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The Hierarchy Based on the Jump Operator*

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Dedicated to Professor S. C. Kleene on the occasion of his 70th birthday

Abstract: The problem of iterating the jump operator into the transfinite is discussed from several points of view.

0. Introduction

This paper is essentially the text of the author's invited address to the Kleene Symposium on June 21, 1978. We dedicate this paper to Professor Stephen C. Kleene on the occasion of his seventieth birthday.

Unless otherwise specified, by a *set* we mean a set of natural numbers. The *jump operator* is a well-known canonical method of passing from a set X to a more complicated set X^*, defined by

$$m \in X^* \leftrightarrow \{m\}(X,m) \simeq 1$$

(see KLEENE and POST (1954)). The purpose of this paper is to report on the work that has been done on the problem of iterating the jump operator into the transfinite. As with all hierarchies, the goal of such an iteration is to classify sets in increasing levels of complexity. It turns out that transfinite iterates of the jump operator correspond closely to the known set-theoretical hierarchies (GÖDEL, 1939; JENSEN, 1972; SILVER, 1971b; DODD and JENSEN, 1976; MITCHELL, 1974). At the same time these iterates lead to interesting questions concerning degrees of unsolvability (SPECTOR, 1955; BOOLOS and PUTNAM, 1968; JOCKUSCH and SIMPSON, 1976; HODES, 1977). Thus is forged an ineluctable bond between recursion theory and set theory.

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1. Finite iterates of the jump operator

These are defined by setting $X^0 = X$ and $X^{n+1} = (X^n)^*$ for all n. In other words, the finite iterates of the jump operator

$$j: \text{sets} \to \text{sets}$$

are defined by composition, $j^n = j \circ \cdots \circ j$ (n times). Note that the nth jump operator j^n still carries sets to sets. However, degrees of unsolvability, as opposed to sets, are relevant as can be seen from the following theorem of KLEENE (1943) and POST (1948).

Theorem 1.1. *Let X be a set. Then $X \leqslant_T \varnothing^n$ if and only if X is Δ^0_{n+1}.*

In other words, the degree of unsolvability of X is less than or equal to that of the nth jump of the empty set, if and only if X is expressible in both $n+1$ quantifier forms in the arithmetical hierarchy. This theorem illustrates the naturalness of the jump operator inasmuch as one application of the jump operator corresponds to one numerical quantifier.

2. Iteration through the constructive ordinals

In this section we define a streamlined variant of Kleene's notation system $\mathcal{O}$ (KLEENE (1938)) and use it to iterate the jump operator through the constructive ordinals.

Let e be an index of a recursive binary relation $\leqslant_e$ on the natural numbers. The *field* of $\leqslant_e$ is defined to be the set of all x such that $x \leqslant_e x$. We write $x <_e y$ to mean that $x \leqslant_e y$ and $x \neq y$. We define $\mathcal{O}$ to be the set of all e such that $<_e$ is a well-ordering of the field of $\leqslant_e$. If $e \in \mathcal{O}$ we write $|e|$ for the order type of $<_e$ and refer to e as a *notation* for the ordinal $|e|$. An ordinal is said to be *constructive* or *recursive* if it has at least one notation in $\mathcal{O}$. The least nonconstructive ordinal is denoted ω_1.

Let ρ be a fixed recursive function of two variables such that

$$x \leqslant_{\rho(e,z)} y \leftrightarrow x \leqslant_e y <_e z$$

for all x,y,z,e. Thus $|\rho(e,z)|$ runs through the ordinals less than $|e|$ as z runs through the field of $\leqslant_e$. For $e \in \mathcal{O}$ we define

$$H_e = \left\{2^z 3^m : z \in \text{field}(\leqslant_e) \text{ and } m \in H^*_{\rho(e,z)}\right\}.$$

For example, if the field of $\leqslant_e$ is the empty set, then $H_e = \varnothing$. If the field of $\leqslant_e$ is a one element set, then $H_e = \{2^z 3^m : m \in \varnothing^*\}$ where z is the unique element of the field. It is easy to check that if $|e|$ is a finite ordinal n, then H_e has the same degree of unsolvability as $\varnothing^n$. More generally, for all

$e \in \mathcal{O}$, H_e can be described as the result of "iterating the jump operator $|e|$ times" along the well-ordering $<_e$. The H-sets were introduced by DAVIS (1950) and MOSTOWSKI (1951).

