Mass Problems and Almost Everywhere Domination

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Abstract

We examine the concept of almost everywhere domination from the viewpoint of mass problems. Let AED and MLR be the set of reals which are almost everywhere dominating and Martin-Löf random, respectively. Let $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$ be the degrees of unsolvability of the mass problems associated with the sets AED, $\mathrm{MLR} \times \mathrm{AED}$, $\mathrm{MLR} \cap \mathrm{AED}$ respectively. Let $\mathcal{P}_w$ be the lattice of degrees of unsolvability of mass problems associated with nonempty Π^0_1 subsets of 2^ω. Let $\mathbf{1}$ and $\mathbf{0}$ be the top and bottom elements of $\mathcal{P}_w$. We show that $\inf(\mathbf{b}_1, \mathbf{1})$ and $\inf(\mathbf{b}_2, \mathbf{1})$ and $\inf(\mathbf{b}_3, \mathbf{1})$ belong to $\mathcal{P}_w$ and that $\mathbf{0} < \inf(\mathbf{b}_1, \mathbf{1}) < \inf(\mathbf{b}_2, \mathbf{1}) < \inf(\mathbf{b}_3, \mathbf{1}) < \mathbf{1}$. Under the natural embedding of the recursively enumerable Turing degrees into $\mathcal{P}_w$, we show that $\inf(\mathbf{b}_1, \mathbf{1})$ and $\inf(\mathbf{b}_3, \mathbf{1})$ but not $\inf(\mathbf{b}_2, \mathbf{1})$ are comparable with some recursively enumerable Turing degrees other than $\mathbf{0}$ and $\mathbf{0}'$. In order to make this paper more self-contained, we exposit the proofs of some recent theorems due to Hirschfeldt, Miller, Nies, and Stephan.

Contents

1 Introduction

In our previous papers [3, 31, 34, 32, 7] we studied the lattice $\mathcal{P}_w$ of degrees of unsolvability of mass problems associated with nonempty Π^0_1 subsets of 2^ω. We showed that $\mathcal{P}_w$ contains many specific, natural degrees in addition to $\mathbf{1}$ and $\mathbf{0}$, the top and bottom elements of $\mathcal{P}_w$. We showed that many specific, natural degrees in $\mathcal{P}_w$ arise from foundationally interesting topics such as reverse mathematics, algorithmic randomness, computational complexity, hyperarithmeticity, and subrecursive hierarchies from Gentzen-style proof theory.

The purpose of the present paper is to exhibit and discuss some relatively new examples of specific, natural degrees in $\mathcal{P}_w$. The new examples arise from *almost everywhere domination*, a concept which was originally introduced by Dobrinen/Simpson [9]. Let B be a Turing oracle. We say that B is almost everywhere dominating if, for all $X \in 2^\omega$ except a set of measure 0, each function computable from X is dominated by some function computable from B. It is known [9] that almost everywhere domination is closely related to the reverse mathematics of measure theory.

In order to succinctly state our results, let MLR $= \{X \in 2^\omega \mid X$ is Martin-Löf random$\}$ and AED $= \{Y \in 2^\omega \mid Y$ is almost everywhere dominating$\}$. For $P, Q \subseteq 2^\omega$ we write $P \times Q = \{X \oplus Y \mid X \in P \text{ and } Y \in Q\}$ and $P \cap Q =$ the intersection of P and Q. With these conventions, let $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$ be the respective degrees of unsolvability of the mass problems associated with AED, MLR $\times$ AED, MLR $\cap$ AED. Trivially $\mathbf{b}_1 \leq \mathbf{b}_2 \leq \mathbf{b}_3$. Our main results may be summarized by saying that the degrees $\inf(\mathbf{b}_1, \mathbf{1})$, $\inf(\mathbf{b}_2, \mathbf{1})$, $\inf(\mathbf{b}_3, \mathbf{1})$ belong to $\mathcal{P}_w$ and

$$\mathbf{0} < \inf(\mathbf{b}_1, \mathbf{1}) < \inf(\mathbf{b}_2, \mathbf{1}) < \inf(\mathbf{b}_3, \mathbf{1}) < \mathbf{1}.$$

The proof of this chain of inequalities uses virtually everything that is currently known about the relationship between Martin-Löf randomness and almost everywhere domination. See Theorems 2.2 and 2.3 below.

Historically, there has been a great deal of interest in the semilattice of recursively enumerable Turing degrees. Therefore, it seems desirable to examine the relationships between recursively enumerable Turing degrees and the various specific, natural degrees in $\mathcal{P}_w$. In order to state our results, let us temporarily identify each recursively enumerable Turing degree $\mathbf{a}$ with its image in $\mathcal{P}_w$ under the natural one-to-one embedding $\mathbf{a} \mapsto \inf(\mathbf{a}, \mathbf{1})$ given in [34, Theorem 5.5]. In particular, we identify $\mathbf{0}'$ and $\mathbf{0}$, the top and bottom recursively enumerable Turing degrees, with $\mathbf{1}$ and $\mathbf{0}$, the top and bottom degrees in $\mathcal{P}_w$. In our papers [31, 34] written in 2004, we remarked that all of the specific, natural degrees in $\mathcal{P}_w$ which were known at that time are incomparable with all recursively enumerable Turing degrees except $\mathbf{0}'$ and $\mathbf{0}$. In this respect it turns out that our new examples of specific, natural degrees in $\mathcal{P}_w$ behave somewhat differently from the old ones. Namely, although $\inf(\mathbf{b}_2, \mathbf{1})$ is again incomparable with all recursively enumerable Turing degrees except $\mathbf{0}'$ and $\mathbf{0}$, this turns out not to be the case for $\inf(\mathbf{b}_1, \mathbf{1})$ and $\inf(\mathbf{b}_3, \mathbf{1})$. See Theorems 3.1 and 3.2 below.

