, infinite sets exist in a non-material, purely mathematical realm. By extending our intuitive understanding of this realm, we will be able to cope with chaos issuing from the Gödel incompleteness phenomenon. The most prominent and frequently cited authority for this kind of Platonism is Gödel himself [5].

There is a good fit between set-theoretical Platonism and certain aspects of 20th century mathematical practice. However, as a philosophical doctrine, set-theoretical Platonism leaves much to be desired. Many of Aristotle's objections to the Platonic forms are still cogent. There are serious questions about how a theory of infinite sets can be applicable to a finite world.

We have mentioned three competing 20th century doctrines: formalism, constructivism, set-theoretical Platonism. None of these doctrines are philosophically satisfactory, and they do not provide much guidance for mathematically oriented scientists and other users of mathematics. As a result, late 20th century

[26]For example, we could consider the following difficult question of Goldbach. Can every even number greater than 2 be expressed as the sum of two prime numbers?

[27]See the essays of Brouwer and Kolmogorov in [22].

mathematicians have developed a split view, a kind of Kantian schizophrenia, which is usually described as "Platonism on weekdays, formalism on weekends". In other words, they accept the existence of infinite sets as a working hypothesis in their mathematical research, but when it comes to philosophical speculation, they retreat to a formalist stance. Thus they have given up hope of an integrated view which accounts for both mathematical knowledge and the applicability of mathematics to physical reality. In this respect, the philosophy of mathematics is in a sorry state.

3.3 The future

From the Renaissance through the 20th century, Aristotle's ideas about the nature of mathematical objects have been neglected and ignored. Now the time seems ripe for a renovation of the philosophy of mathematics, based on Aristotelean and neo-Aristotelean [16] ideas and bolstered by the techniques of modern logic, including the predicate calculus.

The great mathematician David Hilbert anticipated such a renovation in his 1925 essay, On the Infinite [22]. Hilbert was aware that, according to modern physics, the physical universe is finite. Yet infinite sets were playing an increasingly large role in the mathematics of the day. Hilbert therefore recognized that the most vulnerable chink in the armor of mathematics was the infinite. In order to defend what he called "the honor of human understanding", Hilbert proposed to develop a new foundation of mathematics, in which formal theories of infinite sets, such as T_s, would be rigorously justified by reference to the finite. This is Hilbert's program of *finitistic reductionism.*[28]

Although Hilbert did not cite Aristotle, we can imagine that Hilbert would have profited from an examination of Aristotle's distinction between actual and potential infinity. An *actual infinity* is something like an infinite set regarded as a completed totality. A *potential infinity* is more like a finite but indefinitely long, unending series of events. According to Aristotle, actual infinities cannot exist, but potential infinities exist in nature and are manifested to us in various ways, for instance the indefinite cycle of the seasons, or the indefinite divisibility of a piece of gold.

In any case, it turned out that Hilbert had stated his program in too sweeping a fashion. The wholesale finitistic reduction which Hilbert desired cannot be carried out. This follows from Gödel's incompleteness theorem [5, 22]. The remarkable results obtained by Gödel in 1931 caused the philosophical ideas of Hilbert's 1925 essay to fall into disrepute. Hilbert's grand foundational program appeared to be dead, broken beyond hope of repair.

The last 20 years have seen a revival of Hilbert's program. Recent foundational research [20] has revealed that, although T_s is not finitistically reducible, there are other formal theories which *are* finitistically reducible, in the precise

[28]Hilbert is often inaccurately described as a formalist. The details of Hilbert's program will not be presented here, but see [20, 22]. Roughly speaking, a formal theory is said to be *finitistically reducible* if it can be embedded into some very restricted formal theory such as T_a, which is physically meaningful and makes absolutely no reference to actual infinity.

sense envisioned by Hilbert. Moreover, these other formal theories turn out to be adequate for a very large portion of mathematics. They do not encompass actual infinities such as ω, but they do include the main results of arithmetic and geometry and allied disciplines.

This new research has not yet had an impact on the philosophy of mathematics or on mathematical practice. Philosophers and mathematicians are free to choose which directions to pursue and which techniques to emphasize. Only time will reveal the future evolution of the philosophy of mathematics.

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