It is tempting to define the $|e|$th iterate of the jump operator (applied to $\varnothing$) to be the set H_e. Unfortunately, H_e depends on the notation e and not just on the ordinal $|e|$. It can even be shown that there exist e and e' in $\mathcal{O}$ such that $|e| = |e'|$ but H_e and $H_{e'}$ are not recursively isomorphic. Moreover, we know from proof theory that the problem of choosing a "natural" or "canonical" notation for an arbitrary constructive ordinal is far from trival.

Thus we find a serious obstacle to a satisfactory definition of $\varnothing^{\alpha}$, the αth jump, for arbitrary $\alpha < \omega_1$. The following remarkable theorem of SPECTOR (1955) overcomes this obstacle by employing the concept of degrees of unsolvability.

Theorem 2.1. *If* $|e| = |e'|$, *then* $H_e \equiv_T H_{e'}$, *i.e.* H_e *and* $H_{e'}$ *have the same degree of unsolvability.*

Thus we have a well-defined mapping from the constructive ordinals into the degrees of unsolvability,

$$D = \text{degrees} = \text{sets} / \equiv_T,$$

defined by $\mathbf{0}^{\alpha} = \text{degree}(H_e)$ where $|e| = \alpha$.

Similarly, by the well-known procedure of relativization, we may define $\mathbf{d}^{\alpha}$ for all degrees $\mathbf{d}$ and ordinals $\alpha < \omega_1^{\mathbf{d}}$ where $\omega_1^{\mathbf{d}}$ is the least ordinal not recursive in $\mathbf{d}$. In particular we obtain transfinite iterates of the jump operator

$$j^{\alpha} : D \to D$$

for all $\alpha < \omega_1$.

The above definition of $\mathbf{0}^{\alpha}$, given by Spector's theorem, is satisfying in that $\mathbf{0}^{n} = \text{degree}(\varnothing^{n})$ for all finite n, and $\mathbf{0}^{\alpha+1} = \text{jump}(\mathbf{0}^{\alpha})$ whenever this makes sense. However, Spector's theorem leaves open the exact nature of the dependence of $\mathbf{0}^{\lambda}$ on $\{\mathbf{0}^{\alpha} : \alpha < \lambda\}$ when λ is a limit ordinal. Clearly $\mathbf{0}^{\lambda}$ is an upper bound of $\{\mathbf{0}^{\alpha} : \alpha < \lambda\}$, but can we say more? This question will be discussed in a more general setting from two different points of view in Sections 4 and 5 respectively.

3. The degree of Kleene's $\mathcal{O}$

The class of hyperarithmetical sets is defined by

$$\text{HYP} = \{X : X \leq_T H_e \text{ for some } e \in \mathcal{O}\}.$$

Thus a set is hyperarithmetical if and only if its degree of unsolvability is $\leq \mathbf{0}^{\alpha}$ for some $\alpha < \omega_1$. The following theorems of KLEENE (1955, 1959a), SPECTOR (1960), and GANDY (1960) are relevant to the problem of pushing the iteration of the jump operator beyond ω_1.

Theorem 3.1. *Kleene's* $\mathcal{O}$ *is a complete* Π_1^1 *set.*

Theorem 3.2. HYP$=\Delta_1^1$; *i.e. a set is hyperarithmetical if and only if it is expressible in both one-quantifier forms in the analytical hierarchy.*

Theorem 3.3. *Kleene's* $\mathcal{O}$ *is* Σ_1^1 *over* HYP, *i.e. expressible in existential set quantifier form where the set quantifier ranges over* HYP.

These theorems suggest that Kleene's $\mathcal{O}$ should be in some sense the "next natural set" after HYP. This suggestion is confirmed by the following corollary which is expressed in terms of degrees of unsolvability.

Corollary 3.4. *Let X be a set.*

(i) X *is* Δ_1^1 *over* HYP *if and only if* $X \in$ HYP

(ii) X *is* Δ_2^1 *over* HYP *if and only if* $X \leq_T \mathcal{O}$.

(iii) *More generally,* X *is* Δ_{2+n}^1 *over* HYP *if and only if* $X \leq_T$ *the* nth *jump of* $\mathcal{O}$.

