Mass Problems and Hyperarithmeticity

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June 4, 2008

Journal of Mathematical Logic, 7:125–143, 2008.
Submitted December 12, 2006. Accepted April 17, 2008.

Mathematics Subject Classification Codes: 03D28, 03D30, 03D55, 03D60.

Cole's research was partially supported by NSF grants DMS-0245167 and RTG-0353748 and a Schmitt Fellowship at the University of Notre Dame.

Simpson's research was partially supported by NSF grant DMS-0600823.

Abstract

A *mass problem* is a set of Turing oracles. If P and Q are mass problems, we say that P is *weakly reducible* to Q if for all $Y \in Q$ there exists $X \in P$ such that X is Turing reducible to Y. A *weak degree* is an equivalence class of mass problems under mutual weak reducibility. Let $\mathcal{P}_w$ be the lattice of weak degrees of mass problems associated with nonempty Π^0_1 subsets of the Cantor space. The lattice $\mathcal{P}_w$ has been studied in previous publications. The purpose of this paper is to show that $\mathcal{P}_w$ partakes of hyperarithmeticity. We exhibit a family of specific, natural degrees in $\mathcal{P}_w$ which are indexed by the ordinal numbers less than ω_1^{CK} and which correspond to the hyperarithmetical hierarchy. Namely, for each $\alpha < \omega_1^{\mathrm{CK}}$ let $\mathbf{h}_\alpha$ be the weak degree of $0^{(\alpha)}$, the αth Turing jump of 0. If $\mathbf{p}$ is the weak degree of any mass problem P, let $\mathbf{p}^*$ be the weak degree of the mass problem $P^* = \{Y \mid \exists X\,(X \in P \text{ and } \mathrm{BLR}(X) \subseteq \mathrm{BLR}(Y))\}$ where $\mathrm{BLR}(X)$ is the set of functions which are *boundedly limit recursive* in X. Let $\mathbf{1}$ be the top degree in $\mathcal{P}_w$. We prove that all of the weak degrees $\inf(\mathbf{h}^*_\alpha, \mathbf{1})$, $\alpha < \omega_1^{\mathrm{CK}}$, are distinct and belong to $\mathcal{P}_w$. In addition, we prove that certain index sets associated with $\mathcal{P}_w$ are Π^1_1 complete.

Contents

1 Introduction

Let 2^ω be the Cantor space, i.e., the space of infinite sequences of 0's and 1's. In previous papers [6, 8, 38, 34, 36, 37] Simpson and others studied the lattice $\mathcal{P}_w$ of degrees of unsolvability of mass problems associated with nonempty Π^0_1 subsets of 2^ω. Simpson [34, 36, 37, 38] showed that $\mathcal{P}_w$ contains many specific, natural degrees in addition to **1** and **0**, the top and bottom elements of $\mathcal{P}_w$. These specific, natural degrees in $\mathcal{P}_w$ arise from foundationally interesting topics such as reverse mathematics, algorithmic randomness, computational complexity, and subrecursive hierarchies from Gentzen-style proof theory.

In the present paper we exhibit some new examples of specific, natural degrees in $\mathcal{P}_w$. Our new examples arise from hyperarithmetical theory. In order to introduce our new examples, we first review some basic facts about the hyperarithmetical hierarchy. Some useful references for hyperarithmetical theory are Ash/Knight [3], Rogers [29, Chapter 16], Sacks [30, Part A], Shoenfield [31, Sections 7.8–7.11], and Simpson [33, Section VIII.3].

A *recursive ordinal* is an ordinal number which is the order type of a recursive well ordering of the integers. Obviously the recursive ordinals form a countable initial segment of the countable ordinals.[1] Let ω_1^{CK} (pronounced "omega-one-C-K" or "Church-Kleene-omega-one") be the least nonrecursive ordinal. Thus for any ordinal α we have $\alpha < \omega_1^{\mathrm{CK}}$ if and only if α is recursive. For each such α let $0^{(\alpha)}$ be the αth iterated Turing jump of 0. It is known that $0^{(\alpha)}$ is well defined

[1] It is known that an ordinal is recursive if and only if is the order type of a Δ^1_1 well ordering of the integers. An interesting refinement of this result is due to Montalbán [26].

up to Turing degree. Moreover, $0^{(\alpha)}$ is a Σ^0_3 singleton.[2] The Turing degrees $0^{(\alpha)}$, $\alpha < \omega_1^{\mathrm{CK}}$ are collectively known as the *hyperarithmetical hierarchy*. A Turing oracle X is said to be *hyperarithmetical* if it is Turing reducible to $0^{(\alpha)}$ for some $\alpha < \omega_1^{\mathrm{CK}}$. It is known that X is hyperarithmetical if and only if X is Turing reducible to some arithmetical singleton, if and only if X is Δ^1_1.

Given a Turing oracle X, we identify the Turing degree of X with the weak degree of the singleton set $\{X\}$. In particular, for each $\alpha < \omega_1^{\mathrm{CK}}$ let $\mathbf{h}_\alpha$ be the Turing degree of $0^{(\alpha)}$, identified with the weak degree of $\{0^{(\alpha)}\}$. We wish to embed the degrees $\mathbf{h}_\alpha$, $\alpha < \omega_1^{\mathrm{CK}}$, into $\mathcal{P}_w$. There is a general Embedding Lemma due to Simpson [36, Lemma 3.3] (see also [38]) which reads as follows: If $\mathbf{s}$ is the weak degree of a Σ^0_3 set of Turing oracles, then $\inf(\mathbf{s}, \mathbf{1})$ belongs to $\mathcal{P}_w$. Since $\{0^{(\alpha)}\}$ is Σ^0_3, it follows that $\inf(\mathbf{h}_\alpha, \mathbf{1}) \in \mathcal{P}_w$. However, this particular mapping of the hyperarithmetical hierarchy into $\mathcal{P}_w$ is of no interest, because for all $\alpha \geq 1$ we have $\inf(\mathbf{h}_\alpha, \mathbf{1}) = \mathbf{1}$ in view of the Kleene Basis Theorem. To obtain an interesting one-to-one embedding of the hyperarithmetical hierarchy into $\mathcal{P}_w$, we need an additional device: *bounded limit recursiveness*.

Given a Turing oracle X and a function f from integers to integers, we say that f is *boundedly limit recursive* in X, abbreviated $f \in \mathrm{BLR}(X)$, if there exist an approximating function $\widetilde{f}$ and a bounding function $\widehat{f}$ such that $\widetilde{f}$ is Turing computable from X, $\widehat{f}$ is Turing computable, and

$$|\{s \mid \widetilde{f}(n,s) \neq \widetilde{f}(n,s+1)\}| < \widehat{f}(n)$$

with $f(n) = \lim_s \widetilde{f}(n,s)$ for all n. Here we are implicitly citing the Limit Lemma, a well known result which says that f is (unboundedly) limit recursive if and only if f is Turing reducible[3] to $0'$, if and only if f is Δ^0_2. More precisely, one can say that $f \in \mathrm{BLR}(0)$ if and only if f is weakly truth-table reducible to $0'$, if and only if f occurs at the ωth level of the Ershov hierarchy.

We prove below that if S is Σ^0_3 then

$$S^* = \{Y \mid \exists X\,(X \in S \text{ and } \mathrm{BLR}(X) \subseteq \mathrm{BLR}(Y))\}$$

is again Σ^0_3. In particular, for each $\alpha < \omega_1^{\mathrm{CK}}$, $\{0^{(\alpha)}\}^*$ is Σ^0_3. Therefore, letting $\mathbf{h}^*_\alpha$ be the weak degree of $\{0^{(\alpha)}\}^*$, we have $\inf(\mathbf{h}^*_\alpha, \mathbf{1}) \in \mathcal{P}_w$ by the Embedding Lemma [36, 38]. Moreover, we prove below that all of these weak degrees $\inf(\mathbf{h}^*_\alpha, \mathbf{1})$, $\alpha < \omega_1^{\mathrm{CK}}$ are distinct. In this way we obtain our specific, natural, one-to-one embedding of the hyperarithmetical hierarchy into $\mathcal{P}_w$. This is our first main result.

