An objective justification for actual infinity?

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This document is a record of my contribution to a panel discussion which took place on July 27, 2011 as part of the Infinity and Truth Workshop held at the Institute for Mathematical Sciences, National University of Singapore, July 25–29, 2011.

In preparation for the panel discussion, Professor Woodin asked each panelist to formulate a yes/no question to be asked of a benevolent, omniscient mathematician. In addition, each panelist was asked to give reasons for his choice of a question.

Since I don't believe in omniscient mathematicians, I chose to interpret "omniscient mathematician" as "wise and thoughtful philosopher of mathematics." With this change, my yes/no question reads as follows:

> Can there be an *objective* justification for the concept "actual infinity"?

Of course this question would be incomprehensible without some understanding of the key terms "objective" and "actual infinity." Therefore, I shall now explain my views on objectivity in mathematics, and on potential infinity versus actual infinity. After that, I shall point to some relevant results from contemporary foundational research, especially reverse mathematics.

Objectivity in mathematics

Generally speaking, by *objectivity* I mean human understanding of *reality*,[1] "the real world out there," with an eye toward controlling and using aspects of reality for human purposes. I subscribe to Objectivism [4], a well-known modern philosophical system which emphasizes the central role of objectivity.

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[1] Of course reality is not limited to physical reality. For example, the United States government is a real entity but not a physical entity.

My views on objectivity in mathematics are explained in my paper [7], which is the text of an invited talk that I gave at a philosophy of mathematics conference at New York University in April 2009. Briefly, I believe that mathematicians ought to seek objective understanding of the mathematical aspects of reality. This makes it possible to apply mathematics, with varying degrees of success, for the betterment of human life on earth.

Among the highly successful application areas for mathematics are: classical physics, engineering (mechanical engineering, electrical engineering, etc.), modern physics (relativity, quantum theory, etc.), chemistry, microbiology, astronomy. Among the successful application areas are: biology, medicine, agriculture, meteorology. Among the application areas with moderate to low success are: economics, social sciences, psychology, finance.

In all of these application areas, it is crucially important that our mathematical models should be *objective*, i.e., correspond closely to the underlying reality. Otherwise, success will be severely impaired. It would be desirable to place *all* of mathematics on an objective foundation. Failing that, it would be desirable to place at least the applicable parts of mathematics on an objective foundation.

Potential infinity versus actual infinity

The distinction between potential infinity and actual infinity goes back to Aristotle. A detailed, nuanced discussion can be found in Books M and N of the Metaphysics [1, 3], which constitute Aristotle's treatise on the philosophy of mathematics. Aristotle's position is that, while potential infinities have an objective existence in reality, actual infinities do not. This is in the context of a broader argument against Plato's theology.

In modern mathematics, the prime example of *potential infinity* is the natural number sequence 1, 2, 3, …, which manifests itself in reality as iteration, repeated processes, infinite divisibility,[2] etc. Another example in modern mathematics is the full binary tree $\{0,1\}^{<\infty}$, whose infinite paths correspond roughly to the points on the unit interval $[0,1]$.

Contrasted to potential infinity is *actual infinity*, i.e., a completed infinite totality. There are many examples in modern mathematics, including infinite sets such as $\omega = \{0,1,2,\ldots\}$, transfinite ordinals, $[0,1]$, the real line, L_1, $B(H)$, etc. Thus my yes/no question comes down to asking whether certain parts of modern mathematics can have an objective justification.

Insights from reverse mathematics

As regards the distinction between potential and actual infinity, it appears that reverse mathematics can teach us something. Recall from [6, Part A] that *reverse mathematics* is a systematic attempt to classify specific mathematical theorms according to which set existence axioms are needed to prove them.

[2] For example, a piece of metal can be divided indefinitely.

The focus here is mainly on *core mathematics*, i.e., analysis, algebra, number theory, differential equations, probability, geometry, combinatorics, etc. Among the specific core mathematical theorems considered are many which time and again have proved useful in applications.

Reverse mathematics has uncovered a hierarchy of formal systems which are relevant for this classification. Some of the most important formal systems for reverse mathematics are, in order of increasing strength:

$$\mathsf{RCA}_0^*,\ \mathsf{RCA}_0,\ \mathsf{WKL}_0,\ \mathsf{ACA}_0,\ \mathsf{ATR}_0,\ \Pi^1_1\text{-}\mathsf{CA}_0,\ \ldots.$$

For our purposes here, recall [6, Part B] that there is a significant "break point" between the first three systems and the others. Namely, while RCA_0^*, RCA_0, WKL_0 are conservative over primitive recursive arithmetic (= PRA) for Π^0_2 sentences, the other systems ACA_0, ... are much stronger and therefore not conservative over PRA even for Π^0_1 sentences. Moreover, Tait [8] has argued that PRA represents the outer limits of finitism. Recall also that PRA is based on the idea of iteration and so may be viewed as a formal theory of potential infinity.

Now, an important discovery of reverse mathematics is that large parts of contemporary mathematics are formalizable in RCA_0^*, RCA_0, and WKL_0. This seems to include the applicable parts of mathematics. See also my paper [5]. Combining all of the above considerations, we see the possible outline of an objective justification of much of modern mathematics, especially the applicable parts of it. However, the prospects for an objective justification of actual infinity remain much more doubtful. This is the background of my yes/no question.

References

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