Our work in this paper owes much to conversations with Bjørn Kjos-Hanssen, Antonín Kučera, and Joseph S. Miller. In particular, the fact that $\inf(\mathbf{b}_1, \mathbf{1})$ belongs to $\mathcal{P}_w$ was already implicit in Kjos-Hanssen [18], and Miller corrected an error in one of our early proofs of the inequality $\inf(\mathbf{b}_2, \mathbf{1}) < \inf(\mathbf{b}_3, \mathbf{1})$.

The reader who is familiar with the basics of recursion theory will find that this paper is largely self-contained. If E is an expression which may or may not denote a natural number, we write $E\downarrow$ to mean that E *is defined* (i.e., E denotes a natural number), otherwise $E\uparrow$. If E_1 and E_2 are two such expressions, we write $E_1 \simeq E_2$ to mean that E_1 and E_2 are both defined and equal, or both undefined. Throughout this paper, a convenient background reference is our recent paper [33], which includes a fairly thorough exposition of almost everywhere domination and Martin-Löf randomness.

2 Some mass problem inequalities

The purpose of this section is to prove our mass problem inequalities in $\mathcal{P}_w$. The proofs use some recent theorems of Cholak/Greenberg/Miller, Kjos-Hanssen, Hirschfeldt/Miller, Nies, and Stephan concerning almost everywhere domination and Martin-Löf randomness. In order to make this paper more self-contained, we exposit the proofs of the theorems of Hirschfeldt/Miller, Nies, and Stephan respectively in Sections 4, 5, 6 below.

Our notion of reducibility for decision problems is standard. Given $X, Y \in 2^\omega$, we say that X is *Turing reducible* to Y, abbreviated $X \leq_T Y$, if X is computable using Y as an oracle. A *Turing degree* is an equivalence class of elements of 2^ω under mutual Turing reducibility. The Turing degree of X is denoted $\deg_T(X)$.

Our notion of reducibility for mass problems is as follows. Given $P, Q \subseteq 2^\omega$, we say that P is *weakly reducible* to Q, abbreviated $P \leq_w Q$, if for all $Y \in Q$ there exists $X \in P$ such that X is Turing reducible to Y. A *weak degree* is an equivalence class of subsets of 2^ω under mutual weak reducibility. The weak degree of P is denoted $\deg_w(P)$. Weak degrees have sometimes been known as *Muchnik degrees* [23].

Note that for $\mathbf{p} = \deg_w(P)$ and $\mathbf{q} = \deg_w(Q)$ we have $\inf(\mathbf{p}, \mathbf{q}) = \deg_w(P \cup Q)$ and $\sup(\mathbf{p}, \mathbf{q}) = \deg_w(P \times Q)$. Note also that for $X, Y \in 2^\omega$ we have $X \leq_T Y$ if and only if $\{X\} \leq_w \{Y\}$. Here $\{X\}$ denotes the singleton set whose only element is X. Therefore, the Turing degree $\deg_T(X)$ is sometimes identified with the weak degree $\deg_w(\{X\})$.

Definition 2.1.

1. Let $\mathbf{b}_1 = \deg_w(\mathrm{AED})$ where

$$\mathrm{AED} = \{Y \in 2^\omega \mid Y \text{ is almost everywhere dominating}\}.$$

2. Let $\mathbf{b}_2 = \deg_w(\mathrm{MLR} \times \mathrm{AED})$ where

$$\mathrm{MLR} = \{X \in 2^\omega \mid X \text{ is Martin-Löf random}\}.$$

3. Let $\mathbf{b}_3 = \deg_w(\mathrm{MLR} \cap \mathrm{AED})$.

Theorem 2.2. *The weak degrees* $\inf(\mathbf{b}_1, \mathbf{1})$*,* $\inf(\mathbf{b}_2, \mathbf{1})$*,* $\inf(\mathbf{b}_3, \mathbf{1})$ *belong to* $\mathcal{P}_w$*.*

Proof. An important tool in the study of $\mathcal{P}_w$ is the Embedding Lemma [34, Lemma 3.3], [32, Lemma 4]. The Embedding Lemma says: For any $\mathbf{s} = \deg_w(S)$ where S is Σ^0_3, we have $\inf(\mathbf{s}, \mathbf{1}) \in \mathcal{P}_w$. Therefore, in order to prove Theorem 2.2, it suffices to show that AED and MLR are Σ^0_3.

After Dobrinen/Simpson [9], the concept of almost everywhere domination was subsequently explored by Binns/Kjos-Hanssen/Lerman/Solomon [2], Cholak/Greenberg/Miller [5], Kjos-Hanssen [18], Kjos-Hanssen/Miller/Solomon [19], and Simpson [33]. We now know [18, 19] that B is almost everywhere dominating if and only if $0' \leq_{LR} B$. Here $0'$ denotes the Halting Problem, and $\leq_{LR}$ denotes *LR-reducibility*: $A \leq_{LR} B$ if and only if $\forall X$ (if X is random relative to B, then X is random relative to A). Moreover, as shown in [19], *LR*-reducibility is equivalent to *LK-reducibility*: $A \leq_{LK} B$ if and only if $K^B(\tau) \leq K^A(\tau) + O(1)$ for all τ. Here $K^B(\tau)$ denotes the prefix-free Kolmogorov complexity of τ relative to a Turing oracle B. The concepts of *LR*-reducibility and *LK*-reducibility were originally introduced by Nies [24, Section 8]. A convenient reference for these results is Simpson [33].

