Mass problems and intuitionism

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Abstract

Let $\mathcal{P}_w$ be the lattice of Muchnik degrees of nonempty Π^0_1 subsets of 2^ω. The lattice $\mathcal{P}_w$ has been studied extensively in previous publications. In this note we prove that the lattice $\mathcal{P}_w$ is not Brouwerian.

1 Introduction

Definition 1. Let ω denote the set of natural numbers, $\omega = \{0, 1, 2, \ldots\}$. Let ω^ω denote the *Baire space*, $\omega^\omega = \{f \mid f : \omega \to \omega\}$. Following Medvedev [27] and Rogers [32, §13.7] we define a *mass problem* to be an arbitrary subset of ω^ω. For mass problems P and Q we say that P is *Medvedev reducible* or *strongly reducible* to Q, abbreviated $P \leq_s Q$, if there exists a partial recursive functional Ψ such that $\Psi(g) \in P$ for all $g \in Q$. We say that P is *Muchnik reducible* or *weakly reducible* to Q, abbreviated $P \leq_w Q$, if for all $g \in Q$ there exists $f \in P$ such that f is Turing reducible to g. Clearly Medvedev reducibility implies Muchnik reducibility, but the converse does not hold.

Definition 2. A *Medvedev degree* or *degree of difficulty* or *strong degree* is an equivalence class of mass problems under mutual Medvedev reducibility. A *Muchnik degree* or *weak degree* is an equivalence class of mass problems under mutual Muchnik reducibility. We write $\deg_s(P)$ = the Medvedev degree of P. We write $\deg_w(P)$ = the Muchnik degree of P. Let $\mathcal{D}_s$ be the set of

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Medvedev degrees, partially ordered by Medvedev reducibility. There is a natural embedding of the Turing degrees into $\mathcal{D}_s$ given by $\deg_T(f) \mapsto \deg_s(\{f\})$. Let $\mathcal{D}_w$ be the set of Muchnik degrees, partially ordered by Muchnik reducibility. There is a natural embedding of the Turing degrees into $\mathcal{D}_w$ given by $\deg_T(f) \mapsto \deg_w(\{f\})$. Here $\{f\}$ is the singleton set whose only element is f.

Definition 3. Let L be a lattice. For $a, b \in L$ we define $a \Rightarrow b$ to be the unique minimum $x \in L$ such that $\sup(a, x) \geq b$. Note that $a \Rightarrow b$ may or may not exist in L. Following Birkhoff [8, 9] (first two editions) and McKinsey/Tarski [25] we say that L is *Brouwerian* if $a \Rightarrow b$ exists in L for all $a, b \in L$ and L has a top element. It is known (see Birkhoff [9, §IX.12] [10, §II.11] or McKinsey/Tarski [25] or Rasiowa/Sikorski [31, §I.12]) that if L is Brouwerian then L is distributive and has a bottom element and for all $a \leq b$ in L the sublattice

$$\{x \in L \mid a \leq x \leq b\}$$

is again Brouwerian.

Remark 1. Given a Brouwerian lattice L, we may view L as a model of first-order intuitionistic propositional calculus. Namely, for $a, b \in L$ we define $a \wedge b = \sup(a, b)$, $a \vee b = \inf(a, b)$, $a \Rightarrow b$ as above, and $\neg a = (a \Rightarrow 1)$ where 1 is the top element of L. We may also define $a \vdash b$ if and only if $a \geq b$ in L. There is a completeness theorem (see Tarski [52] or McKinsey/Tarski [24, 25, 26] or Rasiowa/Sikorski [31, §IX.3] or Rasiowa [30, §XI.8]) saying that a first-order propositional formula is intuitionistically provable if and only if it evaluates identically to the bottom element in all Brouwerian lattices.