In order to state our second main result, let P_i, $i = 0, 1, \ldots$, be a standard, recursive enumeration of all of the nonempty Π^0_1 subsets of 2^ω. Let $\mathbf{p}_i$ be the weak degree of P_i. Thus by definition $\mathcal{P}_w = \{\mathbf{p}_i \mid i = 0, 1, \ldots\}$. We prove below that the index set $\{i \mid \mathbf{p}_i = \mathbf{1}\}$ is a Π^1_1 complete set of integers. More generally, if j is such that $\mathbf{p}_j > \mathbf{0}$, then $\{i \mid \mathbf{p}_i = \mathbf{p}_j\}$ and $\{i \mid \mathbf{p}_i \geq \mathbf{p}_j\}$ are Π^1_1 complete

[2]By a Σ^0_3 *singleton* we mean a Turing oracle X such that the one-element set $\{X\}$ is Σ^0_3. Some interesting results concerning arithmetical singletons are due to H. Tanaka [40] and Leo Harrington (unpublished).

[3]Note that $0' = 0^{(1)}$ = the Turing jump of 0 = the halting problem.

sets of integers. We leave it as an open question to characterize the j's such that $\{i \mid \mathbf{p}_i \leq \mathbf{p}_j\}$ is a Π^1_1 complete set of integers.

Another open question is to calculate the degree of unsolvability of $\mathrm{Th}(\mathcal{P}_w)$, the first-order theory of the lattice $\mathcal{P}_w$. We conjecture that $\mathrm{Th}(\mathcal{P}_w)$ is recursively isomorphic to $\mathcal{O}^{(\omega)}$, the ωth Turing jump of $\mathcal{O}$. Here $\mathcal{O}$ is Kleene's O, i.e., a Π^1_1 complete set of integers. This conjecture seems reasonable in light of our results stated above plus the known characterizations of $\mathrm{Th}(\mathcal{S})$ (up to recursive isomorphism) for other degree structures $\mathcal{S}$. See for example Simpson [32] and Nies/Shore/Slaman [28].

The idea that $\mathcal{P}_w$ may partake of hyperarithmeticity was first broached by Simpson in his 2005 grant application to the U.S. National Science Foundation, which eventually resulted in NSF grant DMS-0600823.

The reader who is familiar with the basics of recursion theory and hyperarithmetical theory will find that this paper is largely self-contained. We use standard recursion-theoretical notation from Rogers [29].

2 Bounded limit recursiveness

Let $\omega = \{0, 1, 2, \ldots\}$ = the set of nonnegative integers. Let ω^ω be the *Baire space*, i.e., the set of totally defined functions $f : \omega \to \omega$.

Definition 2.1. Let A be a Turing oracle. We define $f : \omega \to \omega$ to be *boundedly limit recursive in* A if there exist an A-recursive *approximating function* $\widetilde{f} : \omega \times \omega \to \omega$ and a recursive *bounding function* $\widehat{f} : \omega \to \omega$ such that for all n, $\{s \mid \widetilde{f}(n,s) \neq \widetilde{f}(n,s+1)\}$ is finite of cardinality $< \widehat{f}(n)$ and $\lim_s \widetilde{f}(n,s) = f(n)$. We write

$$\mathrm{BLR}(A) = \{f \in \omega^\omega \mid f \text{ is boundedly limit recursive in } A\}.$$

Remark 2.2. The next three lemmas provide better understanding of $\mathrm{BLR}(A)$. See also Section 6 below. We write $f \leq_T A$ to mean that f is A-*recursive*, i.e., Turing reducible to A, i.e., computable using A as a Turing oracle. Let A' be the Turing jump of A.

Lemma 2.3. *We have* $f \in \mathrm{BLR}(A)$ *if and only if* $f \leq_T A'$ *with recursively bounded use of* A' *and unbounded use of* A.

Proof. "If". Assume $f \leq_T A'$ with recursively bounded use of A' and unbounded use of A, say $f(n) = \varphi_e^{(2),A}(A' \upharpoonright b(n), n)$ where $b(n)$ is recursive. Let A'_s = the finite subset of A' enumerated in the first s steps of some fixed A-recursive enumeration of A'. Let $\widetilde{f}(n,s) = \varphi_{e,s}^{(2),A\upharpoonright s}(A'_s \upharpoonright b(n), n)$ if this is defined, 0 otherwise. Then $\widetilde{f}(n,s)$ is A-recursive and $f(n) = \lim_s \widetilde{f}(n,s)$. Moreover

$$|\{s \mid \widetilde{f}(n,s) \neq \widetilde{f}(n,s+1)\}| \leq 2b(n) + 1$$

so $f \in \mathrm{BLR}(A)$ via $\widehat{f}(n) = 2b(n) + 2$.

"Only if". Assume $f \in \mathrm{BLR}(A)$ via an A-recursive approximating function $\widetilde{f}$ and a recursive bounding function $\widehat{f}$. We describe how to compute $f(n)$. For each $i < \widehat{f}(n)$ we ask the oracle A' whether

$$|\{s \mid \widetilde{f}(n,s) \neq \widetilde{f}(n,s+1)\}| > i.$$

From the answers to these questions, we immediately read off

$$k(n) = |\{s \mid \widetilde{f}(n,s) \neq \widetilde{f}(n,s+1)\}|.$$

Then $f(n) = \widetilde{f}(n,t)$ for the least t such that

$$|\{s < t \mid \widetilde{f}(n,s) \neq \widetilde{f}(n,s+1)\}| = k(n).$$

Here we have used A' only to compute $k(n)$. Therefore, our use of A' was bounded by $b(n) = \max\{p(n,i) \mid i < \widehat{f}(n)\}$ where $p(n,i)$ is a fixed, primitive recursive function. Since $\widehat{f}(n)$ is recursive, so is $b(n)$. □

Lemma 2.4. *$X \leq_T Y$ implies $\mathrm{BLR}(X) \subseteq \mathrm{BLR}(Y)$, which implies $X' \leq_T Y'$.*

Proof. The first implication is clear from the definition of BLR, because if $X \leq_T Y$ then every X-recursive function is Y-recursive. The second implication is an easy consequence of Lemma 2.3. □

Lemma 2.5. *The following are pairwise equivalent.*

1. $\mathrm{BLR}(X) \subseteq \mathrm{BLR}(Y)$.
2. *For each partial X-recursive function $\psi(n)$ we have $g \in \mathrm{BLR}(Y)$ where*

$$g(n) = \begin{cases} \psi(n) + 1 & \text{if } \psi(n) \text{ is defined,} \\ 0 & \text{otherwise.} \end{cases}$$

3. *There exists $h \in \mathrm{BLR}(Y)$ such that*

$$\forall n \, (\text{if } \varphi_n^{(1),X}(n) \text{ is defined then } h(n) = \varphi_n^{(1),X}(n)).$$

Proof. Clearly $1 \Rightarrow 2$ and $2 \Rightarrow 3$. To prove $3 \Rightarrow 2$, assume that $\psi(n)$ is partial X-recursive. Let e be an index of ψ, i.e., $\varphi_e^{(1),X}(n) \simeq \psi(n)$ for all n. Let $\theta(n) \simeq$ the least $\sigma \subset X$ such that $\varphi_{e,|\sigma|}^{(1),\sigma}(n)$ is defined. Since $\theta(n)$ is partial X-recursive, let $p(n)$ be primitive recursive such that $\theta(n) \simeq \varphi_{p(n)}^{(1),X}(p(n))$. Let h be as in 3, and let $g(n) \simeq \varphi_{e,|h(p(n))|}^{(1),h(p(n))}(n)+1$ if the latter is defined, 0 otherwise. Then g satisfies 2. To prove $2 \Rightarrow 1$, let $f \in \mathrm{BLR}(X)$ be given. For all n and all $i \leq \widehat{f}(n) + 1$, let $\psi_i(n) \simeq$ the ith successive value of $\widetilde{f}(n,s)$ as $s \to \infty$. The functions $\psi_i(n)$ are uniformly partial X-recursive, so by 2 the functions $g_i(n) = \psi_i(n) + 1$ if $\psi_i(n)$ is defined, 0 otherwise, are uniformly $\mathrm{BLR}(Y)$ with uniformly recursive bounding functions $\widehat{g}_i(n)$. Let $g(n) = g_i(n) - 1$ for the least

i such that $g_{i+1}(n) = 0$. Then $f = g \in \mathrm{BLR}(Y)$ via the recursive bounding function $\widehat{g}(n) = \sum\{\widehat{g}_i(n) \mid i < \widehat{f}(n)\}$. $\square$