Thus we are led to define $\mathbf{0}^{\omega_1} =$ degree($\mathcal{O}$) and more generally $\mathbf{0}^{\omega_1+n} =$ degree($\mathcal{O}^n$).

4. Master codes in the constructible hierarchy

The ideas which were introduced above suffice to define $\mathbf{0}^{\alpha}$ for all ordinals $\alpha < \omega_\omega =$ the limit of the first ω admissible ordinals. Further extensions of the jump hierarchy may be defined by other methods. For instance, the ramified analytical hierarchy and the corresponding degrees $\mathbf{0}^{\alpha}, \alpha < \beta_0$, have been discussed by BOYD ET AL. (1969) and JOCKUSCH and SIMPSON (1976). Let us now pass over these piecemeal results and look at a truly far-reaching extension, the constructible hierarchy of GÖDEL (1939).

For our purposes it is convenient to define the constructible hierarchy as follows:

$$L_0 = \mathrm{HF} = \{\text{hereditarily finite sets}\};$$

$$L_{\alpha+1} = \{\text{subsets of } L_\alpha \text{ first order definable over } L_\alpha \text{ with parameters}\};$$

$$L_\lambda = \bigcup \{L_\alpha : \alpha < \lambda\} \text{ for limit ordinals } \lambda.$$

Recall now that by a *set* we mean a subset of ω, the set of natural numbers. A set X is said to be $\Sigma_n(L_\alpha)$ if it is first-order definable over L_α by a Σ_n formula with parameters from L_α (cf. JENSEN (1972)). Thus $X \in L_{\alpha+1}$ if and only if X is $\Sigma_n(L_\alpha)$ for some n. A set X is said to be $\Delta_n(L_\alpha)$ if both X and $\omega - X$ are $\Sigma_n(L_\alpha)$.

We say that a set A is a $\Delta_n(L_\alpha)$ *master code* if

$$\forall X(X \leqslant_T A \leftrightarrow X \text{ is } \Delta_n(L_\alpha)).$$

Thus a $\Delta_n(L_\alpha)$ master code is simply a $\Delta_n(L_\alpha)$ set whose degree of unsolvability is maximum among the degrees of all such sets. The concept of a master code is due to JENSEN (1967, 1972).

By a *Jensen degree* let us mean the degree of a $\Delta_n(L_\alpha)$ master code for some positive integer n and ordinal α. Clearly the Jensen degrees are well-ordered in the natural ordering of degrees of unsolvability. It can also be shown that if A is a $\Delta_n(L_\alpha)$ master code, then the jump of A is a $\Delta_{n+1}(L_\alpha)$ master code. Thus the successor operation on the well-ordering of Jensen degrees is given by the jump operator.

The following theorem of JENSEN (1967) shows that there are no unnecessary gaps in the hierarchy of Jensen degrees.

Theorem 4.1. *Suppose that there exists a set X which is $\Delta_n(L_\alpha)$ but not an element of L_α. Then there exists a $\Delta_n(L_\alpha)$ master code.*

Proof (sketch). We know that X and $\omega - X$ are definable over L by Σ_n formulas with parameters. By the uniformization theorem (JENSEN, 1972) we can choose a canonical collection of Σ_n Skolem functions for these formulas and all their subformulas. Let M be the submodel of L_α generated from the parameters by the Skolem functions. The condensation lemma tells us that there is an ordinal $\beta \leqslant \alpha$ such that L_β is isomorphic to M. By construction X is $\Delta_n(M)$, hence $\Delta_n(L_\beta)$. Hence $\beta = \alpha$ since otherwise we would have $X \in L_\alpha$. Hence the inverse image of M under the Skolem functions yields a subset A of ω which encodes L_α. It can then be shown that A is a $\Delta_n(L_\alpha)$ master code.

An argument of the above type also appeared in a paper of BOOLOS and PUTNAM (1968).

An easy consequence of Jensen's theorem is that the Jensen degrees are well-ordered in order type $\aleph_1^L$, the least constructibly uncountable ordinal. We are therefore justified in defining $\mathbf{0}^\alpha$ for all $\alpha < \aleph_1^L$ to be the αth Jensen degree. It can be shown that this definition of $\mathbf{0}^\alpha$ agrees with the definitions in Sections 1, 2 and 3 above for the α's considered there. Thus we have iterated the jump operator through all the constructibly countable ordinals.