Remark 2. Brouwerian lattices have also been studied under other names and with other notation and terminology. A *pseudo-Boolean algebra* is a lattice L such that the dual of L is Brouwerian; see Rasiowa/Sikorski [31] and Rasiowa [30]. Pseudo-Boolean algebras are also known as *Heyting algebras*; see Balbes/Dwinger [2, Chapter IX], Fourman/Scott [18], and Grätzer [19]. Brouwerian lattices are also known as *Brouwer algebras*; see Sorbi [48, 49], Sorbi/Terwijn [51], and Terwijn [53, 54, 55, 56, 57]. Remarkably, the so-called Brouwerian lattices of Birkhoff [10] (third edition) are dual to those of Birkhoff [8, 9] (first two editions). We adhere to the terminology of Birkhoff [8, 9].

Remark 3. It is known that $\mathcal{D}_s$ and $\mathcal{D}_w$ are Brouwerian lattices. There is a natural homomorphism of $\mathcal{D}_s$ onto $\mathcal{D}_w$ given by $\deg_s(P) \mapsto \deg_w(P)$. This homomorphism preserves the binary lattice operations sup and inf and the top and bottom elements, but it does not preserve the binary if-then operation $\Rightarrow$.

Remark 4. The relationship between mass problems and intuitionism has a considerable history. Indeed, it seems fair to say that the entire subject of mass problems originated from intuitionistic considerations. The impetus came from Kolmogorov 1932 [22, 23] who informally proposed to view Heyting's intuitionistic propositional calculus [20] as a "calculus of problems" ("Aufgabenrechnung"). This idea amounts to what is now known as the BHK or

Brouwer/Heyting/Kolmogorov interpretation of the intuitionistic propositional connectives; see Troelstra/van Dalen [59, §§1.3.1 and 1.5.3]. Elaborating Kolmogorov's idea, Medvedev 1955 [27] introduced $\mathcal{D}_s$ and noted that $\mathcal{D}_s$ is a Brouwerian lattice. Later Muchnik 1963 [28] introduced $\mathcal{D}_w$ and noted that $\mathcal{D}_w$ is a Brouwerian lattice. Some further papers in this line are Skvortsova [47], Sorbi [48, 49, 50], Sorbi/Terwijn [51], and Terwijn [54, 53, 55, 56, 57].

Definition 4. Let 2^ω denote the *Cantor space*, $2^\omega = \{f \mid f : \omega \to \{0,1\}\}$. Following Simpson [40] let $\mathcal{P}_s$ be the sublattice of $\mathcal{D}_s$ consisting of the Medvedev degrees of nonempty Π^0_1 subsets of 2^ω, and let $\mathcal{P}_w$ be the sublattice of $\mathcal{D}_w$ consisting of the Muchnik degrees of nonempty Π^0_1 subsets of 2^ω.

Remark 5. The lattices $\mathcal{P}_s$ and $\mathcal{P}_w$ are mathematically rich and have been studied extensively. See Alfeld [1], Binns [3, 4, 5, 6], Binns/Simpson [7], Cenzer/Hinman [11], Cole/Simpson [13], Kjos-Hanssen/Simpson [21], Simpson [34, 35, 37, 38, 39, 40, 41, 42, 43, 45, 44], Simpson/Slaman [46], and Terwijn [54]. It is known that $\mathcal{P}_w$ contains not only the recursively enumerable Turing degrees [42] but also many specific, natural Muchnik degrees which arise from foundationally interesting topics. Among these foundationally interesting topics are algorithmic randomness [40, 42], reverse mathematics [36, 40, 41, 43], almost everywhere domination [43], hyperarithmeticity [13], diagonal nonrecursiveness [40, 42], subrecursive hierarchies [21, 40], resource-bounded computational complexity [21, 40], and Kolmogorov complexity [21]. Recently Simpson [44] has applied $\mathcal{P}_s$ and $\mathcal{P}_w$ to prove a new theorem in symbolic dynamics.

Remark 6. It is known that $\mathcal{P}_s$ and $\mathcal{P}_w$ are distributive lattices with top and bottom elements. Moreover, the natural lattice homomorphism of $\mathcal{D}_s$ onto $\mathcal{D}_w$ restricts to a natural lattice homomorphism of $\mathcal{P}_s$ onto $\mathcal{P}_w$ preserving top and bottom elements.