Theorem 2.6. *The binary relation* $\mathrm{BLR}(X) \subseteq \mathrm{BLR}(Y)$ *is* Σ^0_3. *Furthermore, if* S *is* Σ^0_3 *then so is*

$$S^* = \{Y \mid \exists X\,(X \in S \textit{ and } \mathrm{BLR}(X) \subseteq \mathrm{BLR}(Y))\}.$$

Proof. By Lemma 2.5, $\mathrm{BLR}(X) \subseteq \mathrm{BLR}(Y)$ if and only if there exists $h \in \mathrm{BLR}(Y)$ such that $\forall n\,(h(n) = \varphi^{(1),X}_n(n)$ if the latter is defined). This holds if and only if there exist (indices for) a totally defined Y-recursive function $\widetilde{h}$ and a totally defined recursive function $\widehat{h}$ such that

$$\forall n\,(|\{s \mid \widetilde{h}(n,s) \neq \widetilde{h}(n,s+1)\}| < \widehat{h}(n))$$

and $\forall n\,\forall s\,\exists t > s\,(\widetilde{h}(n,t) = \varphi^{(1),X\restriction t}_{n,t}(n)$ if the latter is defined). A Tarski/Kuratowski computation (see Rogers [29, Section 14.3]) shows that this statement is Σ^0_3. This proves the first part of the theorem.

In order to prove the second part, we first prove it with S replaced by P, where P is a Π^0_1 subset of ω^ω. Let $T \subseteq \omega^{<\omega}$ be a recursive tree such that

$$P = \{\text{paths through } T\},$$

i.e.,

$$P = \{f \in \omega^\omega \mid \forall j\,(f \restriction j \in T)\}.$$

By Lemma 2.5, $Y \in P^*$ if and only if there exist $f \in P$ and $g \in \mathrm{BLR}(Y)$ such that $\forall i\,(g(2i) = f(i))$ and $\forall n\,(g(2n+1) = \varphi^{(1),f}_n(n)$ if the latter is defined). This holds if and only if there exist (indices for) a totally defined Y-recursive function $\widetilde{g}$ and a totally defined recursive function $\widehat{g}$ such that

$$\forall n\,(|\{s \mid \widetilde{g}(n,s) \neq \widetilde{g}(n,s+1)\}| < \widehat{g}(n))$$

and

$$\forall j\,\forall s\,\exists t > s\,(\langle \widetilde{g}(2i,t) \mid i < j\rangle \in T)$$

and

$$\forall n\,\forall j\,\forall s\,\exists t > s\,(\widetilde{g}(2n+1,t) = \varphi^{(1),\langle \widetilde{g}(2i,t) \mid i<j\rangle}_{n,t}(n) \text{ if the latter is defined}).$$

Thus, by a Tarski/Kuratowski computation, P^* is Σ^0_3.

Now let S be Σ^0_3, say

$$S = \{X \mid \exists i\,\forall m\,\exists n\,R(i,m,n,X)\}$$

where R is recursive. Consider the Π^0_1 set

$$P = \{\langle i\rangle^\frown(X \oplus f) \mid \forall m\,R(i,m,f(m),X)\}.$$

By what we have already proved, P^* is Σ^0_3. But clearly the Turing upward closure of S is the same as the Turing upward closure of P. Hence by Lemma 2.4 we have $S^* = P^*$. It follows that S^* is Σ^0_3. $\square$

Remark 2.7. (1) We could modify Definition 2.1 by requiring $\widehat{f}$ to be not only recursive but also primitive recursive. In this case, Lemma 2.3 would hold with "recursively bounded" replaced by "primitive recursively bounded". Lemma 2.5 and Theorems 2.6, 4.4, 4.5, 4.7 would go through unchanged but with a different meaning. (2) Alternatively, we could modify Definition 2.1 by allowing $\widehat{f}$ to be A-recursive instead of merely recursive. In this case, Lemma 2.3 would hold with "recursively bounded" replaced by "A-recursively bounded". Lemma 2.5 would fail, the definition of S^* would need to be changed accordingly, and Theorems 2.6, 4.4, 4.5, 4.7 would go through with this change.

3 Diagonal nonrecursiveness and 1-genericity

Definition 3.1. A function $g \in \omega^\omega$ is said to be 1-*generic* if for every recursively enumerable set of strings $A \subseteq \omega^{<\omega}$ there exists a string $\sigma \subset g$ such that either $\sigma \in A$ or $\neg \exists \tau\, (\sigma \subseteq \tau \text{ and } \tau \in A)$.

Remark 3.2. There is an extensive literature on 1-genericity. See Kumabe [24] and the references listed there.

Remark 3.3. In the definition of 1-genericity, it suffices to consider only the particular recursive sets of strings

$$A_n = \{\tau \in \omega^{<\omega} \mid \varphi^{(1),\tau}_{n,|\tau|}(n) \text{ is defined}\}.$$

This is because, for any recursively enumerable set $A \subseteq \omega^{<\omega}$, we can find an n such that for all $g \in \omega^\omega$, $\varphi^{(1),g}_n(n)$ is defined if and only if $\exists \tau \subset g\, (\tau \in A)$.

Lemma 3.4. *Given a Turing oracle X, we can find a 1-generic g such that* $\mathrm{BLR}(X) = \mathrm{BLR}(g)$.

Proof. Define $h \in \omega^\omega$ by $h(n) = \varphi^{(1),X}_n(n)+1$ if $\varphi^{(1),X}_n(n)$ is defined, 0 otherwise. Let the sets A_n, $n = 0, 1, 2, \ldots$ be as in Remark 3.3. Define a sequence of strings $\tau_0 \subseteq \tau_1 \subseteq \cdots \subseteq \tau_n \subseteq \cdots$ as follows. Begin with $\tau_0 = \langle\rangle$, the empty string. Assuming that τ_n has already been defined, ask the oracle $0'$ whether there exists $\tau \supseteq \tau_n{}^\frown\langle h(n)\rangle$ such that $\tau \in A_n$. If the answer is yes, let τ_{n+1} = the least such τ in some fixed recursive enumeration of $\omega^{<\omega}$. If the answer is no, let $\tau_{n+1} = \tau_n{}^\frown\langle h(n)\rangle$. Finally let $g = \bigcup_n \tau_n$. By Remark 3.3, g is 1-generic. By construction, the sequence of strings τ_n, $n = 0, 1, 2, \ldots$ is boundedly limit recursive in each of the oracles X and g with bounding functions 4^n and 2^n respectively. Moreover, $h(n) = \tau_{n+1}(|\tau_n|)$ and $\varphi^{(1),g}_n(n) \simeq \varphi^{(1),\tau_{n+1}}_{n,|\tau_{n+1}|}(n)$. It follows by Lemma 2.5 that $\mathrm{BLR}(X) = \mathrm{BLR}(g)$. $\square$

Definition 3.5. A function $f \in \omega^\omega$ is said to be *diagonally nonrecursive*, abbreviated DNR, if $f(n) \not\simeq \varphi^{(1)}_n(n)$ for all n.

Remark 3.6. There is an extensive literature on DNR functions and their Turing degrees. See for instance Jockusch [21], Kjos-Hanssen/Merkle/Stephan [23], and Ambos-Spies/Kjos-Hanssen/Lempp/Slaman [1]. The following lemma appears in Demuth/Kučera [15, Corollary 9].

Lemma 3.7. *If g is* 1*-generic, there is no* DNR *function* $\leq_T g$.