One way to see that AED is Σ^0_3 is to use the characterization in terms of *LK*-reducibility. We know that $B \in \mathrm{AED}$ if and only if $0' \leq_{LK} B$, i.e., $K^B(\tau) \leq K^{0'}(\tau) + O(1)$, i.e., $\exists c \forall \tau\, (K^B(\tau) \leq K^{0'}(\tau) + c)$. But $K^B(\tau) \leq K^{0'}(\tau) + c$ if and only if $\forall \rho\, (U^{0'}(\rho) \simeq \tau \Rightarrow \exists \sigma\, (|\sigma| \leq |\rho| + c \text{ and } U^B(\sigma) \simeq \tau))$. Here U^B is a universal prefix-free oracle machine. The last satement is Π^0_2, so AED is Σ^0_3. The fact that AED is Σ^0_3 has previously been noted by Kjos-Hanssen [18] and Kjos-Hanssen/Miller/Solomon [19]. See also [33, Corollary 5.9].

It remains to show that MLR is Σ^0_3. In fact, MLR is Σ^0_2 in view of the existence of a universal Martin-Löf test. See for instance [20] or [31, Theorem 8.3] or [33, Theorem 3.2]. This completes the proof of Theorem 2.2. □

Theorem 2.3. *In $\mathcal{P}_w$ we have $\mathbf{0} < \inf(\mathbf{b}_1, \mathbf{1}) < \inf(\mathbf{b}_2, \mathbf{1}) < \inf(\mathbf{b}_3, \mathbf{1}) < \mathbf{1}$.*

Proof. Trivially $\mathbf{b}_1 \le \mathbf{b}_2 \le \mathbf{b}_3$, hence $\inf(\mathbf{b}_1, \mathbf{1}) \le \inf(\mathbf{b}_2, \mathbf{1}) \le \inf(\mathbf{b}_3, \mathbf{1})$. Recall from [31, 34] that $\mathbf{1} = \deg_w(\mathrm{PA})$ where

$$\mathrm{PA} = \{X \in 2^\omega \mid X \text{ is a complete extension of Peano Arithmetic}\}.$$

We have $\mathbf{0} < \inf(\mathbf{b}_1, \mathbf{1})$ because by [9] no member of $\mathrm{AED} \cup \mathrm{PA}$ is recursive. In order to prove $\inf(\mathbf{b}_1, \mathbf{1}) < \inf(\mathbf{b}_2, \mathbf{1})$, consider

$$\mathrm{DNR} = \{f \in \omega^\omega \mid f \text{ is diagonally nonrecursive}\}$$

where f is said to be *diagonally nonrecursive* if $\forall n\, (f(n) \not\simeq \varphi^{(1)}_n(n))$.

Lemma 2.4. DNR $\not\le_w$ AED.

Proof. This follows from items 1 and 2 in Lemma 2.7 below. □

The next two lemmas are well known.

Lemma 2.5. DNR $\le_w$ MLR.

Proof. Consider the recursive functional $X \mapsto f^X$ given by $f^X(n) = \sum_{i=0}^{n-1} X(i)2^i$ for all $X \in 2^\omega$. Let $U_n = \{X \in 2^\omega \mid f^X(n) \simeq \varphi^{(1)}_n(n)\}$. Note that U_n is uniformly Σ^0_1 and $\mu(U_n) \le 1/2^n$. If X is random, it follows by Solovay's Lemma (see for instance [33, Lemma 3.7]) that $X \in U_n$ for only finitely many n. Therefore, with finitely many exceptions, f^X is diagonally nonrecursive. This proves the lemma. For refinements, see Jockusch [14, Proposition 3] and Ambos-Spies/Kjos-Hanssen/Lempp/Slaman [1] and Simpson [31]. □

Lemma 2.6. MLR $\le_w$ PA.

Proof. Since MLR is Σ^0_2 and nonempty, we can find a nonempty Π^0_1 set $P \subseteq$ MLR. Since P is a nonempty Π^0_1 subset of 2^ω, we have $P \le_w$ PA in view of Scott/Tennenbaum [30] and Scott [29]. The lemma follows. □

By Lemmas 2.4 and 2.5 and 2.6 we have $\mathrm{MLR} \cup \mathrm{PA} \not\le_w \mathrm{AED}$. From this it follows trivially that $\mathrm{AED} \cup \mathrm{PA} <_w (\mathrm{MLR} \times \mathrm{AED}) \cup \mathrm{PA}$. In other words, $\inf(\mathbf{b}_1, \mathbf{1}) < \inf(\mathbf{b}_2, \mathbf{1})$.

Next we prove $\inf(\mathbf{b}_2, \mathbf{1}) < \inf(\mathbf{b}_3, \mathbf{1})$. We use the forcing construction of Cholak/Greenberg/Miller [5] referring to CGM-genericity.

Lemma 2.7 (Cholak/Greenberg/Miller)**.**

1. *If B is sufficiently* CGM-*generic, then B is almost everywhere dominating.*
2. *If B is sufficiently* CGM-*generic, then* DNR $\not\le_w \{B\}$.

