Implicit definability in arithmetic*

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Abstract

We consider implicit definability over the natural number system $\mathbb{N}, +, \times, =$. We present a new proof of two theorems of Leo Harrington. The first theorem says that there exist implicitly definable subsets of $\mathbb{N}$ which are not explicitly definable from each other. The second theorem says that there exists a subset of $\mathbb{N}$ which is not implicitly definable but belongs to a countable, explicitly definable set of subsets of $\mathbb{N}$. Previous proofs of these theorems have used finite- or infinite-injury priority constructions. Our new proof is easier in that it uses only a non-priority oracle construction, adapted from the standard proof of the Friedberg Jump Theorem.

Keywords: arithmetical hierarchy, arithmetical singletons, implicit definability, hyperarithmetical sets, Turing jump.

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Contents

1 Introduction

Definitions. Let $\mathbb{N} = \{0, 1, 2, \ldots, n, \ldots\}$ = the set of all natural numbers. Let $\mathrm{Pow}(\mathbb{N})$ be the powerset of $\mathbb{N}$, i.e., the set of all subsets of $\mathbb{N}$. A set $X \in \mathrm{Pow}(\mathbb{N})$ is said to be *arithmetical* if it is explicitly definable over the natural number system $\mathbb{N}, +, \times, =$. In other words,

$$X = \{n \in \mathbb{N} \mid (\mathbb{N}, +, \times, =) \models \Phi(n)\}$$

for some first-order formula $\Phi(n)$ in the language $+, \times, =$. Given two sets $X, Y \in \mathrm{Pow}(\mathbb{N})$, we say that X is *arithmetical in* Y if X is explicitly definable from Y, i.e.,

$$X = \{n \in \mathbb{N} \mid (\mathbb{N}, +, \times, Y, =) \models \Phi(n)\}$$

for some first-order formula $\Phi(n)$ in the language $+, \times, Y, =$. We say that X and Y are *arithmetically incomparable* if neither is arithmetical in the other. A set of sets $S \subseteq \mathrm{Pow}(\mathbb{N})$ is said to be *arithmetical* if it is explicitly definable, i.e.,

$$S = \{X \in \mathrm{Pow}(\mathbb{N}) \mid (\mathbb{N}, +, \times, X, =) \models \Phi\}$$

for some first-order sentence Φ in the language $+, \times, X, =$. A set $X \in \mathrm{Pow}(\mathbb{N})$ is called an *arithmetical singleton* or *implicitly arithmetical* if the singleton set $\{X\}$ is arithmetical.

Remark 1. The purpose of this paper is to present a new proof of two theorems of Harrington [6, 7] concerning implicit definability over the natural number system $\mathbb{N}, +, \times, =$. The two theorems read as follows.

1. There exist arithmetical singletons $X, Y \in \mathrm{Pow}(\mathbb{N})$ which are arithmetically incomparable. (See Theorem 4.4 below.)

2. There exists a set $Z \in \mathrm{Pow}(\mathbb{N})$ which belongs to a countable arithmetical set of sets $S \subseteq \mathrm{Pow}(\mathbb{N})$ but is not an arithmetical singleton. (See Theorem 4.5 below.)

We feel that these two theorems deserve to be better known, because they embody significant insight concerning implicit definability in arithmetic.

Remark 2. Before Harrington's work, some early theorems concerning implicit definability in arithmetic were as follows.

1. There exists $X \in \mathrm{Pow}(\mathbb{N})$ which is implicitly arithmetical but not arithmetical. (Namely, let $X = 0^{(\omega)} =$ the Tarski truth set for $\mathbb{N}, +, \times, =$. See Rogers [11, Theorems 14-X and 15-XII].)

2. There exist $X, Y \in \mathrm{Pow}(\mathbb{N})$ such that the pair $X \oplus Y$ is implicitly arithmetical but neither X nor Y is implicitly arithmetical. (Namely, let X and Y be Cohen generic over $\mathbb{N}, +, \times, =$ such that $X \oplus Y$ and $0^{(\omega)}$ are arithmetical in each other. See Feferman [3] or Rogers [11, Exercise 16-72].)

3. Each arithmetical singleton is arithmetical in $0^{(\alpha)}$ for some recursive ordinal α, and each such $0^{(\alpha)}$ is itself an arithmetical singleton. (See for instance Sacks [13, Chapter II].)

4. Every nonempty countable arithmetical set of sets $S \subseteq \mathrm{Pow}(\mathbb{N})$ contains an arithmetical singleton. (This result is due to H. Tanaka [15].)

Remark 3. Harrington's original proof [6] of Theorem 4.4 was based on an infinite-injury priority construction. The same method has been used by Harrington [6] and others to obtain results about ω-REA arithmetical degrees (see M. F. Simpson [14, Chapters 2 and 3] and Odifreddi [10, Chapter XIII]), jump embeddings (see Hinman/Slaman [8]), nonstandard models of arithmetic (see Ash/Knight [1, Chapters 14–19, Theorem 19.19]), and generalized high/low hierarchies (see Montalbán [9]).

Remark 4. Harrington's original proof [7] of Theorem 4.5 was based on a finite-injury priority construction. The same method has been extended into the transfinite by Harrington [7] and Gerdes [5] to obtain other interesting results. In particular, see Remark 12 below. For an application to effectively Borel equivalence relations, see Fokina/Friedman/Törnquist [4].