Remark 7. In view of Remarks 3, 4, 5 and 6, it is natural to ask whether $\mathcal{P}_s$ and $\mathcal{P}_w$ are Brouwerian lattices. The purpose of this note is to show that $\mathcal{P}_w$ is not a Brouwerian lattice. Letting $\mathbf{1}$ denote the top element of $\mathcal{P}_w$, we shall produce a family of Muchnik degrees $\mathbf{p} \in \mathcal{P}_w$ such that $\mathbf{p} \Rightarrow \mathbf{1}$ does not exist in $\mathcal{P}_w$. In other words, $\neg\,\mathbf{p}$ does not exist in $\mathcal{P}_w$.

Remark 8. It remains open whether $\mathcal{P}_s$ is a Brouwerian lattice. Terwijn [54] has shown that the dual of $\mathcal{P}_s$ is not a Brouwerian lattice. It remains open whether the dual of $\mathcal{P}_w$ is a Brouwerian lattice.

2 Proof that $\mathcal{P}_w$ is not Brouwerian

In this section we prove that the lattice $\mathcal{P}_w$ is not Brouwerian.

Definition 5. For $f, g \in \omega^\omega$ we write $f \leq_T g$ to mean that f is *Turing reducible* to g, i.e., f is computable relative to the Turing oracle g. We write g' = the Turing jump of g. In particular $0'$ = the halting problem = the Turing jump of 0. We use standard recursion-theoretic notation from Rogers [32]. We say that f *is majorized by* g if $f(n) < g(n)$ for all n.

We begin with four well known lemmas.

Lemma 1. *Given $f \leq_T 0'$ we can find $g \equiv_T f$ such that $\{g\}$ is Π^0_1.*

Proof. Since $f \leq_T 0'$, it follows by Post's Theorem (see for instance [32, §14.5, Theorem VIII]) that f is Δ^0_2. From this it follows that the singleton set $\{f\}$ is Π^0_2. Let $R \subseteq \omega^\omega \times \omega \times \omega$ be a recursive predicate such that our f is the unique $f \in \omega^\omega$ such that $\forall m\, \exists n\, R(f, m, n)$ holds. Let $g = f \oplus h$ where $h \in \omega^\omega$ is defined by $h(m) =$ the least n such that $R(f, m, n)$ holds. It is easy to verify that $g \equiv_T f$ and $\{g\}$ is Π^0_1. □

Lemma 2. *If $\{f\}$ is Π^0_1 and f is nonrecursive, then f is not majorized by any recursive function.*

Proof. This lemma is equivalent to, for instance, [40, Theorem 4.15]. □

Lemma 3. *For all nonempty Π^0_1 sets $Q \subseteq 2^\omega$ we have $Q \leq_w \{0'\}$.*

Proof. This lemma is a restatement of the well known Kleene Basis Theorem. Namely, every nonempty Π^0_1 subset of 2^ω contains an element which is $\leq_T 0'$. See for instance the proof of [42, Lemma 5.3]. □

Lemma 4. *Let $Q \subseteq 2^\omega$ be nonempty Π^0_1 such that no element of Q is recursive. Then we can find $g \in \omega^\omega$ such that $0 <_T g <_T 0'$ and $Q \not\leq_w \{g\}$.*

Proof. By Lemma 3 it suffices to find $g \in \omega^\omega$ such that $0 <_T g \leq_T 0'$ and $Q \not\leq_w \{g\}$. To construct g we may proceed as in the proof of Lemma 5 below. The construction is easier than in Lemma 5, because we can ignore f. □

Lemma 5. *Let $Q \subseteq 2^\omega$ be nonempty Π^0_1. Let f be such that $0 <_T f <_T 0'$ and $Q \not\leq_w \{f\}$. Then we can find $g \in \omega^\omega$ such that $0 <_T g <_T 0'$ and $Q \not\leq_w \{g\}$ and $f \oplus g \equiv_T 0'$.*

Proof. We adapt the technique of Posner/Robinson [29].

Let $U \subseteq \omega^{<\omega}$ be a recursive tree such that $Q = \{\text{paths through } U\}$. By Lemmas 1 and 2 we may safely assume that f is not majorized by any recursive function.