3. *For each nonrecursive A, if B is sufficiently* CGM*-generic then $A \not\leq_T B$.*

Proof. See Cholak/Greenberg/Miller [5, Section 4]. □

We also use the following result due to Hirschfeldt and Miller, 2006.

Lemma 2.8 (Hirschfeldt/Miller)**.** *There is a nonrecursive, recursively enumerable set A such that $\{A\} \leq_w \mathrm{MLR} \cap \mathrm{AED}$.*

Proof. See Nies [25, Theorem 5.6]. Alternatively, see Section 4 below. □

The next lemma is well known.

Lemma 2.9.

1. *If $A \not\leq_T B$, then $\mu(\{X \in 2^\omega \mid A \not\leq_T X \oplus B\}) = 1$.*
2. *If $\mathrm{PA} \not\leq_w \{B\}$, then $\mu(\{X \in 2^\omega \mid \mathrm{PA} \not\leq_w \{X \oplus B\}\}) = 1$.*

Proof. The first statement is a relativized form of Sacks [28, Theorem 1, page 154]. The second statement is a relativized form of Jockusch/Soare [16, Theorem 5.3]. Both statements follow from the general "non-helping" result in [31, Lemma 7.3]. □

By Lemma 2.8 let A be nonrecursive such that $\{A\} \leq_w \mathrm{MLR} \cap \mathrm{AED}$. Since A is nonrecursive, $\{A\} \cup \mathrm{DNR} \not\leq_w \mathrm{AED}$ by Lemma 2.7. Hence $\{A\} \cup \mathrm{PA} \not\leq_w \mathrm{AED}$ by Lemmas 2.5 and 2.6. But then, by Lemma 2.9, $\{A\} \cup \mathrm{PA} \not\leq_w \mathrm{MLR} \times \mathrm{AED}$, since $\mu(\mathrm{MLR}) = 1$. It now follows by our choice of A that $(\mathrm{MLR} \cap \mathrm{AED}) \cup \mathrm{PA} \not\leq_w \mathrm{MLR} \times \mathrm{AED}$. From this it follows trivially that $(\mathrm{MLR} \times \mathrm{AED}) \cup \mathrm{PA} <_w (\mathrm{MLR} \cap \mathrm{AED}) \cup \mathrm{PA}$. In other words, $\inf(\mathbf{b}_2, \mathbf{1}) < \inf(\mathbf{b}_3, \mathbf{1})$.

It remains to prove $\inf(\mathbf{b}_3, \mathbf{1}) < \mathbf{1}$. We use the following results of Nies 2006 and Stephan 2002.

Lemma 2.10 (Nies)**.** *There exists $B \in \mathrm{MLR} \cap \mathrm{AED}$ such that $0' \not\leq_T B$.*

Proof. See Nies [26, Theorem VI.18]. Alternatively, see Section 5 below. □

Lemma 2.11 (Stephan)**.** *If $B \in \mathrm{MLR}$ and $0' \not\leq_T B$, then $\mathrm{PA} \not\leq_w \{B\}$.*

Proof. See Stephan [35]. Alternatively, see Section 6 below. □

By Lemma 2.10 let $B \in \mathrm{MLR} \cap \mathrm{AED}$ be such that $0' \not\leq_T B$. By Lemma 2.11 we have $\mathrm{PA} \not\leq_w \{B\}$. Thus $\mathrm{PA} \not\leq_w \mathrm{MLR} \cap \mathrm{AED}$. It follows trivially that $(\mathrm{MLR} \cap \mathrm{AED}) \cup \mathrm{PA} <_w \mathrm{PA}$. In other words, $\inf(\mathbf{b}_3, \mathbf{1}) < \mathbf{1}$.

This completes the proof of Theorem 2.3. □

3 Comparison with r.e. Turing degrees

In this section we discuss the relationship between the weak degrees $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$ and the recursively enumerable Turing degrees. Recall from [34, Theorem 5.5] that there is a natural one-to-one embedding of the recursively enumerable Turing degrees into $\mathcal{P}_w$ given by $\mathbf{a} \mapsto \inf(\mathbf{a}, \mathbf{1})$.

Theorem 3.1. *There is no recursively enumerable Turing degree* $\mathbf{0} < \mathbf{a} < \mathbf{0}'$ *such that* $\inf(\mathbf{a}, \mathbf{1})$ *is comparable with* $\inf(\mathbf{b}_2, \mathbf{1})$.

Proof. Let $\mathbf{a} = \deg_T(A)$ where A is recursively enumerable and $0 <_T A <_T 0'$. Since A is nonrecursive, we have $\{A\} \cup \text{DNR} \not\leq_w \text{AED}$ by Lemma 2.7, hence $\{A\} \cup \text{PA} \not\leq_w \text{AED}$ by Lemmas 2.5 and 2.6, hence $\{A\} \cup \text{PA} \not\leq_w \text{MLR} \times \text{AED}$ by Lemma 2.9. From this it follows trivially that $\{A\} \cup \text{PA} \not\leq_w (\text{MLR} \times \text{AED}) \cup \text{PA}$. In other words, $\inf(\mathbf{a}, \mathbf{1}) \not\leq \inf(\mathbf{b}_2, \mathbf{1})$. On the other hand, since A is recursively enumerable and not Turing complete, we have $\text{DNR} \not\leq_w \{A\}$ by the Arslanov Completeness Criterion [14], hence $\text{MLR} \cup \text{PA} \not\leq_w \{A\}$ by Lemmas 2.5 and 2.6. From this it follows trivially that $(\text{MLR} \times \text{AED}) \cup \text{PA} \not\leq_w \{A\} \cup \text{PA}$. In other words, $\inf(\mathbf{b}_2, \mathbf{1}) \not\leq \inf(\mathbf{a}, \mathbf{1})$. This completes the proof. □