Remark 5. Our new proof of Theorems 4.4 and 4.5 does not use a priority construction of any kind. Instead our proof is based on a direct oracle construction, adapted from the standard proof of the Friedberg Jump Theorem. In this sense our proof of Theorems 4.4 and 4.5 is much easier than the proofs in [1, 5, 6, 7, 8, 9, 10, 14]. On the other hand, our proof uses the Recursion Theorem in exactly the same way as Harrington used it. Harrington [6] has referred to this way of using the Recursion Theorem as "the shiny little box which was first opened by Sacks [12]."

Remark 6. Beyond Theorems 4.4 and 4.5, we believe we can extend our non-priority oracle method farther into the transfinite to obtain relatively easy proofs of at least some of the other results of Harrington [7] and Gerdes [5]. However, we reserve that extension for a future paper. In this paper we limit ourselves to providing relatively easy proofs of Theorems 4.4 and 4.5.

Remark 7. The plan of this paper is as follows. In §2 we review some basic recursion-theoretic notions. In §3 we prove a rudimentary version of Theorems 4.4 and 4.5. In §4 we prove Theorems 4.4 and 4.5.

2 Recursion-theoretic background

In this section we review some basic notions from recursion theory which are needed for our proof of Theorems 4.4 and 4.5. A good reference for this material is Rogers [11].

Natural numbers are denoted $e, i, j, k, l, m, n, \ldots$. The set of all natural numbers is denoted $\mathbb{N}$. Instead of working with $\mathrm{Pow}(\mathbb{N})$, the set of all subsets $X \subseteq \mathbb{N}$, we work with $\mathbb{N}^{\mathbb{N}}$, the set of all functions $X : \mathbb{N} \to \mathbb{N}$. The space $\mathbb{N}^{\mathbb{N}}$ with the product topology is known as *the Baire space*. Points in $\mathbb{N}^{\mathbb{N}}$ are denoted $X, Y, Z, \ldots$. Subsets of $\mathbb{N}^{\mathbb{N}}$ are denoted $P, Q, \ldots$.

Recall that a point $X \in \mathbb{N}^{\mathbb{N}}$ or a set $P \subseteq \mathbb{N}^{\mathbb{N}}$ is arithmetical if and only if it is Π^0_n for some $n \geq 1$. The hierarchy Π^0_n where $n = 1, 2, \ldots$ is known as *the arithmetical hierarchy*. See for instance [11, Chapters 14–16]. (It is known [15] that every arithmetical set is in arithmetical one-to-one correspondence with a Π^0_1 set. However, we shall not need this result here.) A Π^0_n *singleton* is a point X such that the singleton set $\{X\}$ is Π^0_n. Thus X is an arithmetical singleton if and only if it is a Π^0_n singleton for some $n \geq 1$. A *ranked point* is a point X such that $X \in P$ for some countable Π^0_1 set P.

Points in $\mathbb{N}^{\mathbb{N}}$ may be viewed as *Turing oracles*. See for instance [11, Chapters 9–13]. Relativizing to a Turing oracle $A \in \mathbb{N}^{\mathbb{N}}$, a point $X \in \mathbb{N}^{\mathbb{N}}$ or a set $P \subseteq \mathbb{N}^{\mathbb{N}}$ is said to be $\Pi^{0,A}_n$ if it is Π^0_n relative to A, and *arithmetical* *in* A if it is $\Pi^{0,A}_n$ for some n. In particular, a set P is topologically closed if and only if it is $\Pi^{0,A}_1$ for some A. A point X such that the singleton set $\{X\}$ is $\Pi^{0,A}_n$ is called a $\Pi^{0,A}_n$ *singleton*.

For $A \in \mathbb{N}^{\mathbb{N}}$ we write $\{e\}^A(i) = j$ to mean that the eth Turing machine with oracle A and input i halts with output j. We write $\{e\}^A(i)\downarrow$ (respectively $\uparrow$) to mean that the eth Turing machine with oracle A and input i halts (respectively, does not halt). Thus $\{e\}^A(i)\downarrow$ if and only if $\exists j\,(\{e\}^A(i) = j)$. For $A, B \in \mathbb{N}^{\mathbb{N}}$ we write $A \le_{\mathrm{T}} B$ to mean that A is *Turing reducible to* B, i.e., $\exists e\,\forall i\,(A(i) = \{e\}^B(i))$. We write $A \equiv_{\mathrm{T}} B$ to mean that A is *Turing equivalent to* B, i.e., $A \le_{\mathrm{T}} B$ and $B \le_{\mathrm{T}} A$. We define $A \oplus B \in \mathbb{N}^{\mathbb{N}}$ by the equations $(A\oplus B)(2i) = A(i)$ and $(A\oplus B)(2i+1) = B(i)$. Thus $A \oplus B \le_{\mathrm{T}} C$ if and only if $A \le_{\mathrm{T}} C$ and $B \le_{\mathrm{T}} C$.

For $A \in \mathbb{N}^{\mathbb{N}}$ we write A' = the *Turing jump* of A, defined by

$$A'(e) = \begin{cases} 1 & \text{if } \{e\}^A(e)\downarrow, \\ 0 & \text{if } \{e\}^A(e)\uparrow. \end{cases}$$

We write $A^{(n)}$ = the nth Turing jump of A, defined inductively by letting $A^{(0)} = A$ and $A^{(n+1)} = (A^{(n)})'$ for all n. Recall that A is arithmetical in B if and only if $\exists n\,(A \le_{\mathrm{T}} B^{(n)})$. For use in the proof of Theorems 3.5 and 4.5, note that for each $n \ge 1$, a set $P \subseteq \mathbb{N}^{\mathbb{N}}$ is Π^0_n if and only if $\exists e\,\forall X\,(X \in P \Leftrightarrow X^{(n)}(e) = 0)$. See for instance [11, §14.5].