5. Degree theoretic characterizations

In this section we discuss the Jensen degrees $\mathbf{0}^\alpha, \alpha < \aleph_1^L$ which were defined above and reexamine them from an algebraic point of view. Specifically, we consider the algebraic structure

$$\mathfrak{D} = \langle D, \cup, \leqslant, \mathbf{0}, j \rangle$$

where D is the set of all degrees of unsolvability, $\mathbf{0}$ is the least element of D, and j is the jump operator. This is an upper semilattice (with admittedly some extra structure) and it is natural to ask whether the degrees $\mathbf{0}^\alpha, \alpha < \aleph_1^L$ can be characterized in lattice theoretic terms. We may begin by defining

$$\mathbf{0}^0 = \mathbf{0}$$

and

$$\mathbf{0}^{\alpha+1} = \text{jump}(\mathbf{0}^\alpha)$$

for successor ordinals $\alpha + 1$, but what is the lattice theoretic aspect of the dependence of $\mathbf{0}^\lambda$ on $\{\mathbf{0}^\alpha : \alpha < \lambda\}$ for limit ordinals $\lambda < \aleph_1^L$? If we define the ideal

$$I_\lambda = \{\mathbf{d} \in D: \quad (\exists \alpha < \lambda) \quad \mathbf{d} \leqslant \mathbf{0}^\alpha\},$$

then clearly $\mathbf{0}^\lambda$ is an upper bound for I_λ, but we would like to characterize $\mathbf{0}^\lambda$ somehow as the "least natural" upper bound for I_λ. This is accomplished by the following theorem.

Theorem 5.1. *Let λ be a limit ordinal less than $\aleph_1^L$. Define ν_λ to be the least ordinal ν such that the set of all degrees of the form $(\mathbf{a} \cup \mathbf{b})^\nu$ with*

$$I_\lambda = \{\mathbf{d} \in D: \mathbf{d} \leqslant \mathbf{a} \text{ and } \mathbf{d} \leqslant \mathbf{b}\} \tag{$*$}$$

has a least element. Then ν_λ exist and $\mathbf{0}^\lambda$ is that least element. Moreover, there exist degrees $\mathbf{a}$ and $\mathbf{b}$ such that $(*)$ holds and $\mathbf{0}^\lambda = (\mathbf{a} \cup \mathbf{b})^{\nu_\lambda}$ and ν_λ is recursive in $\mathbf{a} \cup \mathbf{b}$.

For $\lambda < \omega_1$ the above theorem is essentially due Enderton and Putnam (1970) and Sacks (1971). In this case it turns out that $\nu_\lambda = 2$ and an even more perspicuous characterization of $\mathbf{0}^\lambda$ is possible.

For $\lambda < \beta_0$ the above theorem is due to Jockusch and Simpson (1976). Here β_0 is the least ordinal β such that L_β is a model of ZF minus the power set axiom, and ν_λ can assume any finite value $n \geqslant 2$.

The full Theorem 5.1 appears in the thesis of Hodes (1977) who acknowledges some help from Abramson in the proof of the "moreover" clause. The main tool in Hodes' proof is a notion of forcing due to Sacks (1971) in which a condition is a recursively pointed perfect tree P whose degree lies in I_λ. The recursive pointedness means that P is recursive in each of its infinite branches. The "moreover" clause is obtained with the aid of another notion of forcing due to STEEL (1978).

The noteworthy point about Theorem 5.1 is that the next Jensen degree, $\mathbf{0}^\lambda$, is defined in purely algebraic terms from the ideal generated by the previous Jensen degrees. At first glance, the given definition of $\mathbf{0}^\lambda$ may appear circular in that it uses the concept of νth jump where ν is essentially arbitrary. Indeed, it often happens that $\nu_\lambda > \lambda$. But, the "moreover" clause tells us that the result of the given definition remains unchanged if we restrict attention to $\mathbf{a}, \mathbf{b}, \nu$ such that ν is recursive in $\mathbf{a} \cup \mathbf{b}$. For such $\mathbf{a}, \mathbf{b}, \nu$ the definition of $(\mathbf{a} \cup \mathbf{b})^\nu$ is unproblematical (cf. Section 2 above). Thus we really do have a definition of $\mathbf{0}^\lambda$ in terms of simpler concepts.

6. Open problems

The purpose of this section is to discuss several problems which are suggested by results stated in previous sections.