Proof. Fix an index e. Consider the recursively enumerable set of strings

$$A = \{\tau \mid \exists n\,(\varphi_e^{(1),\tau}(n)\downarrow = \varphi_n^{(1)}(n)\downarrow)\}.$$

Because g is 1-generic, there are two cases.

Case 1. There exists $\tau \subset g$ such that $\tau \in A$. Then for some n we have $\varphi_e^{(1),\tau}(n)\downarrow = \varphi_n^{(1)}(n)\downarrow$, hence $\varphi_e^{(1),g}(n)\downarrow = \varphi_n^{(1)}(n)\downarrow$, hence $\varphi_e^{(1),g}$ is not DNR.

Case 2. For some $\sigma \subset g$, there is no $\tau \supseteq \sigma$ belonging to A. Fix such a σ. Define a partial recursive function $\theta(n)$ as follows. Given n, look for $\tau \supseteq \sigma$ such that $\varphi_e^{(1),\tau}(n)\downarrow$. If and when we find such a τ, define $\theta(n) = \varphi_e^{(1),\tau}(n)$. If no such τ is found, let $\theta(n)$ be undefined.

Let n be an index of θ. We claim that $\varphi_e^{(1),\tau}(n)\uparrow$ for all $\tau \supseteq \sigma$. Otherwise, there exists $\tau \supseteq \sigma$ such that $\varphi_e^{(1),\tau}(n)\downarrow$ and $\theta(n) \simeq \varphi_e^{(1),\tau}(n)$, by definition of $\theta(n)$. But then

$$\varphi_e^{(1),\tau}(n)\downarrow = \theta(n)\downarrow = \varphi_n^{(1)}(n)\downarrow,$$

hence $\tau \in A$, contradicting the case hypothesis. This proves our claim.

Our claim implies that $\varphi_e^{(1),g}(n)\uparrow$. Hence $\varphi_e^{(1),g}$ is not totally defined, hence not DNR.

We have now shown that $\varphi_e^{(1),g}$ is not DNR for any e. □

Theorem 3.8. *Given a Turing oracle X, we can find a Turing oracle Y such that* $\mathrm{BLR}(X) = \mathrm{BLR}(Y)$ *and there is no* DNR *function* $\leq_T Y$.

Proof. This is immediate from Lemmas 3.4 and 3.7. □

4 Embedding hyperarithmeticity into $\mathcal{P}_w$

The Embedding Lemma of Simpson [36, 38] reads as follows.

Lemma 4.1 (Embedding Lemma)**.** *Let $S \subseteq \omega^\omega$ be Σ_3^0. Let $\mathbf{s}$ be the weak degree of S. Then $\inf(\mathbf{s}, \mathbf{1})$ belongs to $\mathcal{P}_w$.*

Proof. See [36, Lemma 3.3] or [38, Lemma 4]. □

Remark 4.2. Simpson [34, 36, 37, 38] has used the Embedding Lemma to obtain many examples of specific, natural degrees in $\mathcal{P}_w$. In particular, let $\mathbf{d}$ be the weak degree of DNR, the set of diagonally nonrecursive functions, and let $\mathbf{r}_n$ be the weak degree of R_n, the set of n-random sequences of 0's and 1's. Using the Embedding Lemma 4.1, Simpson [34, 36] has shown that $\mathbf{d}$ and $\mathbf{r}_1$ and $\inf(\mathbf{r}_2, \mathbf{1})$ belong to $\mathcal{P}_w$ and that

$$\mathbf{0} < \mathbf{d} < \mathbf{r}_1 < \inf(\mathbf{r}_2, \mathbf{1}) < \mathbf{1},$$

where $\mathbf{0}$ and $\mathbf{1}$ are the bottom and top degrees in $\mathcal{P}_w$.

We now extend this methodology as follows.

Definition 4.3. For any set S of Turing oracles, let

$$S^* = \{Y \mid \exists X\,(X \in S \text{ and } \mathrm{BLR}(X) \subseteq \mathrm{BLR}(Y))\}.$$

If $\mathbf{s}$ is the weak degree of S, let $\mathbf{s}^*$ be the weak degree of S^*. By Lemma 2.4, $\mathbf{s}^*$ depends only on the weak degree $\mathbf{s}$ and not on its representative S.

Theorem 4.4. *If S is Σ^0_3, then the weak degree $\inf(\mathbf{s}^*, \mathbf{1})$ belongs to $\mathcal{P}_w$.*

Proof. By Theorem 2.6 S^* is Σ^0_3. Therefore by Lemma 4.1 $\inf(\mathbf{s}^*, \mathbf{1}) \in \mathcal{P}_w$. □

Theorem 4.5. *If $\emptyset \neq S \subseteq \{X \mid X' \geq_T 0''\}$, then the weak degree $\inf(\mathbf{s}^*, \mathbf{1})$ is incomparable with each of the weak degrees $\mathbf{d}$ and $\mathbf{r}_1$ and $\inf(\mathbf{r}_2, \mathbf{1})$.*

Proof. Since S is nonempty, Theorem 3.8 gives $Y \in S^*$ such that there is no DNR function $\leq_T Y$. It follows that $\mathbf{d} \not\leq \mathbf{s}^*$, hence $\mathbf{d} \not\leq \inf(\mathbf{s}^*, \mathbf{1})$. On the other hand, since $S \subseteq \{X \mid X' \geq_T 0''\}$, it follows by Lemma 2.4 that $S^* \subseteq \{Y \mid Y' \geq_T 0''\}$, and this set is known to be of measure 0. (See for instance [37, proof of Theorem 4.5].) In addition, recall that $\mathbf{1}$ is the weak degree of PA, the set of complete extensions of Peano Arithmetic, and it is known that the set $\{Y \in 2^\omega \mid \exists X\,(X \in \mathrm{PA} \text{ and } X \leq_T Y)\}$ is also of measure 0. (See for instance the general non-helping result in [34, Lemma 7.3].) Since R_2 is of measure 1, we can find $Y \in \mathrm{R}_2$ not belonging to either of these sets of measure 0. It follows that $\mathbf{r}_2 \not\geq \inf(\mathbf{s}^*, \mathbf{1})$, hence $\inf(\mathbf{r}_2, \mathbf{1}) \not\geq \inf(\mathbf{s}^*, \mathbf{1})$. The proof is now finished, because $\mathbf{d} < \mathbf{r}_1 < \inf(\mathbf{r}_2, \mathbf{1})$. □

Remark 4.6. The effect of Theorems 4.4 and 4.5 is to supplement the Embedding Lemma 4.1 by providing another general method for discovering specific, natural degrees in $\mathcal{P}_w$. We shall now use this technique to embed the hyperarithmetical hierarchy into $\mathcal{P}_w$. Recall from the Introduction that $0^{(\alpha)}$ is the αth Turing jump of 0, and $\mathbf{h}_\alpha$ is the weak degree of the singleton set $\{0^{(\alpha)}\}$.