We write $A^{(\omega)}$ = the ωth Turing jump of A, defined by

$$A^{(\omega)}(i) = \begin{cases} A^{(n)}(e) & \text{if } i = 3^n 5^e, \\ 0 & \text{otherwise}. \end{cases}$$

Thus $A^{(\omega)} = \bigoplus_n A^{(n)}$ and $A^{(n)} \le_{\mathrm{T}} A^{(\omega)}$ uniformly in n.

Let $0 \in \mathbb{N}^{\mathbb{N}}$ denote the constant zero function. Thus $0^{(n)}$ = the nth Turing jump of 0, and $0^{(\omega)}$ = the ωth Turing jump of 0. Note also that X is arithmetical if and only if $X \le_{\mathrm{T}} 0^{(n)}$ for some n.

3 A rudimentary version of Harrington's theorems

The purpose of this section is to prove a rudimentary version of Harrington's theorems, with "arithmetical" replaced by Π^0_n for a fixed n. Our rudimentary versions of Theorems 4.4 and 4.5 are Theorems 3.4 and 3.5 respectively.

Lemma 3.1. Given a $\Pi^{0,A'}_1$ set P we can find a $\Pi^{0,A}_1$ set Q and a homeomorphism $F : P \cong Q$ such that $X \oplus A \equiv_{\mathrm{T}} F(X) \oplus A$ uniformly for all $X \in P$.

Proof. Since P is a $\Pi^{0,A'}_1$ set, it follows that P is a $\Pi^{0,A}_2$ set, say $P = \{X \mid \forall i\,\exists j\,R(X,i,j)\}$ where R is an A-recursive predicate. Define $F : P \cong Q = F(P)$ by letting $F(X) = X \oplus \widehat{X}$ where $\widehat{X}(i) =$ the least j such that $R(X,i,j)$ holds. Clearly Q is a $\Pi^{0,A}_1$ set and $X \oplus A \equiv_{\mathrm{T}} F(X) \oplus A$ uniformly for all $X \in P$. ☐

Lemma 3.2. Given a $\Pi^{0,A'}_1$ set P we can find a $\Pi^{0,A}_1$ set Q and a homeomorphism $H : P \cong Q$ such that $X \oplus A' \equiv_{\mathrm{T}} H(X) \oplus A' \equiv_{\mathrm{T}} (H(X) \oplus A)'$ uniformly for all $X \in P$.

In order to prove Lemma 3.2, we first present some general remarks concerning strings, trees, and treemaps.

Notation (strings)**.** Let $\mathbb{N}^* = \bigcup_{l\in\mathbb{N}} \mathbb{N}^l =$ the set of *strings*, i.e., finite sequences of natural numbers. For $\sigma = \langle n_0, n_1, \ldots, n_{l-1}\rangle \in \mathbb{N}^*$ we write $\sigma(i) = n_i$ for all $i < |\sigma| = l =$ the *length* of σ. For $\sigma,\tau \in \mathbb{N}^*$ we write $\sigma^\frown\tau =$ the *concatenation*, σ followed by τ, defined by the conditions $|\sigma^\frown\tau| = |\sigma| + |\tau|$, $(\sigma^\frown\tau)(i) = \sigma(i)$ for all $i < |\sigma|$, and $(\sigma^\frown\tau)(|\sigma| + i) = \tau(i)$ for all $i < |\tau|$. We write $\sigma \subseteq \tau$ if $\sigma^\frown\rho = \tau$ for some ρ. If $|\sigma| \geq n$ we write $\sigma\restriction n = \langle\sigma(0), \sigma(1), \ldots, \sigma(n-1)\rangle =$ the unique $\rho \subseteq \sigma$ such that $|\rho| = n$. For $X \in \mathbb{N}^{\mathbb{N}}$ we write $X\restriction n = \langle X(0), X(1), \ldots, X(n-1)\rangle =$ the unique $\sigma \subset X$ such that $|\sigma| = n$. If $|\sigma| = |\tau| = n$ we define $\sigma \oplus \tau \in \mathbb{N}^*$ by the conditions $|\sigma \oplus \tau| = 2n$ and $(\sigma \oplus \tau)(2i) = \sigma(i)$ and $(\sigma \oplus \tau)(2i + 1) = \tau(i)$ for all $i < n$.

Definition (trees)**.** A *tree* is a set $T \subseteq \mathbb{N}^*$ such that

$$\forall\rho\,\forall\sigma\,((\rho \subseteq \sigma \text{ and } \sigma \in T) \Rightarrow \rho \in T).$$

For any tree T we write

$$[T] = \{\text{paths through } T\} = \{X \mid \forall n\,(X\restriction n \in T)\}.$$

Remark 8. It is well known (see for instance [11, Chapter 15]) that the following statements are pairwise equivalent.

1. P is a $\Pi^{0,A}_1$ set.
2. $P = [T]$ for some $\Pi^{0,A}_1$ tree T.
3. $P = [T]$ for some A-recursive tree T.
4. $P = \{X \mid X \oplus A \in [T]\}$ for some recursive tree T.