For integers $e \in \omega$ and strings $\sigma \in \omega^{<\omega}$ we write

$$\Phi_e(\sigma) = \langle \varphi^{(1),\sigma}_{e,|\sigma|}(i) \mid i < j \rangle$$

where $j =$ the least i such that either $\varphi^{(1),\sigma}_{e,|\sigma|}(i)\uparrow$ or $i \geq |\sigma|$. Note that the mapping $\Phi_e : \omega^{<\omega} \to \omega^{<\omega}$ is recursive and *monotonic*, i.e., $\sigma \subseteq \tau$ implies $\Phi_e(\sigma) \subseteq \Phi_e(\tau)$. Moreover, for all $g, h \in \omega^\omega$ we have $g \geq_T h$ if and only if $\exists e\, (\Phi_e(g) = h)$. Here we are writing

$$\Phi_e(g) = \bigcup_{n=0}^{\infty} \Phi_e(g \upharpoonright n)\,.$$

In order to prove Lemma 5, we shall inductively define an increasing sequence of strings $\tau_e \in \omega^{<\omega}$, $e = 0, 1, 2, \ldots$. We shall then let $g = \bigcup_{e=0}^{\infty} \tau_e$. In presenting the construction, we shall identify strings with their Gödel numbers.

Stage 0. Let $\tau_0 = \langle\rangle =$ the empty string.

Stage $e+1$. Assume that τ_e has been defined. The definition of τ_{e+1} will be given in a finite number of substages.

Substage 0. Let $\sigma_{e,0} = \tau_e$.

Substage $i+1$. Assume that $\sigma_{e,i}$ has been defined. Let $n_{e,i} =$ the least n such that either

$$(1) \qquad \exists \sigma < f(n) \, [\, \sigma_{e,i}{}^\frown\langle n \rangle \subseteq \sigma \text{ and } \Phi_e(\sigma_{e,i}) \subset \Phi_e(\sigma) \in U \,]$$

or

$$(2) \qquad \neg\, \exists \sigma \, [\, \sigma_{e,i}{}^\frown\langle n \rangle \subseteq \sigma \text{ and } \Phi_e(\sigma_{e,i}) \subset \Phi_e(\sigma) \in U \,] \, .$$

Note that $n_{e,i}$ exists, because otherwise $f(n)$ would be majorized by the recursive function $l_{e,i}(n) =$ least σ such that $\sigma_{e,i}{}^\frown\langle n \rangle \subseteq \sigma$ and $\Phi_e(\sigma_{e,i}) \subset \Phi_e(\sigma) \in U$. If (1) holds with $n = n_{e,i}$ let $\sigma_{e,i+1} = l_{e,i}(n_{e,i})$. If (2) holds with $n = n_{e,i}$ let $\tau_{e+1} = \sigma_{e,i}{}^\frown\langle n_{e,i}, 0'(e) \rangle$. This completes our description of the construction.

We claim that, within each stage $e+1$, (2) holds for some i. Otherwise, we would have infinite increasing sequences of strings

$$\sigma_{e,0} \subset \sigma_{e,1} \subset \cdots \subset \sigma_{e,i} \subset \sigma_{e,i+1} \subset \cdots$$

and

$$\Phi_e(\sigma_{e,0}) \subset \Phi_e(\sigma_{e,1}) \subset \cdots \subset \Phi_e(\sigma_{e,i}) \subset \Phi(\sigma_{e,i+1}) \subset \cdots$$

with $\Phi_e(\sigma_{e,i}) \in U$ for all i. Moreover, these sequences would be recursive relative to f, namely $\sigma_{e,i+1} = l_{e,i}(n_{e,i})$ where $n_{e,i} =$ least n such that (1) holds. Thus, letting $h = \bigcup_{i=0}^{\infty} \Phi_e(\sigma_{e,i})$, we would have $h \in Q$ and $h \leq_T f$. Thus $Q \leq_w \{f\}$, a contradiction. This proves our claim.