Theorem 4.7. *For each $\alpha < \omega_1^{\mathrm{CK}}$ we have $\inf(\mathbf{h}^*_\alpha, \mathbf{1}) \in \mathcal{P}_w$. For each $\alpha < \omega_1^{\mathrm{CK}}$ except 0, $\inf(\mathbf{h}^*_\alpha, \mathbf{1})$ is incomparable with $\mathbf{d}$ and $\mathbf{r}_1$ and $\inf(\mathbf{r}_2, \mathbf{1})$ in the lattice ordering of $\mathcal{P}_w$. For all $\alpha < \beta < \omega_1^{\mathrm{CK}}$ we have $\inf(\mathbf{h}^*_\alpha, \mathbf{1}) < \inf(\mathbf{h}^*_\beta, \mathbf{1})$, in fact $\inf(\mathbf{h}^*_\alpha, \mathbf{d}) < \inf(\mathbf{h}^*_\beta, \mathbf{d})$.*

Proof. By [30, Section II.4] or [33, Section VIII.3], the singleton set $\{0^{(\alpha)}\}$ is Σ^0_3. Hence, by Theorem 4.4, $\inf(\mathbf{h}^*_\alpha, \mathbf{1}) \in \mathcal{P}_w$. For $\alpha > 0$ we obviously have $0^{(\alpha)} \geq_T 0'$, hence by Theorem 4.5 $\inf(\mathbf{h}^*_\alpha, \mathbf{1})$ is incomparable with $\mathbf{d}$ and $\mathbf{r}_1$ and $\inf(\mathbf{r}_2, \mathbf{1})$. For $\alpha < \beta < \omega_1^{\mathrm{CK}}$ we obviously have $0^{(\alpha)} <_T 0^{(\beta)}$, hence $\mathbf{h}_\alpha < \mathbf{h}_\beta$, hence $\mathbf{h}^*_\alpha \leq \mathbf{h}^*_\beta$, hence $\inf(\mathbf{h}^*_\alpha, \mathbf{d}) \leq \inf(\mathbf{h}^*_\beta, \mathbf{d})$. To obtain a strict inequality, apply Theorem 3.8 to get Y such that $\mathrm{BLR}(0^{(\alpha)}) = \mathrm{BLR}(Y)$ and there is no DNR function $\leq_T Y$. Then $Y \in \{0^{(\alpha)}\}^*$ and, since $\mathrm{BLR}(0^{(\alpha)}) \subsetneqq \mathrm{BLR}(0^{(\beta)})$, there is no $Z \leq_T Y$ such that $Z \in \{0^{(\beta)}\}^*$. This shows that $\mathbf{h}^*_\alpha \not\geq \inf(\mathbf{h}^*_\beta, \mathbf{d})$. It now follows that $\inf(\mathbf{h}^*_\alpha, \mathbf{d}) < \inf(\mathbf{h}^*_\beta, \mathbf{d})$, hence $\inf(\mathbf{h}^*_\alpha, \mathbf{1}) < \inf(\mathbf{h}^*_\beta, \mathbf{1})$. □

5 Index sets associated with $\mathcal{P}_w$

In this section we prove that certain index sets associated with weak reducibility in $\mathcal{P}_w$ are Π^1_1 complete. The proof is based on the following lemma concerning Π^0_1 subsets of ω^ω. Let $\mathrm{REC} = \{g \in \omega^\omega \mid g \text{ is recursive}\}$.

Lemma 5.1. *Given Π^0_1 sets $P, Q \subseteq \omega^\omega$ such that $Q \cap \mathrm{REC} = \emptyset$, we can effectively find a Π^0_1 set $H(P,Q) \subseteq \omega^\omega$ such that P is homeomorphic to $H(P,Q)$ and there do not exist $g \in Q$ and $h \in H(P,Q)$ such that $g \leq_T h$.*

Proof. The proof of Lemma 5.1 will involve a construction and two sublemmas. We begin with some general remarks concerning treemaps.

Definition 5.2. A *treemap* is a function $F : \omega^{<\omega} \to \omega^{<\omega}$ such that

$$F(\sigma)^\frown\langle i\rangle \subseteq F(\sigma^\frown\langle i\rangle)$$

for all $\sigma \in \omega^{<\omega}$ and all $i \in \omega$. A *tree* is a set $T \subseteq \omega^{<\omega}$ such that

$$\forall\sigma\,\forall\tau\,((\sigma \subset \tau \text{ and } \tau \in T) \Rightarrow \sigma \in T).$$

Given a treemap F and a tree T, we have another tree

$$F(T) = \{\tau \mid \exists\sigma\,(\sigma \in T \text{ and } \tau \subseteq F(\sigma))\}.$$

Remark 5.3. Given $\tau \in F(T)$, let σ be mininal such that $\tau \subseteq F(\sigma)$. Then $\sigma \in T$ and σ is a *substring* of τ, i.e., $\sigma = \langle \tau(j_0), \tau(j_1), \ldots, \tau(j_{m-1})\rangle$ for some $j_0 < j_1 < \cdots < j_{m-1} < |\tau|$ where $m = |\sigma|$. Therefore, in the definition of $F(T)$, the quantifier $\exists\sigma$ can be replaced by a bounded quantifier, namely

$$F(T) = \{\tau \mid (\exists\sigma \text{ substring of } \tau)\,(\sigma \in T \text{ and } \tau \subseteq F(\sigma))\}.$$

From this it follows that, for example, if F and T are recursive then so is $F(T)$.

Remark 5.4. The sets

$$P = \{\text{paths through } T\}$$

and

$$F(P) = \{\text{paths through } F(T)\}.$$

are closed subsets of ω^ω and we have a *homeomorphism* $F : P \to F(P)$, i.e., a continuous, open, one-to-one mapping of P onto $F(P)$, defined by

$$F(f) = \bigcup\{F(\sigma) \mid \sigma \subset f\}$$

for all $f \in P$. In particular, if $T \subseteq \omega^{<\omega}$ is a recursive tree and $F : \omega^{<\omega} \to \omega^{<\omega}$ is a recursive treemap, then P and $F(P)$ are Π^0_1 subsets of ω^ω, and $F : P \to F(P)$ is a recursive homeomorphism of P onto $F(P)$.

We now begin the proof of Lemma 5.1. Given a Π^0_1 set $Q \subseteq \omega^\omega$ such that $Q \cap \mathrm{REC} = \emptyset$, let $U \subseteq \omega^{<\omega}$ be a recursive tree such that

$$Q = \{g \in \omega^\omega \mid g \text{ is a path through } U\}.$$

We shall use U to define a treemap H_Q.

In constructing H_Q, we shall sometimes identify strings $\sigma \in \omega^{<\omega}$ with their Gödel numbers $\#(\sigma) = \prod_{i<|\sigma|} p_i^{\sigma(i)+1}$, where $p_0, p_1, p_2, \ldots$ are the prime numbers $2, 3, 5, \ldots$ in increasing order. For integers e and strings σ, we write $\Phi_e(\sigma) = \langle \varphi_{e,|\sigma|}^{(1),\sigma}(j) \mid j < n \rangle$ where n = least j such that $\varphi_{e,|\sigma|}^{(1),\sigma}(j)\uparrow$. Note that $\Phi_e : \omega^{<\omega} \to \omega^{<\omega}$ is recursive and *monotonic*, i.e., $\sigma \subseteq \tau$ implies $\Phi_e(\sigma) \subseteq \Phi_e(\tau)$.

The construction of H_Q is as follows. We define $H_Q(\sigma)$ by induction on $\sigma \in \omega^{<\omega}$. Begin with $H_Q(\langle\rangle) = \langle\rangle$ where $\langle\rangle$ is the empty string. Assume that $H_Q(\sigma)$ has already been defined. In order to define $H_Q(\sigma^\frown\langle i\rangle)$, let $e = |\sigma|$ and let $\tau_0 = H_Q(\sigma)^\frown\langle i\rangle$. Given τ_n, let τ_{n+1} = the least τ such that $\tau_n \subset \tau$ and $\Phi_e(\tau_n) \subset \Phi_e(\tau) \in U$. If no such τ exists, let τ_{n+1} be undefined. This must happen for some n, because otherwise we would have $\Phi_e(\tau_n) \subset \Phi_e(\tau_{n+1}) \in U$ for all n, hence $\bigcup_n \Phi_e(\tau_n)$ would be a recursive path through U, i.e., a member of $Q \cap \mathrm{REC}$, contradicting the assumption that $Q \cap \mathrm{REC} = \emptyset$. We then let $H_Q(\sigma^\frown\langle i\rangle) = \tau_n$ for the least n such that τ_{n+1} is undefined.