Definition (treemaps)**.** Let T be a tree. A *treemap* is a function $F : T \to \mathbb{N}^*$ such that

$$F(\sigma^\frown\langle i\rangle) \supseteq F(\sigma)^\frown\langle i\rangle$$

for all $\sigma \in T$ and all $i \in \mathbb{N}$ such that $\sigma^\frown\langle i\rangle \in T$. We then have another tree

$$F(T) = \{\tau \mid \exists\sigma\,(\sigma \in T \text{ and } \tau \subseteq F(\sigma))\}.$$

Thus $P = [T]$ and $F(P) = [F(T)]$ are closed sets in the Baire space, and we have a homeomorphism $F : P \cong F(P)$ defined by $F(X) = \bigcup_{n\in\mathbb{N}} F(X{\upharpoonright}n)$ for all $X \in P$. Note also that the composition of two treemaps is a treemap. A treemap $F : T \to \mathbb{N}^*$ is said to be A-*recursive* if it is the restriction to T of a partial A-recursive function.

Remark 9. Let T be a tree and let $F : T \to \mathbb{N}^*$ be a treemap. Given $\tau \in F(T)$ let $\sigma \in T$ be minimal such that $\tau \subseteq F(\sigma)$. Then σ is a *substring* of τ, i.e., $\sigma = \langle \tau(j_0), \tau(j_1), \ldots, \tau(j_{l-1})\rangle$ for some $j_0 < j_1 < \cdots < j_{l-1} < |\tau|$. Thus, in the definition of $F(T)$, the quantifier $\exists\sigma$ may be replaced by a bounded quantifier,

$$F(T) = \{\tau \mid (\exists\sigma \text{ substring of } \tau)\,(\sigma \in T \text{ and } \tau \subseteq F(\sigma))\}.$$

This implies that, for instance, if F and T are A-recursive then so is $F(T)$.

We are now ready to prove Lemma 3.2.

Proof of Lemma 3.2. Given A we construct a particular A'-recursive treemap $G : \mathbb{N}^* \to \mathbb{N}^*$. We define $G(\sigma)$ by induction on $|\sigma|$ beginning with $G(\langle\rangle) = \langle\rangle$. If $G(\sigma)$ has been defined, let $e = |\sigma|$ and for each i let $G(\sigma^\frown\langle i\rangle) =$ the least $\tau \supseteq G(\sigma)^\frown\langle i\rangle$ such that $\{e\}^{\tau\oplus A}_{|\tau|}(e)\downarrow$ if such a τ exists, otherwise $G(\sigma^\frown\langle i\rangle) = G(\sigma)^\frown\langle i\rangle$. Clearly G is an A'-recursive treemap, and our construction of G implies that for all e and X, $\{e\}^{G(X)\oplus A}(e)\downarrow$ if and only if $\{e\}^{G(X{\upharpoonright}e+1)\oplus A}_{|G(X{\upharpoonright}e+1)|}(e)\downarrow$. Thus $X \oplus A' \equiv_{\mathrm{T}} G(X) \oplus A' \equiv_{\mathrm{T}} (G(X) \oplus A)'$ uniformly for all X.

Let G be the A'-recursive treemap which was constructed above. Let P be a $\Pi^{0,A'}_1$ set. By Remarks 8 and 9 we know that the restriction of G to P maps P homeomorphically onto another $\Pi^{0,A'}_1$ set $G(P)$. Applying Lemma 3.1 to $G(P)$ we obtain a $\Pi^{0,A}_1$ set Q and a homeomorphism $F : G(P) \cong Q$ such that $Y \oplus A \equiv_{\mathrm{T}} F(Y) \oplus A$ uniformly for all $Y \in G(P)$. Thus $H = F \circ G$ is a homeomorphism of P onto Q, and for all $X \in P$ we have $G(X) \oplus A \equiv_{\mathrm{T}} F(G(X)) \oplus A = H(X) \oplus A$ uniformly, hence $X \oplus A' \equiv_{\mathrm{T}} H(X) \oplus A' \equiv_{\mathrm{T}} (H(X) \oplus A)'$ uniformly, Q.E.D. □

Remark 10. Our proof of Lemma 3.2 via treemaps is similar to the proof of [2, Lemma 5.1]. Within our proof of Lemma 3.2, the construction of the specific treemap G is the same as the standard proof of the Friedberg Jump Theorem as expounded for instance in [11, §13.3].

Lemma 3.3. Given a $\Pi^{0,0^{(n)}}_1$ set P_n we can find a Π^0_1 set P_0 and a homeomorphism $H^n_0 : P_n \cong P_0$ such that $X_n \oplus 0^{(n)} \equiv_{\mathrm{T}} X_0 \oplus 0^{(n)} \equiv_{\mathrm{T}} X^{(n)}_0$ uniformly for all $X_n \in P_n$ and $X_0 = H^n_0(X_n) \in P_0$.

