Turing degrees of hyperjumps

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Abstract

The Posner-Robinson Theorem states that for any reals Z and A such that $Z \oplus 0' \leq_{\mathrm{T}} A$ and $0 <_{\mathrm{T}} Z$, there exists B such that $A \equiv_{\mathrm{T}} B' \equiv_{\mathrm{T}} B \oplus Z \equiv_{\mathrm{T}} B \oplus 0'$. Consequently, any nonzero Turing degree $\deg_{\mathrm{T}}(Z)$ is a Turing jump relative to some B. Here we prove the hyperarithmetical analog, based on an unpublished proof of Slaman, namely that for any reals Z and A such that $Z \oplus \mathcal{O} \leq_{\mathrm{T}} A$ and $0 <_{\mathrm{HYP}} Z$, there exists B such that $A \equiv_{\mathrm{T}} \mathcal{O}^B \equiv_{\mathrm{T}} B \oplus Z \equiv_{\mathrm{T}} B \oplus \mathcal{O}$. As an analogous consequence, any nonhyperarithmetical Turing degree $\deg_{\mathrm{T}}(Z)$ is a hyperjump relative to some B.

Contents

1 Introduction

Our starting point is the *Friedberg Jump Theorem*:

Theorem 1.1 (Friedberg Jump Theorem)**.** *[10, Theorem 13.3.IX, pg. 265] Suppose A is a real such that $0' \leq_{\mathrm{T}} A$. Then there exists B such that*

$$A \equiv_{\mathrm{T}} B' \equiv_{\mathrm{T}} B \oplus 0'.$$

There are several refinements of the Friedberg Jump Theorem. One such extension shows that B can be taken to be an element of any special Π^0_1 class $P \subseteq \{0,1\}^{\mathbb{N}}$. Here *special* means that P is nonempty and has no recursvie elements.

Theorem 1.2. *[6, following Theorem 3.1, pg. 37] Suppose $P \subseteq \{0,1\}^{\mathbb{N}}$ is a special Π^0_1 class and A is a real such that $0' \leq_{\mathrm{T}} A$. Then there exists $B \in P$ such that*

$$A \equiv_{\mathrm{T}} B' \equiv_{\mathrm{T}} B \oplus 0'.$$

Another refinement is the *Posner-Robinson Theorem*:

Theorem 1.3 (Posner-Robinson Theorem)**.** *[8, Theorem 1, pg. 715] [5, Theorem 3.1, pg. 1228] Suppose Z and A are reals such that $Z \oplus 0' \leq_{\mathrm{T}} A$ and $0 <_{\mathrm{T}} Z$. Then there exists B such that*

$$A \equiv_{\mathrm{T}} B' \equiv_{\mathrm{T}} B \oplus Z \equiv_{\mathrm{T}} B \oplus 0'.$$

In this paper we prove hyperarithmetical analogs of Theorem 1.2 and Theorem 1.3. The hyperarithmetical analog of Theorem 1.1 is due to Macintyre [7, Theorem 3, pg. 9]. In these hyperarithmetical analogs, the Turing jump operator $X \mapsto X'$ is replaced by the hyperjump operator $X \mapsto \mathcal{O}^X$ and Π^0_1 classes are replaced by Σ^1_1 classes. A feature of [7, Theorem 3, pg. 9] and of our results is that they involve Turing degrees rather than hyperdegrees, so for instance $\mathcal{O}^B$ is not only hyperarithmetically equivalent to A, but in fact Turing equivalent to A.

Here is an outline of this paper:

In §2 we prove the following basis theorem for uncountable Σ^1_1 classes $K \subseteq \{0,1\}^{\mathbb{N}}$.

Theorem 2.1. *Suppose $K \subseteq \{0,1\}^{\mathbb{N}}$ is an uncountable Σ^1_1 class and Z and A are reals such that $Z \oplus \mathcal{O} \leq_{\mathrm{T}} A$ and $0 <_{\mathrm{HYP}} Z$. Then there exists $B \in K$ such that*

$$A \equiv_{\mathrm{T}} \mathcal{O}^B \equiv_{\mathrm{T}} B \oplus \mathcal{O}$$

and $Z \nleq_{\mathrm{HYP}} B$.

In §3 we prove the following analog of Theorem 1.3, which is essentially due to Slaman [13].

Theorem 3.1. *Suppose Z and A are reals such that $Z \oplus \mathcal{O} \leq_{\mathrm{T}} A$ and $0 <_{\mathrm{HYP}} Z$. Then there exists B such that*

$$A \equiv_{\mathrm{T}} \mathcal{O}^B \equiv_{\mathrm{T}} B \oplus Z \equiv_{\mathrm{T}} B \oplus \mathcal{O}.$$

The remainder of this section fixes notation and terminology.

$g{:}\subseteq A \to B$ denotes a partial function with domain $\operatorname{dom} g \subseteq A$ and codomain B. For $a \in A$, if $a \in \operatorname{dom} g$ then we say '$g(a)$ converges' or '$g(a)$ is defined' and write $g(a)\downarrow$. Otherwise, we say '$g(a)$ diverges' or '$g(a)$ is undefined' and write $g(a)\uparrow$. If f and g are two partial functions

$\subseteq A \to B$ and $a \in A$, then $f(a) \simeq g(a)$ means $(f(a)\downarrow \wedge g(a)\downarrow \wedge f(a) = g(a)) \vee (f(a)\uparrow \wedge g(a)\uparrow)$. We write $f(a)\downarrow = b$ to mean that $f(a)\downarrow$ and $f\colon a \mapsto b$.