Sublemma 5.5. *H_Q is a treemap, and $H_Q \leq_T 0'$.*

Proof. This is clear from the construction of H_Q. □

Sublemma 5.6. *There do not exist $f \in \omega^\omega$ and $g \in Q$ such that $H_Q(f) \geq_T g$.*

Proof. Suppose $H_Q(f) \geq_T g$. Let e be such that $\varphi_e^{(1),H_Q(f)} = g$. Let $\sigma^\frown\langle i\rangle = f \upharpoonright e+1$ and consider the definition of $H_Q(\sigma^\frown\langle i\rangle)$. Since τ_{n+1} is undefined and $\tau_n = H_Q(\sigma^\frown\langle i\rangle) = H_Q(f \upharpoonright e+1)$, there is no $\tau \supset H_Q(f \upharpoonright e+1)$ such that $\Phi_e(\tau) \supset \Phi_e(H_Q(f \upharpoonright e+1))$ and $\Phi_e(\tau) \in U$. In particular, $\varphi_e^{(1),H_Q(f)}$ is not a path through U. In other words, $g \notin Q$. This proves the sublemma. □

Now let $P \subseteq \omega^\omega$ be Π_1^0. By Remarks 5.3 and 5.4 and Sublemma 5.5, $H_Q(P)$ is $\Pi_1^{0,0'}$ and H_Q is a homeomorphism of P onto $H_Q(P)$. It follows that $H_Q(P)$ is Π_2^0, say

$$H_Q(P) = \{h \in \omega^\omega \mid \forall m\, \exists n\, R(m,n,h)\}$$

where $R \subseteq \omega \times \omega \times \omega^\omega$ is recursive. Define

$$H(P,Q) = \{h \oplus k \in \omega^\omega \mid \forall m\, (k(m) = \text{least } n \text{ such that } R(m,n,h))\}.$$

Clearly $H(P,Q)$ is Π_1^0 and $h \mapsto h \oplus k$ is a homeomorphism of $H_Q(P)$ onto $H(P,Q)$. Moreover, for all $h \oplus k \in H(P,Q)$ we have $h \oplus k \equiv_T h \in H_Q(P)$, hence by Sublemma 5.6 there is no $g \in Q$ such that $g \leq_T h \oplus k$. Note also that our construction was such that, given Π_1^0 indices of P and Q, we can effectively find a $\Pi_1^{0,0'}$ index of $H_Q(P)$, a Π_2^0 index of $H_Q(P)$, and a Π_1^0 index of $H(P,Q)$.

This finishes the proof of Lemma 5.1. □

We now return to the study of Π_1^0 subsets of 2^ω.

Theorem 5.7. *Let P_i, $i = 0, 1, \ldots$ be a standard recursive enumeration of the nonempty Π^0_1 subsets of 2^ω. For each i, let $\mathbf{p}_i$ be the weak degree of P_i. If j is such that $\mathbf{p}_j > \mathbf{0}$, then the index sets $\{i \mid \mathbf{p}_i = \mathbf{p}_j\}$ and $\{i \mid \mathbf{p}_i \geq \mathbf{p}_j\}$ are Π^1_1 complete sets of integers.*

Proof. Let S_e, $e = 0, 1, \ldots$ be a standard recursive enumeration of the Π^0_1 subsets of ω^ω. As a consequence of the Kleene Normal Form Theorem, it is well known that the index set $\{e \mid S_e = \emptyset\}$ is a Π^1_1 complete set of integers. We shall reduce this Π^1_1 complete set to each of the index sets in question. Fix an index j such that $\mathbf{p}_j > \mathbf{0}$, i.e., $P_j \cap \mathrm{REC} = \emptyset$. Given an index e, Lemma 5.1 tells us that we can effectively find an index $h(e, j)$ such that $S_{h(e,j)} = H(S_e, P_j)$. Also, the proof of the Embedding Lemma 4.1 (see also Simpson [36, Lemma 3.3] and [38, Lemma 4]) tells us that we can effectively find an index $f(e, j)$ such that $P_{f(e,j)}$ is weakly equivalent to $S_e \cup P_j$. Combining these two results, we obtain:

> Given an index e, we can effectively find an index $i = f(h(e, j), j)$ such that P_i is weakly equivalent to $H(S_e, P_j) \cup P_j$.

Now, if $S_e = \emptyset$ then $H(S_e, P_j) = \emptyset$, hence P_i is weakly equivalent to P_j. On the other hand, if $S_e \neq \emptyset$ then $H(S_e, P_j) \neq \emptyset$ and for all $h \in H(S_e, P_j)$ there is no $g \in P_j$ such that $g \leq_T h$, hence P_j is not weakly reducible to P_i. Thus we see that the Π^1_1 complete set $\{e \mid S_e = \emptyset\}$ is reducible to both $\{i \mid \mathbf{p}_i = \mathbf{p}_j\}$ and $\{i \mid \mathbf{p}_i \geq \mathbf{p}_j\}$ via the reduction $e \mapsto f(h(e, j), j)$. This proves our theorem. □

Corollary 5.8. *The index set $\{i \mid \mathbf{p}_i = \mathbf{1}\}$ is Π^1_1 complete.*

Proof. This is the special case $\mathbf{p}_j = \mathbf{1}$ of Theorem 5.7. □

Remark 5.9. All of the special cases $\mathbf{p}_j \geq \mathbf{d}$ of Theorem 5.7 can be given an alternative proof based on Theorems 2.6 and 3.8. An open question is to characterize the indices j such that $\{i \mid \mathbf{p}_i \leq \mathbf{p}_j\}$ is Π^1_1 complete.

The following corollary of Theorem 5.7 is originally due to Simpson/Slaman [39] who proved it using a different method. Let $\mathcal{P}_s$ be the lattice of strong degrees of nonempty Π^0_1 subsets of 2^ω.

Corollary 5.10. *Each nonzero weak degree in $\mathcal{P}_w$ includes infinitely many strong degrees in $\mathcal{P}_s$.*

Proof. We begin with some background information. If P and Q are mass problems, we say that P is *strongly reducible* to Q if there exists a partial recursive functional Ψ such that $\Psi(Y) \in P$ for all $Y \in Q$. A *strong degree* is an equivalence class of mass problems under strong reducibility. Obviously strong reducibility implies weak reducibility, but the converse does not hold. In the case of Π^0_1 subsets of 2^ω, the strong reducibility relation

$$\{(i, j) \mid P_i \text{ is strongly reducible to } P_j\}$$

is easily seen to be arithmetical, in fact Σ^0_3. (See for instance Simpson [34, Corollary 4.9] or Cenzer/Hinman [9].) On the other hand, Theorem 5.7 implies that the weak reducibility relation

$$\{(i,j) \mid P_i \text{ is weakly reducible to } P_j\}$$

is Π^1_1 complete, hence not arithmetical, hence not Σ^0_3.

To prove Corollary 5.10, let $\mathbf{p} > \mathbf{0}$ be a weak degree in $\mathcal{P}_w$. We must show that there exists an infinite set $I \subseteq \omega$ such that (a) for all $i \in I$, P_i is of weak degree $\mathbf{p}$, and (b) for all $i, j \in I$, if $i \neq j$ then P_i and P_j are of different strong degrees. If such an I did not exist, then there would be a finite set of indices $j_1, \ldots, j_n$ such that for all i, P_i is of weak degree $\mathbf{p}$ if and only if P_i is of the same strong degree as one of $P_{j_1}, \ldots, P_{j_n}$. Hence the index set

$$\{i \mid P_i \text{ is of weak degree } \mathbf{p}\}$$

would be Σ^0_3, contradicting Theorem 5.7 which says that this set is Π^1_1 complete. This proves the corollary. □

6 More on bounded limit recursiveness

This section is tangential to the rest of the paper. The purpose of this section is to compare our notion of bounded limit recursiveness, Definition 2.1 above, with other notions that have appeared in the recursion-theoretical literature.

Remark 6.1. Rogers [29, Chapters 8 and 9] and Downey/Jockusch/Stob [17] have considered *truth-table reducibility* and *weak truth-table reducibility.* For $X, Y \in 2^\omega$ we write $X \le_{tt} Y$ to mean that X is truth-table computable from Y. For $f \in \omega^\omega$ and $Y \in 2^\omega$ we write $f \le_{wtt} Y$ to mean that f is weakly truth-table computable from Y. We sometimes identify a set $A \subseteq \omega$ with its *characteristic function* $\chi_A \in 2^\omega$ defined by $\chi_A(n) = 1$ if $n \in A$, 0 if $n \notin A$.