Theorem 3.2. *There are recursively enumerable Turing degrees* $\mathbf{0} < \mathbf{a} < \mathbf{0}'$ *and* $\mathbf{0} < \mathbf{c} < \mathbf{0}'$ *such that* $\mathbf{0} < \inf(\mathbf{a}, \mathbf{1}) < \inf(\mathbf{b}_3, \mathbf{1}) < \mathbf{1}$ *and* $\mathbf{0} < \inf(\mathbf{b}_1, \mathbf{1}) < \inf(\mathbf{c}, \mathbf{1}) < \mathbf{1}$.

Proof. By Lemma 2.8 due to Hirschfeldt and Miller, let $A >_T 0$ be recursively enumerable such that $\{A\} \leq_w \text{MLR} \cap \text{AED}$. By Simpson [33, Example 6.8] or Cholak/Greenberg/Miller [5, Theorem 2.1], let $C <_T 0'$ be recursively enumerable and almost everywhere dominating. Let $\mathbf{a} = \deg_T(A)$ and $\mathbf{c} = \deg_T(C)$. Clearly $\mathbf{0} < \inf(\mathbf{a}, \mathbf{1}) \leq \inf(\mathbf{b}_3, \mathbf{1}) < \mathbf{1}$ and $\mathbf{0} < \mathbf{a} < \mathbf{0}'$ and $\mathbf{0} < \inf(\mathbf{b}_1, \mathbf{1}) \leq \inf(\mathbf{c}, \mathbf{1}) < \mathbf{1}$ and $\mathbf{0} < \mathbf{c} < \mathbf{0}'$. By Theorems 2.3 and 3.1 it follows that $\inf(\mathbf{a}, \mathbf{1}) < \inf(\mathbf{b}_3, \mathbf{1})$ and $\inf(\mathbf{b}_1, \mathbf{1}) < \inf(\mathbf{c}, \mathbf{1})$. □

4 A theorem of Hirschfeldt and Miller

In this section we exposit the proof of the following theorem of Hirschfeldt and Miller 2006, generalizing a much earlier theorem of Kučera [21].

Theorem 4.1 (Hirschfeldt/Miller). *Let* $S \subseteq 2^\omega$ *be* Σ^0_3 *of measure* 0. *Then we can find a nonrecursive, recursively enumerable set* A *such that* $A \leq_T X$ *for all random* $X \in S$.

Proof. We follow the writeup of Nies [25, Theorem 5.6].

We first prove the theorem for Π^0_2 sets. Let $P \subseteq 2^\omega$ be Π^0_2 of measure 0. Write $P = \bigcap_n V_n$ where V_n is uniformly Σ^0_1 and $\lim_n \mu(V_n) = 0$.

The construction of A is as follows. At stage s, for each $e < s$ such that $W_{e,s} \cap A_s = \emptyset$, look for $n \in W_{e,s}$ such that $n > 2e$ and $\mu(V_{n,s}) < 1/2^e$ and put the least such n into A_{s+1}.

Clearly A is recursively enumerable. Moreover, for each e, at most one n gets into A for the sake of W_e, and for this n we have $n > 2e$. Hence A has at most e members $\leq 2e$. Thus $\overline{A}$ is infinite.

We claim that if W_e is infinite then $W_e \cap A \neq \emptyset$. To see this, fix $n \in W_e$ so large that $n > 2e$ and $\mu(V_n) < 1/2^e$. Let s be such that $n \in W_{e,s}$. We have $\mu(V_{n,s}) < 1/2^e$, so by construction $W_{e,s} \cap A_{s+1} \neq \emptyset$, Q.E.D.

It follows from the previous claim that A is nonrecursive. Indeed, A is *simple* in the sense of Post (compare Rogers [27, Section 8.1]).

We claim that $A \leq_T X$ for all random $X \in P$. To see this, note that by construction

$$\sum_{n \in A_{s+1} \setminus A_s} \mu(V_{n,s}) \;<\; \sum_e 1/2^e \;=\; 2 \;<\; \infty\,.$$

Since X is random, it follows by Solovay's Lemma (see [33, Lemma 3.7]) that $X \notin V_{n,s}$ for all but finitely many pairs n, s such that $n \in A_{s+1} \setminus A_s$. Let s_0 be so large that $(\forall s > s_0)\,(\forall n \in A_{s+1} \setminus A_s)\,(X \notin V_{n,s})$. Given n, since $X \in V_n$ we can effectively find $s > s_0$ such that $X \in V_{n,s}$. We then have $n \in A \Longleftrightarrow n \in A_s$. Thus $A \leq_T X$, Q.E.D. This proves the theorem for Π^0_2 sets.