From the previous claim it follows that τ_e is defined for all $e = 0, 1, 2, \ldots$. By construction, the sequence $\langle \tau_0, \tau_1, \ldots, \tau_e, \tau_{e+1}, \ldots \rangle$, is recursive relative to $0'$. Moreover, $0'$ is recursive relative to $\langle \tau_0, \tau_1, \ldots, \tau_e, \tau_{e+1}, \ldots \rangle$, because for all e we have $0'(e) = \tau_{e+1}(|\tau_{e+1}| - 1)$.

Finally let $g = \bigcup_{e=0}^{\infty} \tau_e$. Clearly $g \leq_T 0'$.

We claim that the sequence $\langle \tau_0, \tau_1, \ldots, \tau_e, \tau_{e+1}, \ldots \rangle$ is $\leq_T f \oplus g$. Namely, given τ_e, we may use f and g as oracles to compute τ_{e+1} as follows. We begin with $\sigma_{e,0} = \tau_e$. Given $\sigma_{e,i}$ we use the oracle g to compute $n_{e,i} = g(|\sigma_{e,i}|)$. Then, using the oracle f, we ask whether there exists $\sigma < f(n_{e,i})$ such that $\sigma_{e,i}{}^\frown\langle n_{e,i} \rangle \subseteq \sigma$ and $\Phi_e(\sigma_{e,i}) \subset \Phi_e(\sigma) \in U$. If so, we compute $\sigma_{e,i+1} =$ the least such σ. If not, we use the oracle g to compute $\tau_{e+1} = g \upharpoonright |\sigma_{e,i}| + 2$. This proves our claim.

From the previous claim it follows that $0' \leq_T f \oplus g$. Hence $0' \equiv_T f \oplus g$.

We claim that $Q \not\leq_w \{g\}$. To see this, let e be such that $\Phi_e(g) = \bigcup_{e=0}^{\infty} \Phi_e(\tau_e)$ is a total function. Consider what happened at stage $e+1$ of the construction. Consider the least i such that (2) holds, i.e., $\tau_{e+1} = \sigma_{e,i}{}^\frown\langle n_{e,i}, 0'(e) \rangle$. Since (2) holds, there does not exist σ such that $\sigma_{e,i}{}^\frown\langle n_{e,i}\rangle \subseteq \sigma$ and $\Phi_e(\sigma_{e,i}) \subset \Phi_e(\sigma) \in U$. In particular, letting τ be an initial segment of g such that $\sigma_{e,i}{}^\frown\langle n_{e,i}\rangle \subseteq \tau$ and $\Phi_e(\sigma_{e,i}) \subset \Phi_e(\tau)$, we have $\Phi_e(\tau) \notin U$. Hence $\Phi_e(g) \notin Q$. This proves our claim.

From the two previous claims, it follows that $0 <_T g <_T 0'$. The proof of Lemma 5 is now finished. □

Remark 9. By a similar argument we can prove the following. Let $S \subseteq \omega^\omega$ be Σ^0_3. Let $f \in \omega^\omega$ be of hyperimmune Turing degree such that $S \not\leq_w \{f\}$. Let $h \in \omega^\omega$ be such that $f \oplus 0' \leq_T h$. Then we can find $g \in \omega^\omega$ such that $0 <_T g <_T h$ and $S \not\leq_w \{g\}$ and $f \oplus g \equiv_T g' \equiv_T g \oplus 0' \equiv_T h$.

Lemma 6. *Let $P \subseteq 2^\omega$ be nonempty Π^0_1. Let $S \subseteq \omega^\omega$ be Σ^0_3. Then*

$$\deg_w(P \cup S) \in \mathcal{P}_w\,.$$

Proof. This is Simpson's Embedding Lemma. See [42, Lemma 3.3] or [45]. □

We are now ready to prove our main result.

Theorem 1. *$\mathcal{P}_w$ is not Brouwerian.*

Proof. Let PA be the set of completions of Peano Arithmetic. Recall from Simpson [40] that $\deg_w(\text{PA}) = \mathbf{1}$ = the top element of $\mathcal{P}_w$. By Lemma 4 let f be such that $0 <_T f <_T 0'$ and $\text{PA} \not\leq_w \{f\}$. Let

$$\mathbf{p} = \deg_w(\text{PA} \cup \{f\})$$

and note that $\mathbf{p} < \mathbf{1}$. By Lemmas 1 and 6 we have $\mathbf{p} \in \mathcal{P}_w$.