Theorem 6.2. *For $f \in \omega^\omega$ we have $f \in \mathrm{BLR}(0)$ if and only if $f \le_{wtt} 0'$, if and only if f occurs at level $\le \omega$ of the Ershov hierarchy.*

Proof. The proof is straightforward. See Lemma 2.3 above, plus Rogers [29, Exercise 9-45, pages 158–159], plus Downey/Jockusch/Stob [17, paragraph preceding Definition 1.1], plus Arslanov [2]. Note also that, according to [17, Definition 1.1], a Turing oracle X is *array recursive* if and only if there exists $g \le_{wtt} 0'$ which eventually dominates all $f \le_T X$. □

Remark 6.3. For arbitrary Turing oracles Y, it is not always the case that $f \in \mathrm{BLR}(Y)$ if and only if $f \le_{wtt} Y'$. For example, let $f \in \omega^\omega$ be so fast-growing that $f \not\le_{wtt} Z$ for all $Z \in 2^\omega$. In particular $f \not\le_{wtt} Y'$ for all Turing oracles Y, but by Lemma 3.4 we have $f \in \mathrm{BLR}(Y)$ for some Y.

On the other hand, there is the following result for $\{0,1\}$-valued functions.

Theorem 6.4. *For $X \subseteq \omega$ we have $X \in \mathrm{BLR}(Y)$ if and only if $X \le_{tt} Y'$.*

Proof. Let g be the characteristic function of X. Note that $g \in 2^\omega$. For the "if" part of the theorem, assume $X \le_{tt} Y'$, i.e., $g \le_{tt} Y'$. This means that there is a recursive mapping $n \mapsto \langle \tau_{n,i} \mid i < k_n \rangle$ such that for all n, $g(n) = 1$ if and only if $(\exists i < k_n)\,(\tau_{n,i} \subset Y')$. Let $Y' = \bigcup_s Y'_s$ where $Y'_0 \subseteq Y'_1 \subseteq \cdots \subseteq Y'_s \subseteq Y'_{s+1} \subseteq \cdots$

is a Y-recursive sequence of finite sets. Let $\widetilde{g}(n,s) = 1$ if $(\exists i < k_n)\,(\tau_{n,i} \subset Y'_s)$, 0 otherwise, and let $\widehat{g}(n) = \max\{|\tau_{n,i}| \mid i < k_n\}$. Clearly $g \in \mathrm{BLR}(Y')$ via the Y-recursive approximating function $\widetilde{g}$ and the recursive bounding function $\widehat{g}$.

For the "only if" part of the theorem, assume $g \in \mathrm{BLR}(Y')$ via a Y-recursive approximating function $\widetilde{g}$ and a recursive bounding function $\widehat{g}$. We may safely assume that $\widetilde{g}(n,s) < 2$ for all n and s, and $\widetilde{g}(n,0) = 0$ for all n. We now describe how to compute g from Y'. Given n, for each $i < \widehat{g}(n)$ ask the oracle Y' whether $|\{s \mid \widetilde{g}(n,s) \neq \widetilde{g}(n,s+1)\}| > i$. Upon receiving the answers to these questions, we know the number $|\{s \mid \widetilde{g}(n,s) \neq \widetilde{g}(n,s+1)\}|$. Then $g(n) = 0$ if this number is even, 1 if it is odd. Thus $X \leq_{tt} Y'$, Q.E.D. □

Corollary 6.5. *For arbitrary Turing oracles X and Y, we have $X' \leq_{tt} Y'$ if and only if $X' \in \mathrm{BLR}(Y)$.*

Proof. This is immediate from Theorem 6.4 upon recalling that $X' \subseteq \omega$. □

Corollary 6.6. *For arbitrary Turing oracles X and Y, the binary relation $X' \leq_{tt} Y'$ is Σ^0_3 in X and Y.*

Proof. Let h be the characteristic function of X'. By Corollary 6.5, $X' \leq_{tt} Y'$ if and only $X' \in \mathrm{BLR}(Y)$, i.e., $h \in \mathrm{BLR}(Y)$. This holds if and only if there exist (indices for) a totally defined Y-recursive function $\widetilde{h}$ and a totally defined recursive function $\widehat{h}$ such that

$$\forall n\,(|\{s \mid \widetilde{h}(n,s) \neq \widetilde{h}(n,s+1)\}| < \widehat{h}(n))$$

and

$$\forall n\,(n \in X' \Rightarrow \forall s\,\exists t\,(t > s \text{ and } \widetilde{h}(n,t) = 1))$$

and

$$\forall n\,(n \notin X' \Rightarrow \forall s\,\exists t\,(t > s \text{ and } \widetilde{h}(n,t) = 0)).$$

By a Tarski/Kuratowski computation, this statement is Σ^0_3 in X and Y. □

Definition 6.7. We say that X is *jump-traceable by Y* (Simpson [35]) if there exist recursive functions $f(n)$ and $g(n)$ such that $\forall n\,(\varphi^{(1),X}_n(n)\downarrow \Rightarrow \varphi^{(1),X}_n(n) \in W^Y_{f(n)})$ and $\forall n\,(|W^Y_{f(n)}| \leq g(n))$.

Lemma 6.8. *Assume that $\mathrm{BLR}(X) \subseteq \mathrm{BLR}(Y)$. Then X is jump-traceable by Y and $X' \leq_{tt} Y'$.*

Proof. Assume $\mathrm{BLR}(X) \subseteq \mathrm{BLR}(Y)$. By Lemma 2.5 the function

$$h(n) = h^X(n) = \begin{cases} \varphi^{(1),X}_n(n) + 1 & \text{if } \varphi^{(1),X}_n(n)\downarrow, \\ 0 & \text{otherwise} \end{cases}$$

belongs to $\mathrm{BLR}(Y)$, say $h(n) = \lim_s \widetilde{h}(n,s)$ and

$$|\{s \mid \widetilde{h}(n,s) \neq \widetilde{h}(n,s+1)\}| < \widehat{h}(n)$$

where $\widetilde{h}$ is Y-recursive and $\widehat{h}$ is recursive. By the S-m-n Theorem let $f(n)$ be recursive such that $W^Y_{f(n)} = \{\widetilde{h}(n,s) \mid s = 0,1,2,\ldots\}$ for all n. Then $h(n) \in W^Y_{f(n)}$ and $|W^Y_{f(n)}| \leq \widehat{h}(n)$, hence X is jump-traceable by Y. To show that $X' \leq_{tt} Y'$, note that $X' \in \mathrm{BLR}(X)$, hence $X' \in \mathrm{BLR}(Y)$, hence $X' \leq_{tt} Y'$ by Theorem 6.4. □

Remark 6.9. Lemma 6.8 may suggest that, in general, $\mathrm{BLR}(X) \subseteq \mathrm{BLR}(Y)$ if and only if X is jump-traceable by Y and $X' \leq_{tt} Y'$. We do not endorse such a conjecture, but we shall now prove some additional results in this direction.

Lemma 6.10. *Assume that X is jump-traceable by Y and $(X \oplus Y)' \leq_{tt} Y'$. Then $\mathrm{BLR}(X) \subseteq \mathrm{BLR}(Y)$.*