Suppose now that S is Σ^0_3 of measure 0. Write $S = \bigcup_i P_i$ where P_i is uniformly Π^0_2. Write $P_i = \bigcap_n V_{i,n}$ where $V_{i,n}$ is uniformly Σ^0_1 and $\lim_n \mu(V_{i,n}) = 0$ for each i. This implies that $\lim_n \sum_i \mu(V_{i,n})/2^i = 0$, so we can build A as before replacing $\mu(V_{n,s})$ by $\sum_i \mu(V_{i,n,s})/2^i$. The construction insures that

$$\sum_{n \in A_{s+1} \setminus A_s} \; \sum_i \; \mu(V_{i,n,s})/2^i \;<\; \sum_e 1/2^e \;=\; 2\,,$$

hence for each i

$$\sum_{n \in A_{s+1} \setminus A_s} \mu(V_{i,n,s}) \;<\; 2^{i+1} \;<\; \infty\,,$$

hence for any random X we have by Solovay's Lemma $X \notin V_{i,n,s}$ for all but finitely many n, s such that $n \in A_{s+1} \setminus A_s$. It follows as before that $A \leq_T X$ for all random $X \in P_i$. This proves the theorem. □

Remark 4.2. Hirschfeldt, Miller and Nies describe the proof of Theorem 4.1 as a "cost function" construction. In the case of a Π^0_2 set, the cost of putting n into A at stage s is $\mu(V_{n,s})$. In the case of a Σ^0_3 set, the cost of putting n into A at stage s is $\sum_i \mu(V_{i,n,s})/2^i$. The construction insures that the total cost of building A is finite, so that Solovay's Lemma can be applied.

The following corollary is originally due to Kučera [21].

Corollary 4.3 (Kučera)**.** *Let $X_1, \ldots, X_n$ be random and $\leq_T 0'$. Then we can find a nonrecursive, recursively enumerable set A such that $A \leq_T X_i$ for all $i = 1, \ldots, n$.*

Proof. Let $S = \{X_1, \ldots, X_n\}$ and apply Theorem 4.1. □

The following lemma is well known.

Lemma 4.4. $\mu(\{X \in 2^\omega \mid X' \equiv_T X \oplus 0'\}) = 1$.

Proof. It suffices to show that $X' \equiv_T X \oplus 0'$ whenever X is random relative to $0'$. (Generalizations of this result are in Kautz's thesis [17, Theorem III.2.1].) Consider the sets $U_n = \{X \in 2^\omega \mid n \in X'\}$. Obviously these sets are uniformly

Σ^0_1. Let $f(n) =$ least s such that $\mu(U_n \setminus U_{n,s}) \leq 1/2^n$. Note that $f \leq_T 0'$. Thus $U_n \setminus U_{n,f(n)}$ is uniformly $\Sigma^{0,0'}_1$ and these sets form a Solovay test relative to $0'$. Now assume that X is random relative to $0'$. By Solovay's Lemma relative to $0'$, we have $X \notin U_n \setminus U_{n,f(n)}$ for all but finitely many n. In other words, for all but finitely many n, $n \in X'$ if and only if $X \in U_{n,f(n)}$. Since $f \leq_T 0'$, it follows that $X' \leq_T X \oplus 0'$. This completes the proof. □

Theorem 4.5 (Hirschfeldt/Miller). *We can find a nonrecursive, recursively enumerable set A such that $A \leq_T X$ for all X such that X is random and almost everywhere dominating.*

Proof. By Theorem 4.1 with $S =$ AED, it suffices to show that AED is Σ^0_3 and of measure 0. We have seen in the proof of Theorem 2.2 that AED is Σ^0_3. To show that $\mu(\text{AED}) = 0$, recall from [19] and [33, Section 8] that $\text{AED} \subseteq \{X \mid X' \geq_T 0''\}$. Thus AED is disjoint from the intersection of two sets: $\{X \mid X' \equiv_T X \oplus 0'\}$ and $\{X \mid X \oplus 0' \not\geq_T 0''\}$. The first set is of measure 1 by Lemma 4.4, and the second set is of measure 1 by Lemma 2.9. This completes the proof. □

Remark 4.6. According to a theorem of Nies (see Section 5 below), there are uncountably many X's as in Theorem 4.5 which are $\not\geq_T 0'$. On the other hand, a result of Hirschfeldt/Nies/Stephan [13, Corollary 3.6] says that if X is random and $\not\geq_T 0'$ then any recursively enumerable $A \leq_T X$ is K-trivial [24], i.e., low-for-random [22, 24, 33]. In particular, any A as in Theorem 4.5 is low-for-random.

5 A theorem of Nies

In this section we present a new proof of a theorem of Nies [26, Theorem VI.18] refining the Jockusch/Shore Pseudojump Inversion Theorem [15, Theorem 2.1]. By a *pseudojump operator* we mean an operator $J_e : 2^\omega \to 2^\omega$ given by $J_e(X) = X \oplus W^X_e$ for all $X \in 2^\omega$.

Theorem 5.1 (Nies). *Let $P \subseteq 2^\omega$ be Π^0_1 of positive measure. For any pseudojump operator J_e and any Turing oracle $A \geq_T 0'$, we can find $B \in P$ such that $J_e(B) \equiv_T B \oplus 0' \equiv_T A$.*

Proof. Our proof is based on a sketch given by Kučera at an American Institute of Mathematics workshop on algorithmic randomness, August 7–11, 2006. The idea is to combine the proofs of the Pseudojump Inversion Theorem and the Kučera/Gács Theorem.

Let us write $Q_e = \{X \in 2^\omega \mid \varphi^{(1),X}_e(e)\uparrow\}$. Note that Q_e, $e = 0, 1, 2, \ldots$ is a standard, recursive enumeration of all Π^0_1 subsets of 2^ω. Given a Π^0_1 set $Q \subseteq 2^\omega$, an *index* of Q is any integer e such that $Q = Q_e$. Let us say that a Π^0_1 set $P \subseteq 2^\omega$ is *rich* if $\mu(P) > 0$ and there exists a recursive function h such that for all e, if $\emptyset \neq Q_e \subseteq P$ then $\mu(Q_e) \geq 1/2^{h(e)}$.