Proof. The proof is by induction on n. For $n = 0$ there is nothing to prove. For the inductive step, given a $\Pi^{0,0^{(n+1)}}_1$ set P_{n+1} apply Lemma 3.2 with $A = 0^{(n)}$ to obtain a $\Pi^{0,0^{(n)}}_1$ set P_n and a homeomorphism $H_n : P_{n+1} \cong P_n$ such that $X_{n+1} \oplus 0^{(n+1)} \equiv_{\mathrm{T}} H_n(X_{n+1}) \oplus 0^{(n+1)} \equiv_{\mathrm{T}} (H_n(X_{n+1}) \oplus 0^{(n)})'$ uniformly for all $X_{n+1} \in P_{n+1}$. Then apply the inductive hypothesis to P_n to find a Π^0_1 set P_0 and a homeomorphism $H^n_0 : P_n \cong P_0$ such that $X_n \oplus 0^{(n)} \equiv_{\mathrm{T}} X_0 \oplus 0^{(n)} \equiv_{\mathrm{T}} X^{(n)}_0$ uniformly for all $X_n \in P_n$. Letting $H^{n+1}_0 = H_n \circ H^n_0$ it follows that $X_{n+1} \oplus 0^{(n+1)} \equiv_{\mathrm{T}} X_0 \oplus 0^{(n+1)} \equiv_{\mathrm{T}} X^{(n+1)}_0$ uniformly for all $X_{n+1} \in P_{n+1}$ and $X_0 = H^{n+1}_0(X_{n+1}) \in P_0$, Q.E.D. $\square$

We now use Lemma 3.3 to prove a rudimentary version of Harrington's theorems.

Theorem 3.4. Given n we can find Π^0_1 singletons X, Y such that $X \not\leq_{\mathrm{T}} Y^{(n)}$ and $Y \not\leq_{\mathrm{T}} X^{(n)}$.

Proof. It is well known [11, §13.3] that there exist incomparable Turing degrees between 0 and $0'$. Relativizing to $0^{(n)}$, let X_n, Y_n be such that $0^{(n)} \leq_{\mathrm{T}} X_n \leq_{\mathrm{T}} 0^{(n+1)}$ and $0^{(n)} \leq_{\mathrm{T}} Y_n \leq_{\mathrm{T}} 0^{(n+1)}$ and $X_n \not\leq_{\mathrm{T}} Y_n$ and $Y_n \not\leq_{\mathrm{T}} X_n$. Note that X_n and Y_n are $\Delta^{0,0^{(n)}}_2$, hence X_n and Y_n are $\Pi^{0,0^{(n)}}_2$ singletons. Therefore, by the proof of Lemma 3.1 we may safely assume that X_n and Y_n are $\Pi^{0,0^{(n)}}_1$ singletons. Apply Lemma 3.3 to $P_n = \{X_n, Y_n\}$ to get $X_0 = H^n_0(X_n)$ and $Y_0 = H^n_0(Y_n)$. Note that $P_0 = \{X_0, Y_0\}$ is a Π^0_1 set, hence X_0 and Y_0 are Π^0_1 singletons. Since $X_n \not\leq_{\mathrm{T}} Y_n \oplus 0^{(n)} \equiv_{\mathrm{T}} Y^{(n)}_0$ and $X_n \oplus 0^{(n)} \equiv_{\mathrm{T}} X_0 \oplus 0^{(n)}$ we have $X_0 \not\leq_{\mathrm{T}} Y^{(n)}_0$, and similarly $Y_0 \not\leq_{\mathrm{T}} X^{(n)}_0$. Letting $X = X_0$ and $Y = Y_0$ we obtain our theorem. $\square$

Theorem 3.5. Given n we can find a countable Π^0_1 set P such that some $Z \in P$ is not a Π^0_n singleton.

Proof. Let P_n be a countable Π^0_1 set such that some $Z_n \in P_n$ is not isolated in P_n. (For instance, let $P_n = \{X \mid \forall i \forall j\, (X(i) \neq 0 \neq X(j) \Rightarrow i = j)\}$

and let $Z_n = 0$.) Treating P_n as a $\Pi^{0,0^{(n)}}_1$ set, apply Lemma 3.3. Then P_0 is a countable Π^0_1 set and, because $H^n_0 : P_n \cong P_0$ is a homeomorphism, $Z_0 = H^n_0(Z_n)$ is not isolated in P_0. We claim that Z_0 is not a Π^0_n singleton. Otherwise, let e be such that $\{Z_0\} = \{X \mid X^{(n)}(e) = 0\}$. Since $Z^{(n)}_0(e) = 0$ and $Z_0 \in P_0$ and $X^{(n)}_0 \equiv_{\mathrm{T}} X_n \oplus 0^{(n)}$ uniformly for all $X_n \in P_n$ and $X_0 = H^n_0(X_n) \in P_0$, there exists j such that $X^{(n)}_0(e) = 0$ for all $X_n \in P_n$ such that $X_n{\upharpoonright}j = Z_n{\upharpoonright}j$. But Z_n is not isolated in P_n, so there exists $X_n \in P_n$ such that $X_n{\upharpoonright}j = Z_n{\upharpoonright}j$ and $X_n \neq Z_n$. Thus $X^{(n)}_0(e) = 0$ and $X_0 \neq Z_0$, a contradiction. Letting $P = P_0$ and $Z = Z_0$ we obtain our theorem. □

4 Proof of Harrington's theorems

In order to prove the full version of Harrington's theorems, we need to show that Lemma 3.3 holds with n replaced by ω. To this end we first draw out some effective uniformities which are implicit in the proofs of Lemmas 3.1 and 3.2.

Notation. Let W^A_e for $e = 0, 1, 2, \ldots$ be a standard enumeration of all A-recursively enumerable subsets of $\mathbb{N}^*$. Then

$$T^A_e = \{\sigma \in \mathbb{N}^* \mid (\forall n \leq |\sigma|)\,(\sigma{\upharpoonright}n \notin W^A_e)\}$$

for $e = 0, 1, 2, \ldots$ is a standard enumeration of all $\Pi^{0,A}_1$ trees. Hence $P^A_e = [T^A_e]$ for $e = 0, 1, 2, \ldots$ is a standard enumeration of all $\Pi^{0,A}_1$ sets.