It is well known (see for instance [40, Remark 3.9]) that $\mathcal{D}_w$ is a complete lattice. This means that for all $\mathcal{A} \subseteq \mathcal{D}_w$ the least upper bound $\sup(\mathcal{A})$ and the greatest lower bound $\inf(\mathcal{A})$ exist in $\mathcal{D}_w$. Therefore, within $\mathcal{D}_w$, let

$$\mathbf{q} = \inf(\{\mathbf{x} \in \mathcal{P}_w \mid \sup(\mathbf{p}, \mathbf{x}) = \mathbf{1}\})$$

and note that $\sup(\mathbf{p}, \mathbf{q}) = \mathbf{1}$ in $\mathcal{D}_w$. In other words, $\mathbf{q} \geq (\mathbf{p} \Rightarrow \mathbf{1})$ in $\mathcal{D}_w$.

We claim that $\mathbf{q} \notin \mathcal{P}_w$. Otherwise, let $\mathbf{q} = \deg_w(Q)$ where $Q \subseteq 2^\omega$ is nonempty Π^0_1. Since $\sup(\mathbf{p}, \mathbf{q}) = \mathbf{1}$, we have $\text{PA} \leq_w \{f \oplus h\}$ for all $h \in Q$. Since $\text{PA} \not\leq_w \{f\}$, it follows that $Q \not\leq_w \{f\}$. By Lemma 5 let g be such that $0 <_T g <_T 0'$ and $Q \not\leq_w \{g\}$ and $f \oplus g \equiv_T 0'$. Let

$$\mathbf{q}_0 = \deg_w(Q \cup \{g\})$$

and note that $\mathbf{q}_0 < \mathbf{q}$. By Lemmas 1 and 6 we have $\mathbf{q}_0 \in \mathcal{P}_w$. By Lemma 3 we have $\text{PA} \leq_w \{0'\} \equiv_w \{f \oplus g\}$, hence $\sup(\mathbf{p}, \mathbf{q}_0) = \mathbf{1}$ contradicting the definition of $\mathbf{q}$. This proves our claim.

Because $\mathbf{q} \notin \mathcal{P}_w$ it follows that $\mathbf{p} \Rightarrow \mathbf{1}$ does not exist in $\mathcal{P}_w$. Thus $\mathcal{P}_w$ is not Brouwerian. □

Remark 10. The same proof shows that for all $\mathbf{q} > \mathbf{0}$ in $\mathcal{P}_w$ we can find $\mathbf{p} < \mathbf{q}$ in $\mathcal{P}_w$ such that $\mathbf{p} \Rightarrow \mathbf{q}$ does not exist in $\mathcal{P}_w$. On the other hand, we know at least a few nontrivial instances where $\mathbf{p} \Rightarrow \mathbf{q}$ exists in $\mathcal{P}_w$. For example, letting $\mathbf{r}$ be the Muchnik degree of the set of 1-random reals, Theorem 8.12 of Simpson [40] tells us that $\mathbf{r} < \mathbf{1}$ in $\mathcal{P}_w$ and $\mathbf{r} \Rightarrow \mathbf{1}$ exists in $\mathcal{P}_w$. In fact, $\mathbf{r} \Rightarrow \mathbf{1}$ in $\mathcal{P}_w$ is equal to $\mathbf{r} \Rightarrow \mathbf{1}$ in $\mathcal{D}_w$, which is equal to $\mathbf{1}$. We do not know any instances of $\mathbf{p}, \mathbf{q} \in \mathcal{P}_w$ where $\mathbf{p} \Rightarrow \mathbf{q}$ exists in $\mathcal{P}_w$ and both $\mathbf{p}$ and $\mathbf{p} \Rightarrow \mathbf{q}$ are $< \mathbf{q}$ in $\mathcal{P}_w$.

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