An obvious problem, suggested by Sections 4 and 5, is to extend the degree theoretic hierarchy $\mathbf{0}^\alpha$ to ordinals $\alpha \geqslant \aleph_1^L$. Clearly some unusual hypothesis is called for here since if $V = L$, then the degrees $\mathbf{0}^\alpha, \alpha < \aleph_1^L$ have no upper bound. If we assume the existence of, say, a Ramsey cardinal, then the work of DODD and JENSEN (1976) establishes the existence of a hierarchy of master codes beyond $\aleph_1^L$. The first such master code is of course Silver's remarkable set $0^\#$ (SILVER, 1971a). It therefore seems reasonable to define $\mathbf{0}^{\aleph_1^L}$ to be the degree of $0^\#$. Similarly, we may define $\mathbf{0}^\alpha$ for $\alpha < \aleph_1^K$ to be the degree of the αth master code in the Dodd–Jensen core model K (DODD and JENSEN, 1976). This much is clear. What is not clear is how to characterize these degrees algebraically from below in the style of Theorem 5.1.

We therefore pose the following test problem: find a natural algebraic characterization of the degree of $0^\#$ within the degree-theoretic structure $\mathfrak{D}$. A fact which may be relevant here is that there exists a Σ_1^1 set of degrees of unsolvability whose determinacy is provably equivalent to the existence of $0^\#$. This result is due to HARRINGTON (1979).

In order to be completely honest with the reader, we must now pause to point out that the problem just stated is highly speculative because the existence of $0^\#$ is not firmly established. Indeed, several prominent set theorists have seriously attempted to refute the existence of $0^\#$. So far it is known that

(i) $0^\#$ does not exist in the models of set theory considered by GÖDEL (1939) and COHEN (1966);

(ii) the consistency of the existence of $0^\#$ with set theory cannot be proved in set theory, even if large cardinal axioms of the kinds considered by LÉVY (1971) and SILVER (1970) are assumed;

(iii) the hypothesis of the nonexistence of $0^{\#}$ is an attractive one with far-reaching consequences, e.g. the solution of the singular cardinals problem (DEVLIN and JENSEN, 1975).

However, the nonexistence of $0^{\#}$ has not yet been proved and indeed may be unprovable.

We now turn to another set of problems. It is known from JOCKUSCH and SIMPSON (1976) that many specific Jensen degrees have natural algebraic characterizations within $\mathcal{D}$. For instance, let α_n be the least ordinal α such that L_α is a model of the Δ_n^1 comprehension axiom of second order arithmetic. Thus $\alpha_0 = \omega, \alpha_1 = \omega_1, \alpha_2 = \omega_1^{E_1}$, and in general α_{3+n} = the first Σ_{2+n} admissible ordinal greater than ω. The results of JOCKUSCH and SIMPSON (1976) show that the degrees $\mathbf{0}^{\alpha_n}$ as well as $\mathbf{0}^{\beta_0}$ have simple algebraic characterizations. Question: Can we do the same for some of the other specific Jensen degrees which arise from the theories of recursion in higher types (KLEENE, 1959b, 1963) and nonmonotonic inductive definitions (RICHTER and ACZEL, 1974)? A good test case here is the Jensen degree $\mathbf{0}^\sigma$ where $\sigma = \omega_1^{E_1^\#}$ = the least Σ_1^1 reflecting ordinal (RICHTER and ACZEL, 1974) = the least non-Gandy ordinal (ABRAMSON and SACKS, 1976). This degree $\mathbf{0}^\sigma$ can also be characterized as the largest degree of a set which is Σ_1^1 inductively definable, i.e. recursively enumerable in $E_1^\#$ (HINMAN, 1978, Theorem VI.6.14). Does $\mathbf{0}^\sigma$ have a natural algebraic characterization within $\mathcal{D}$?

It is perhaps worth remarking that $\mathbf{0}^\sigma$ and most other specific Jensen degrees (and also the degree of $0^{\#}$ if it exists) are already known to be first-order definable in $\mathcal{D}$. This follows from the general definability theorem of SIMPSON (1977, Theorem 3.12). However, the first-order definitions of $\mathbf{0}^\sigma$ which are known at this writing look extremely artificial from the algebraic and degree-theoretic viewpoint. What we lack is a degree-theoretically natural characterization of $\mathbf{0}^\sigma$.

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