Remark 11. If F is an A-recursive treemap and T is a $\Pi^{0,A}_1$ tree, then $F(T)$ is again a $\Pi^{0,A}_1$ tree. Moreover, this holds uniformly in the sense that there is a primitive recursive function f such that $T^A_{f(e)} = F(T^A_e)$ and $P^A_{f(e)} = F(P^A_e)$ for all e, and we can compute a primitive recursive index of f knowing only an A-recursive index of F.

The next two lemmas are refinements of Lemmas 3.1 and 3.2 respectively.

Lemma 4.1 (refining Lemma 3.1)**.** There is a primitive recursive function f with the following property. Given e we can effectively find an A-recursive treemap $F : T^{A'}_e \to T^A_{f(e)}$ which induces a homeomorphism $F : P^{A'}_e \cong P^A_{f(e)}$. It follows that $X \oplus A \equiv_{\mathrm{T}} F(X) \oplus A$ uniformly for all $X \in P^{A'}_e$.

Proof. Let $T = T^{A'}_e$ and $P = P^{A'}_e$. Since $T^{A'}_e$ is uniformly $\Pi^{0,A'}_1$, it is uniformly $\Pi^{0,A}_2$, say $T = T^{A'}_e = \{\sigma \mid \forall i\, \exists j\, R(\sigma, e, i, A{\upharpoonright}j)\}$ where $R \subseteq \mathbb{N}^* \times \mathbb{N} \times \mathbb{N} \times \mathbb{N}^*$ is a fixed primitive recursive predicate. Let $(-,-)$ be a fixed primitive recursive one-to-one mapping of $\mathbb{N} \times \mathbb{N}$ onto $\mathbb{N}$ such that $m \le (m,n)$ and $n \le (m,n)$ for all m and n. Define $Q = [\widehat{T}]$ where $\widehat{T} = \{\sigma \oplus \tau \mid |\sigma| = |\tau|$ and $(\forall (n,i) < |\tau|)\,(\tau((n,i)) =$ the least j such that $R(\sigma{\restriction}n, e, i, A{\restriction}j))\}$. Thus $Q = \{X \oplus \widehat{X} \mid X \in P\}$ where $\widehat{X}((n,i)) =$ the least j such that $R(X{\restriction}n, e, i, A{\restriction}j)$. Moreover, we have an A-recursive treemap $F : T \to \widehat{T}$ given by $F(\sigma) = \sigma \oplus \widehat{\sigma}$ for all $\sigma \in T$, where $|\sigma| = |\widehat{\sigma}|$ and $(\forall (n,i) < |\sigma|)\,(\widehat{\sigma}((n,i)) =$ the least j such that $R(\sigma{\restriction}n, e, i, A{\restriction}j))$. Although we cannot expect to have $F(T) = \widehat{T}$, we nevertheless have $F : [T] \cong [\widehat{T}]$, i.e., $F : P \cong F(P) = Q$, and $F(X) = X \oplus \widehat{X}$ and $X \oplus A \equiv_{\mathrm{T}} F(X) \oplus A$ uniformly for all $X \in P$. The definition of $\widehat{T}$ shows that $\widehat{T}$ is uniformly A-recursive, hence uniformly $\Pi^{0,A}_1$, so we can find a fixed primitive recursive function f such that $T^A_{f(e)} = \widehat{T^{A'}_e}$ for all e and A. □

Lemma 4.2 (refining Lemma 3.2)**.** There is a primitive recursive function h with the following property. Given e we can effectively find an A'-recursive treemap $H : T^{A'}_e \to T^A_{h(e)}$ which induces a homeomorphism $H : P^{A'}_e \cong P^A_{h(e)}$ such that $X \oplus A' \equiv_{\mathrm{T}} H(X) \oplus A' \equiv_{\mathrm{T}} (H(X) \oplus A)'$ uniformly for all $X \in P^{A'}_e$.

Proof. Let G be the specific A'-recursive treemap which was constructed in the proof of Lemma 3.2. By Remark 11 we can find a primitive recursive function g such that for all e we have $G(T^{A'}_e) = T^{A'}_{g(e)}$ and the restriction of G to $T^{A'}_e$ is a treemap from $T^{A'}_e$ to $T^{A'}_{g(e)}$ which induces a homeomorphism $G : P^{A'}_e \cong P^{A'}_{g(e)}$. By construction of G we have $X \oplus A' \equiv_{\mathrm{T}} G(X) \oplus A' \equiv_{\mathrm{T}} (G(X) \oplus A)'$ uniformly for all $X \in P^{A'}_e$. Now applying Lemma 4.1 we obtain an A-recursive treemap $F : T^{A'}_{g(e)} \to T^A_{f(g(e))}$ which induces a homeomorphism $F : P^{A'}_{g(e)} \cong P^A_{f(g(e))}$ such that $Y \oplus A \equiv_{\mathrm{T}} F(Y) \oplus A$ uniformly for all $Y \in P^A_{g(e)}$. Thus the treemap $H = F \circ G : T^{A'}_e \to T^A_{f(g(e))}$ induces a homeomorphism $F \circ G = H : P^{A'}_e \cong P^A_{f(g(e))}$ such that $X \oplus A' \equiv_{\mathrm{T}} H(X) \oplus A' \equiv_{\mathrm{T}} (H(X) \oplus A)'$ uniformly for all $X \in P^{A'}_e$. Our lemma follows upon defining $h(e) = f(g(e))$. □

We now show that Lemma 3.3 holds with n replaced by ω.