Proof. Consider the partial X-recursive function $\theta^X(n) \simeq$ the least $\sigma \subset X$ such that $\varphi^{(1),\sigma}_{n,|\sigma|}(n)\downarrow$. Clearly $X' = \mathrm{dom}(\theta^X) = \{n \mid \theta^X(n)\downarrow\}$. Since X is jump-traceable by Y, let $f(n)$ and $g(n)$ be recursive functions such that $\forall n\,(\theta^X(n)\downarrow \Rightarrow \theta^X(n) \in W^Y_{f(n)})$ and $\forall n\,(|W^Y_{f(n)}| \leq g(n))$. We may safely assume that for all $\sigma \in W^Y_{f(n)}$ we have $\varphi^{(1),\sigma}_{n,|\sigma|}(n)\downarrow$ and $\varphi^{(1),\rho}_{n,|\rho|}(n)\uparrow$ for all $\rho \subset \sigma$. Now for each n and each $i < g(n)$ let $\sigma_{n,i} \simeq$ the ith member of $W^Y_{f(n)}$ in order of Y-recursive enumeration. As in the proof of Lemma 6.8, let $h^X(n) = \varphi^{(1),X}_n(n)+1$ if $\varphi^{(1),X}_n(n)\downarrow$, 0 otherwise. We shall now describe how to compute $h^X(n)$ from Y'. Since $(X \oplus Y)' \leq_{tt} Y'$, we can ask questions of the oracle $(X \oplus Y)'$ via the oracle Y'. Given n, for each $i < g(n)$ ask the oracle Y' whether $\sigma_{n,i}\downarrow \subset X$. After receiving the answers to these questions, we know the unique $i < g(n)$ such that $\sigma_{n,i}\downarrow \subset X$ if such an i exists. We can then use the oracle Y to find this $\sigma = \sigma_{n,i}$. We then have $h^X(n) = \varphi^{(1),\sigma}_{n,|\sigma|}(n)+1$ if σ exists, 0 otherwise. Since our use of Y' in computing $h^X(n)$ was recursively bounded, we have $h^X \in \mathrm{BLR}(Y)$ by Lemma 2.3. It then follows by Lemma 2.5 that $\mathrm{BLR}(X) \subseteq \mathrm{BLR}(Y)$, Q.E.D. □

Theorem 6.11. *Assume that $X \geq_T Y$. Then $\mathrm{BLR}(X) \subseteq \mathrm{BLR}(Y)$ if and only if X is jump-traceable by Y and $X' \leq_{tt} Y'$.*

Proof. The "only if" follows from Lemma 6.8. The "if" follows from Lemma 6.10 upon noting that $X \geq_T Y$ and $X' \leq_{tt} Y'$ imply $(X \oplus Y)' \leq_{tt} Y'$. □

We say that $X \subseteq \omega$ is *Y-recursively enumerable* if X is the range of a Y-recursive function.

Theorem 6.12. *Assume that $X \geq_T Y$ and X is Y-recursively enumerable. Then the following are pairwise equivalent.*

1. $\mathrm{BLR}(X) \subseteq \mathrm{BLR}(Y)$.
2. *X is jump-traceable by Y.*
3. $X' \leq_{tt} Y'$.

Proof. Lemma 6.8 gives $1 \Rightarrow 2$ and $1 \Rightarrow 3$. To prove $2 \Rightarrow 1$, assume X is jump-traceable by Y. Let $\theta^X(n)$, $h^X(n)$, $f(n)$, $g(n)$, $\sigma_{n,i}$ be as in the proof of Lemma 6.10. Since X is Y-recursively enumerable, let $X = \bigcup_s X_s$ where $X_0 \subseteq X_1 \subseteq \cdots \subseteq X_s \subseteq X_{s+1} \subseteq \cdots$ is a Y-recursive sequence of finite sets. We identify the sets X and X_s with their characteristic functions. We shall now describe how to compute $h^X(n)$ from Y'. Given n, for each $i < g(n)$ ask the oracle Y' whether $\exists s\, (\sigma_{n,i} \downarrow\subset X_s)$ and whether $\exists s\, \exists t\, (s < t$ and $\sigma_{n,i} \downarrow\subset X_s$ and $\sigma_{n,i} \not\subset X_t)$. After receiving the answers to these $2g(n)$ questions, we know the unique $i < g(n)$ such that $\sigma_{n,i} \downarrow\subset X$ if such an i exists. We can then use the oracle Y to find this $\sigma = \sigma_{n,i}$. We then have $h^X(n) = \varphi^{(1),\sigma}_{n,|\sigma|}(n) + 1$ if σ exists, 0 otherwise. Thus $h^X \in \mathrm{BLR}(Y)$. It follows by Lemma 2.5 that $\mathrm{BLR}(X) \subseteq \mathrm{BLR}(Y)$. This finishes the proof of $2 \Rightarrow 1$. To prove $3 \Rightarrow 2$, recall the proof by Nies (see (ii)⇒(i) of [27, Theorem 4.1]) that if X is recursively enumerable and $X' \leq_{tt} 0'$ then X is jump-traceable. The same proof relativizes to Y to give our implication $3 \Rightarrow 2$. □

We now draw some additional corollaries.

Definition 6.13. We say that X is *superlow* (see Mohrherr [25], Nies [27], Simpson [35]) if $X' \leq_{tt} 0'$. We say that X is *jump-traceable* (see Nies [27], Simpson [35], Figueira/Nies/Stephan [20], Cholak/Downey/Greenberg [11], Downey/Greenberg [16]) if there exist recursive functions $f(n)$, $g(n)$ such that $\forall n\, (\varphi^{(1),X}_n(n) \downarrow \Rightarrow \varphi^{(1),X}_n(n) \in W_{f(n)})$ and $\forall n\, (|W_{f(n)}| < g(n))$.

Corollary 6.14. *The set* $\{X \mid X$ *is superlow*$\}$ *is* Σ^0_3.

Proof. This is the special case $Y = 0$ of Corollary 6.6. □

Corollary 6.15. *For all* X, $\mathrm{BLR}(X) \subseteq \mathrm{BLR}(0)$ *if and only if* X *is jump-traceable and superlow.*

Proof. This is the special case $Y = 0$ of Theorem 6.11. □

Remark 6.16. Nies [27] has shown that, in general, jump-traceability does not imply superlowness, and superlowness does not imply jump-traceability. In the following corollary, the equivalence $2 \Leftrightarrow 3$ is due to Nies [27, Theorem 4.1].

Corollary 6.17. *If* X *is recursively enumerable, the following are pairwise equivalent.*

1. $\mathrm{BLR}(X) \subseteq \mathrm{BLR}(0)$.
2. X *is jump-traceable.*
3. X *is superlow.*

Proof. This is the special case $Y = 0$ of Theorem 6.12. □

Definition 6.18. We say that Y is *superhigh* (see Mohrherr [25], Binns/Kjos-Hanssen/Lerman/Solomon [7], Cholak/Greenberg/Miller [12], Kjos-Hanssen [22], Simpson [35]) if $0'' \leq_{tt} Y'$. We say that Y is *generalized superhigh* (see Barmpalias/Lewis/Soskova [4]) if $(Y \oplus 0')' \leq_{tt} Y'$.

Corollary 6.19. *For all Y the following are pairwise equivalent.*

1. $\mathrm{BLR}(Y \oplus 0') \subseteq \mathrm{BLR}(Y)$.
2. $Y \oplus 0'$ *is jump-traceable by* Y.
3. Y *is generalized superhigh.*

Proof. This is the special case $X = Y \oplus 0'$ of Theorem 6.12. □

Corollary 6.20. *The set* $\{Y \mid Y$ *is generalized superhigh*$\}$ *is* Σ^0_3.

Proof. The method of Theorem 2.6 shows that, for any Σ^0_3 binary relation $S(X,Y)$, the set

$$\{Y \mid \exists X\,(S(X,Y) \text{ and } \mathrm{BLR}(X) \subseteq \mathrm{BLR}(Y))\}$$

is Σ^0_3. Letting $S(X,Y)$ be the binary relation $X = Y \oplus 0'$, we see that the set $\{Y \mid \mathrm{BLR}(Y \oplus 0') \subseteq \mathrm{BLR}(Y)\}$ is Σ^0_3. Our result then follows in view of Corollary 6.19. □

Remark 6.21. We do not know whether the set $\{Y \mid Y$ is superhigh$\}$ is Σ^0_3, nor whether Y being superhigh is equivalent to $\mathrm{BLR}(0') \subseteq \mathrm{BLR}(Y)$. More generally, for $1 \leq \alpha < \omega_1^{\mathrm{CK}}$ we do not know whether $0^{(\alpha+1)} \leq_{tt} Y'$ is equivalent to $\mathrm{BLR}(0^{(\alpha)}) \subseteq \mathrm{BLR}(Y)$. If this were the case, it would follow by Theorems 2.6 and 4.7 that $\{Y \mid 0^{(\alpha+1)} \leq_{tt} Y'\}$ is Σ^0_3 and of weak degree $\mathbf{h}^*_\alpha$.

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