We claim that every Π^0_1 set $P \subseteq 2^\omega$ of positive measure includes a Π^0_1 set which is rich. To see this, let n be such that $\mu(P) > 1/2^n$. Write $Q_{e,s} = \{X \in$

$2^\omega \mid \varphi_{e,s}^{(1),X}(e)\uparrow\}$ and note that $Q_{e,s}$, $s=0,1,2,\ldots$ is a uniformly recursive descending sequence of clopen sets such that $Q_e=\bigcap_s Q_{e,s}$. Define a recursive ascending sequence of clopen sets V_s, $s=0,1,2,\ldots$ as follows. Begin with $V_0=\emptyset$. Given V_s define $V_{s+1}=V_s\cup\bigcup_{e<s}(Q_{e,s}\setminus V_s)$ where the union is taken over all $e<s$ such that $\mu(Q_{e,s}\setminus V_s)\le 1/2^{n+e+1}$. Finally let $\widetilde{P}=P\setminus V$ where $V=\bigcup_s V_s$. Clearly $\widetilde{P}$ is Π^0_1. Moreover $\mu(V)\le\sum_e 1/2^{n+e+1}=1/2^n$, hence $\mu(\widetilde{P})\ge\mu(P)-1/2^n>0$. If $\emptyset\ne Q_e\subseteq\widetilde{P}$ then for all $s>e$ we have $\mu(Q_{e,s}\setminus V_s)>1/2^{n+e+1}$, hence $\mu(Q_e)\ge 1/2^{n+e+1}$. Thus $\widetilde{P}$ is rich via $h(e)=n+e+1$. This proves our claim.

To prove the theorem, let $P\subseteq 2^\omega$ be Π^0_1 of positive measure. By our claim, we may safely assume that P is rich. Under this assumption we shall carry out the proof of the Pseudojump Inversion Theorem "within P" to produce $B\in P$ with the desired properties.

For strings $\sigma\in 2^{<\omega}$ let us write $N_\sigma=\{X\in 2^\omega\mid\sigma\subset X\}$. For each string $\rho\in 2^{<\omega}$ we define a string $f(\rho)\in 2^{<\omega}$ and a nonempty Π^0_1 set $Q_\rho\subseteq P\cap N_{f(\rho)}$. Begin with $f(\langle\rangle)=\langle\rangle$ and $Q_{\langle\rangle}=P$. Assume inductively that $f(\rho)$ and Q_ρ have already been defined.

In order to control the pseudojump $J_e(B)$, define a string $f^*(\rho)\supseteq f(\rho)$ and a nonempty Π^0_1 set $Q^*_\rho\subseteq Q_\rho\cap N_{f^*(\rho)}$ as follows. Let $m=|\rho|$. If $\exists X\,(X\in Q_\rho$ and $m\notin W_e^X)$, let $Q^*_\rho=\{X\in Q_\rho\mid m\notin W_e^X\}$ and let $f^*(\rho)=f(\rho)$. Otherwise, consider the least $\sigma\supseteq f(\rho)$ such that $Q_\rho\cap N_\sigma\ne\emptyset$ and $m\in W_e^\sigma$, and let $Q^*_\rho=Q_\rho\cap N_\sigma$ and $f^*(\rho)=\sigma$. Thus $f^*(\rho)$ "decides" whether $m\in W_e^B$ or not.

Because P is rich, given an index of Q^*_ρ we can effectively find k such that $\mu(Q^*_\rho)\ge 1/2^k$. But then there are at least two strings $\tau\supset f^*(\rho)$ of length $k+1$ such that $Q^*_\rho\cap N_\tau\ne\emptyset$. Let $f(\rho^\frown\langle 0\rangle)$ and $f(\rho^\frown\langle 1\rangle)$ be the lexicographically leftmost and rightmost such τ. Let $Q_{\rho^\frown\langle 0\rangle}=Q^*_\rho\cap N_{f(\rho^\frown\langle 0\rangle)}$ and $Q_{\rho^\frown\langle 1\rangle}=Q^*_\rho\cap N_{f(\rho^\frown\langle 1\rangle)}$.

We have now defined $f(\rho)$, $f^*(\rho)$, Q_ρ, Q^*_ρ for all ρ. It is straightforward to check that $f(\rho)$ and $f^*(\rho)$ and the indices of Q_ρ and Q^*_ρ are uniformly computable from $0'$.

Given $A\in 2^\omega$ let $B=f(A)=\bigcup_m f(A\restriction m)$. Then $m\in W_e^B$ if and only if $m\in W_e^{f^*(A\restriction m)}$. Moreover, it is straightforward to show that $f(A\restriction m)$ and $f^*(A\restriction m)$ and the indices of $Q_{A\restriction m}$ and $Q^*_{A\restriction m}$ are uniformly computable from each of the Turing oracles $J_e(B)=B\oplus W_e^B$ and $B\oplus 0'$ and $A\oplus 0'$. From this, the desired conclusions follow easily. □

Theorem 5.2 (Nies)**.** *For any pseudojump operator J_e and any Turing oracle $A\ge_T 0'$, we can find a random B such that $J_e(B)\equiv_T B\oplus 0'\equiv_T A$.*

Proof. In Theorem 5.1 let P be a nonempty Π^0_1 set such that $\forall X\,(X\in P\Rightarrow X$ is random). □

Remark 5.3. Theorem 5.2 is a common generalization of several known theorems. If we omit the conclusion that B is random, we get the Jockusch/Shore Pseudojump Inversion Theorem [15, Theorem 2.1]. If we keep the conclusion that B is random but let J_e be the identity operator, we get the Kučera/Gács Theorem [33, Theorem 3.5]. If we let J_e be the Turing jump operator, we get the Friedberg Jump Inversion Theorem (see Rogers [27, Section 13.3]) with the additional conclusion that B is random.