Lemma 4.3. Given a $\Pi^{0,0^{(\omega)}}_1$ set P_ω we can effectively find a Π^0_1 set P_0 and a homeomorphism $H^\omega_0 : P_\omega \cong P_0$ such that $X_\omega \oplus 0^{(\omega)} \equiv_{\mathrm{T}} X_0 \oplus 0^{(\omega)} \equiv_{\mathrm{T}} X^{(\omega)}_0$ uniformly for all $X_\omega \in P_\omega$ and $X_0 = H^\omega_0(X_\omega) \in P_0$.

Proof. Since P_ω is a $\Pi^{0,0^{(\omega)}}_1$ set, Remark 8 gives a recursive tree T such that $P_\omega = \{X \mid X \oplus 0^{(\omega)} \in [T]\}$. Moreover, from the definition of $0^{(\omega)}$ we know that $0^{(\omega)}{\restriction}n$ is computable from $0^{(n)}$ uniformly for all n. Thus, letting

$T_\omega = \{\sigma \mid \sigma \oplus 0^{(\omega)}{\upharpoonright}|\sigma| \in T\}$, we see that $P_\omega = [T_\omega]$ and $\{\sigma \mid |\sigma| \leq n,\, \sigma \in T_\omega\} \leq_{\mathrm{T}} 0^{(n)}$ uniformly for all n. Define

$$T_{e,n} = \{\sigma \mid |\sigma| \leq n\} \cup \{\sigma \mid |\sigma| > n,\, \sigma{\upharpoonright}n \in T_\omega,\, \sigma \in T_e^{\langle n\rangle^\frown 0^{(n)}}\}.$$

Thus $T_{e,n}$ is a $\Pi_1^{0,0^{(n)}}$ tree, hence $P_{e,n} = [T_{e,n}]$ is $\Pi_1^{0,0^{(n)}}$ uniformly for all n.

In the vein of Lemma 4.2, we claim there is a primitive recursive function h^* with the following property. Given e and n we can effectively find a $0^{(n+1)}$-recursive treemap

$$H_{e,n} : T_{e,n+1} \to T_{h^*(e),n}$$

which induces a homeomorphism $H_{e,n} : P_{e,n+1} \cong P_{h^*(e),n}$ such that $X \oplus 0^{(n+1)} \equiv_{\mathrm{T}} H_{e,n}(X) \oplus 0^{(n+1)} \equiv_{\mathrm{T}} (H_{e,n}(X) \oplus 0^{(n)})'$ uniformly for all $X \in P_{e,n+1}$, and in addition $H_{e,n}(\sigma) = \sigma$ for all σ such that $|\sigma| \leq n$.

To prove our claim, let r be a 3-place primitive recursive function such that $T^{0^{(n)}}_{r(e,n,\sigma)} = \{\tau \mid \sigma^\frown\tau \in T_{e,n}\}$ for all e, n, σ. We can then write

$$T_{e,n+1} = \{\sigma \mid |\sigma| \leq n\} \cup \{\sigma^\frown\tau \mid |\sigma| = n,\, \tau \in T^{0^{(n+1)}}_{r(e,n+1,\sigma)}\}.$$

Since n is uniformly computable from $\langle n\rangle^\frown 0^{(n)}$, let h^* be a primitive recursive function such that

$$T_{h^*(e),n} = \{\sigma \mid |\sigma| \leq n\} \cup \{\sigma^\frown\tau \mid |\sigma| = n,\, \tau \in T^{0^{(n)}}_{h(r(e,n+1,\sigma))}\}$$

where h is as in Lemma 4.2. For all σ and τ such that $|\sigma| = n$ and $\tau \in T^{0^{(n+1)}}_{r(e,n+1,\sigma)}$ let $H_{e,n}(\sigma^\frown\tau) = \sigma^\frown H(\tau)$ where $H : T^{0^{(n+1)}}_{r(e,n+1,\sigma)} \to T^{0^{(n)}}_{h(r(e,n+1,\sigma))}$ is as in Lemma 4.2. Clearly $h^*(e)$ and $H_{e,n}$ have the required properties, so our claim is proved.

Let h^* and $H_{e,n}$ be as in the above claim. By the Recursion Theorem (see [11, Chapter 11]) let e^* be a *fixed point* of h^*, so that $T^A_{h^*(e^*)} = T^A_{e^*}$ for all A, hence $T_{h^*(e^*),n} = T_{e^*,n}$ for all n. Let $H_n = H_{e^*,n}$ and $T_n = T_{e^*,n}$ and $P_n = P_{e^*,n} = [T_n]$ for all n. As in the proof of Lemma 3.3 we have uniformly for each $s > n$ a $0^{(s)}$-recursive treemap $H_n^s = H_n \circ \cdots \circ H_{s-1} : T_s \to T_n$ which induces a homeomorphism $H_n^s : P_s \cong P_n$ such that $X \oplus 0^{(s)} \equiv_{\mathrm{T}} H_n^s(X) \oplus 0^{(s)} \equiv_{\mathrm{T}} (H_n^s(X))^{(s-n)}$ uniformly for all $X \in P_s$, and in addition $H_n^s(\sigma) = \sigma$ for all σ such that $|\sigma| \leq n$. We also have for each n a $0^{(\omega)}$-recursive treemap $H_n^\omega : T_\omega \to T_n$ which induces a homeomorphism $H_n^\omega : P_\omega \cong P_n$, namely $H_n^\omega(\sigma) = H_n^{|\sigma|}(\sigma)$ if $|\sigma| > n$ and $H_n^\omega(\sigma) = \sigma$ if $|\sigma| \leq n$. Note also that for all $n < s < t < \omega$ we have $H_n^t = H_n^s \circ H_s^t$ and $H_n^\omega = H_n^s \circ H_s^\omega$. Finally, given $X_\omega \in P_\omega$ let $X_n = H_n^\omega(X_\omega)$ for all n. Then $X_\omega{\upharpoonright}n = X_n{\upharpoonright}n$ and $X_n \oplus 0^{(n)} \equiv_{\mathrm{T}} X_0 \oplus 0^{(n)} \equiv_{\mathrm{T}} X_0^{(n)}$ uniformly for all n and all $X_\omega \in P_\omega$, hence

$X_\omega \oplus 0^{(\omega)} \equiv_{\mathrm{T}} X_0 \oplus 0^{(\omega)} \equiv_{\mathrm{T}} X_0^{(\omega)}$ uniformly for all $X_\omega \in P_\omega$. This completes the proof. ☐

We now present Harrington's construction of arithmetically incomparable arithmetical singletons.

Theorem 4.4. There is a pair of arithmetically incomparable Π^0_1 singletons.

Proof. As in the proof of Theorem 3.4, let X_ω, Y_ω be such that $0^{(\omega)} \leq_{\mathrm{T}} X_\omega \leq_{\mathrm{T}} 0^{(\omega+1)}$ and $0^{(\omega)} \leq_{\mathrm{T}} Y_\omega \leq_{\mathrm{T}} 0^{(\omega+1)}$ and $X_\omega \not\leq_{\mathrm{T}} Y_\omega$ and $Y_\omega \not\leq_{\mathrm{T}} X_\omega$. Note that X_ω and Y_ω are $\Delta_2^{0,0^{(\omega)}}$ and hence $\Pi_2^{0,0^{(\omega)}}$ singletons. Therefore, by the proof of Lemma 3.1 we may safely assume that X_ω and Y_ω are $\Pi_1^{0,0^{(\omega)}}$ singletons. Apply Lemma 4.3 to $P_\omega = \{X_\omega, Y_\omega\}$ to get a Π^0_1 set P_0 and a homeomorphism $H_0^\omega : P_\omega \cong P_0$. Let $X_0 = H_0^\omega(X_\omega)$ and $Y_0 = H_0^\omega(Y_\omega)$. Since $P_0 = \{X_0, Y_0\}$ it follows that X_0 and Y_0 are Π^0_1 singletons. Since $X_\omega \not\leq_{\mathrm{T}} Y_\omega \oplus 0^{(\omega)} \equiv_{\mathrm{T}} Y_0^{(\omega)}$ and $X_\omega \oplus 0^{(\omega)} \equiv_{\mathrm{T}} X_0 \oplus 0^{(\omega)}$ we have $X_0 \not\leq_{\mathrm{T}} Y_0^{(\omega)}$, and similarly $Y_0 \not\leq_{\mathrm{T}} X_0^{(\omega)}$. In particular X_0 and Y_0 are arithmetically incomparable, Q.E.D. ☐

Finally we present Harrington's construction of a ranked point which is not an arithmetical singleton. This refutes a conjecture which had been known as McLaughlin's Conjecture and which was suggested by the result of H. Tanaka [15] mentioned in Remark 2 above.

Theorem 4.5. There is a countable Π^0_1 set P such that some $Z \in P$ is not an arithmetical singleton.

Proof. As in the proof of Theorem 3.5, let P_ω be a countable Π^0_1 set such that some $Z_\omega \in P_\omega$ is not isolated in P_ω. Apply Lemma 4.3 and note that P_0 is a countable Π^0_1 set and $Z_0 = H_0^\omega(Z_\omega) \in P_0$ is not isolated in P_0. We claim that Z_0 is not an arithmetical singleton. Otherwise, let i be such that $\{Z_0\} = \{X \mid X^{(\omega)}(i) = 0\}$. Since $Z_0^{(\omega)}(i) = 0$ and $Z_0 \in P_0$ and $X_0^{(\omega)} \equiv_{\mathrm{T}} X_\omega \oplus 0^{(\omega)}$ uniformly for all $X_\omega \in P_\omega$ and $X_0 = H_0^\omega(X_\omega) \in P_0$, there exists j such that $X_0^{(\omega)}(i) = 0$ for all $X_\omega \in P_\omega$ such that $Z_\omega{\upharpoonright}j \subset X_\omega$. But Z_ω is not isolated in P_ω, so there exists $X_\omega \in P_\omega$ such that $Z_\omega{\upharpoonright}j \subset X_\omega$ and $X_\omega \neq Z_\omega$. Thus $X_0^{(\omega)}(i) = 0$ and $X_0 \neq Z_0$, a contradiction. Letting $P = P_0$ and $Z = Z_0$ we obtain our theorem. ☐

Remark 12. Modifying the proof of Lemma 4.3, it is easy to replace ω by a small recursive ordinal such as $\omega + \omega$ or $\omega \cdot \omega$ or ω^ω. Harrington [7] and Gerdes [5] have shown that Lemma 4.3 and consequently Theorems 4.4 and 4.5 hold generally with ω replaced by any recursive ordinal.

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