Corollary 5.4 (Nies)**.** *For any $A \geq_T 0'$ we can find a random $B <_T A$ such that $B \oplus 0' \equiv_T A$ and A is low-for-random relative to B.*

Proof. By Theorem 5.2, it suffices to produce a psuedojump operator J_e such that for all B, $J_e(B) >_T B$ and $J_e(B)$ is low-for-random relative to B. Such an operator is obtained by uniformly relativizing the Kučera/Terwijn [22] construction of a nonrecursive, recursively enumerable set which is low-for-random. See also the exposition in Simpson [33, Section 6]. □

Corollary 5.5 (Nies)**.** *We can find a random $B <_T 0'$ such that $0'$ is low-for-random relative to B.*

Proof. This is the special case of Corollary 5.4 in which we let $A = 0'$. □

Corollary 5.6 (Nies)**.** *There are uncountably many random $B \not\geq_T 0'$ such that $0'$ is low-for-random relative to B.*

Proof. This follows from Corollary 5.4 by considering uncountably many $A >_T 0'$. □

Corollary 5.7 (Nies)**.** *We can find a random $B <_T 0'$ which is almost everywhere dominating.*

Corollary 5.8 (Nies)**.** *There are uncountably many random $B \not\geq_T 0'$ such that B is almost everywhere dominating.*

Proof. Corollaries 5.7 and 5.8 are immediate from Corollaries 5.5 and 5.6 plus the following fact: If $0'$ is low-for-random relative to B then B is almost everywhere dominating. This fact is due to Kjos-Hanssen/Miller/Solomon [19]. See also the exposition in Simpson [33, Section 5]. □

6 A theorem of Stephan

In this section we exposit the proof of the following theorem of Stephan [35].

Theorem 6.1 (Stephan)**.** *If B is random and $0' \not\leq_T B$, then $\mathrm{PA} \not\leq_w \{B\}$.*

Proof. We shall define a recursively bounded, partial recursive function ψ with the following property: For all random B, if $B \geq_T$ some total function extending ψ then $B \geq_T 0'$. This suffices to prove the theorem, because clearly every $\{B\} \geq_w \mathrm{PA}$ computes a total extension of every recursively bounded, partial recursive function (see for instance [31, Theorem 4.10]).

Recall that $0'$ is a recursively enumerable set. If $n \notin 0'$ let $\psi(\langle e, n\rangle)$ be undefined for all e. If $n \in 0'$, say $n \in 0'_{s+1} \setminus 0'_s$, then for each $i < 2^n$ compute the rational numbers

$$r_{e,n,i} = \mu(\{X \mid \varphi_{e,s}^{(1),X}(\langle e,n\rangle) \simeq i\})$$

and define $\psi(\langle e,n\rangle) \simeq i$ chosen so as to minimize $r_{e,n,i}$. Note that ψ is partial recursive, and $\psi(\langle e,n\rangle)\downarrow$ if and only if $n \in 0'$. Moreover, if $\psi(\langle e,n\rangle)\downarrow$ then $\psi(\langle e,n\rangle) < 2^n$, so ψ is recursively bounded. In addition $\mu(V_{e,n}) \leq 1/2^n$ where

$$V_{e,n} = \{X \in 2^\omega \mid \exists s\,(n \in 0'_{s+1} \setminus 0'_s \text{ and } \varphi_{e,s}^{(1),X}(\langle e,n\rangle) \simeq \psi(\langle e,n\rangle))\}.$$

Assume now that B is random and computes a total extension of ψ. Let e be such that $\varphi_e^{(1),B}$ is total and extends ψ. Define $f(n) =$ least s such that $\varphi_{e,s}^{(1),B}(\langle e,n\rangle)\downarrow$. Since $\varphi_e^{(1),B}$ is total, f is total and $\leq_T B$. Since B is random, it follows by Solovay's Lemma that $B \notin V_{e,n}$ for all but finitely many n. In other words, for all but finitely many n, if $n \in 0'_{s+1} \setminus 0'_s$ then $\varphi_{e,s}^{(1),B}(\langle e,n\rangle) \not\simeq \psi(\langle e,n\rangle)$. But then, since $\psi(\langle e,n\rangle)\downarrow$ and $\varphi_e^{(1),B}$ extends ψ, it follows that $\varphi_{e,s}^{(1),B}(\langle e,n\rangle)\uparrow$, i.e., $f(n) > s$. We now see that, for all but finitely many n, if $n \in 0'$ then $n \in 0'_{f(n)}$. Thus $0' \leq_T f \leq_T B$. This completes the proof. □

Remark 6.2. A similar proof yields the following more detailed result. Given a recursively enumerable set A, we can find recursively enumerable sets A_1 and A_2 such that $A_1 \equiv_T A_2 \equiv_T A$ and $A_1 \cap A_2 = \emptyset$ and, for all random B, if $(\exists Z \leq_T B)\,(Z \supseteq A_1 \text{ and } Z \cap A_2 = \emptyset)$ then $A \leq_T B$.

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