$\mathbb{N}^\mathbb{N}$ and $\{0,1\}^\mathbb{N}$ denote the Baire and Cantor spaces, respectively, whose elements we sometimes call *reals*. We identify $\{0,1\}^\mathbb{N}$ and the powerset $\mathcal{P}(\mathbb{N})$ in the usual manner.

If S is a set, then S^* is the set of strings of elements from S. If $s_0, \ldots, s_{n-1} \in S$, then $\sigma = \langle s_0, \ldots, s_{n-1}\rangle \in S^*$ denotes the string of *length* $|\sigma| := n$ defined by $\sigma(k) = s_k$. If $\langle s_0, \ldots, s_{n-1}\rangle, \langle t_0, \ldots, t_{m-1}\rangle \in S^*$, then their *concatenation* is $\langle s_0, \ldots, s_{n-1}\rangle^\frown\langle t_0, \ldots, t_{m-1}\rangle := \langle s_0, \ldots, s_{n-1}, t_0, \ldots, t_{m-1}\rangle$. If $\sigma, \tau \in S^*$, then σ is an *initial segment* of τ (equivalently, τ is an extension of σ) written $\sigma \subseteq \tau$, if $\tau \restriction |\sigma| = \sigma$. If $f\colon \mathbb{N} \to S$ then $\sigma \in S^*$ is an *initial segment* of f (equivalently, f is an *extension* of σ), written $\sigma \subset f$, if $f \restriction |\sigma| = \sigma$. $\sigma, \tau \in S^*$ are *incompatible* if neither is an initial segment of the other. If $\leq$ is a partial order on S, then the *lexicographical ordering* $\leq_{\text{lex}}$ on S^* is defined by setting $\sigma \leq_{\text{lex}} \tau$ if $\sigma \subseteq \tau$ or, where k is the least index at which $\sigma(k) \neq \tau(k)$, then $\sigma(k) < \tau(k)$.

$\varphi^{(k)}_e$ denotes the e-th partial recursive function $\subseteq\mathbb{N}^k \to \mathbb{N}$; e is called an *index* of $\varphi^{(k)}_e$. Likewise, if $f \in \mathbb{N}^\mathbb{N}$ then $\varphi^{(k),f}_e$ denotes the e-th partial function $\varphi^{(k),f}_e\colon\subseteq\mathbb{N}^k \to \mathbb{N}$ which is partial recursive in f; e is again called an *index* of $\varphi^{(k),f}_e$, while f is called an *oracle* of $\varphi^{(k),f}_e$.

$\leq_\text{T}$ denotes Turing reducibility while $\equiv_\text{T}$ denotes Turing equivalence. $\leq_\text{HYP}$ denotes hyperarithmetical reducibility while $\equiv_\text{HYP}$ denots hyperarithmetical equivalence. For $X \in \{0,1\}^\mathbb{N}$, X' denotes the Turing jump of X and $\mathcal{O}^X$ denotes the hyperjump of X. $\mathcal{O}$ denotes Kleene's $\mathcal{O}$. For $f, g \in \mathbb{N}^\mathbb{N}$, their *join* $f \oplus g \in \mathbb{N}^\mathbb{N}$ is defined by $(f \oplus g)(2n) = f(n)$ and $(f \oplus g)(2n+1) = g(n)$.

P_e denotes the e-th Π^0_1 set $\{f \in \mathbb{N}^\mathbb{N} \mid \varphi^{(1),f}_e(0)\downarrow\} \subseteq \mathbb{N}^\mathbb{N}$. P^*_e denotes the e-th Σ^1_1 class $\{X \in \{0,1\}^\mathbb{N} \mid \exists f\ (f \oplus X \in P_e)\}$.

2 A Basis Theorem for Σ^1_1 Classes

The following theorem includes the Gandy Basis Theorem [11, Theorem III.1.4, pg. 54], the Kreisel Basis Theorem for Σ^1_1 Classes [11, Theorem III.7.2, pg. 75], and Macintyre's Hyperjump Inversion Theorem [7, Theorem 3, pg. 9].

Theorem 2.1. *Suppose $K \subseteq \{0,1\}^\mathbb{N}$ is an uncountable Σ^1_1 class and Z and A are reals such that $Z \oplus \mathcal{O} \leq_\text{T} A$ and $0 <_\text{HYP} Z$. Then there exists $B \in K$ such that*

$$A \equiv_\text{T} \mathcal{O}^B \equiv_\text{T} B \oplus \mathcal{O}$$

and $Z \nleq_\text{HYP} B$.

To prove Theorem 2.1 we use Gandy-Harrington forcing (first introduced by Harrington in an unpublished manuscript [2]; see, e.g., [11, Theorem IV.6.3, pg. 108]), forming a descending sequence of uncountable Σ^1_1 classes

$$K = K_0 \supseteq K_1 \supseteq \cdots \supseteq K_n \supseteq \cdots$$

where an element of the intersection $\bigcap^\infty_{n=0} K_n$ has the desired property. Unlike in the case of Π^0_1 subsets of $\{0,1\}^\mathbb{N}$, compactness cannot be used to easily show that the intersection $\bigcap^\infty_{n=0} K_n$ is nonempty. Instead, some care must be taken to show that this is the case.

Proposition 2.2.

(a) *Given a* Σ^1_1 *predicate* $K \subseteq \{0,1\}^{\mathbb{N}} \times \mathbb{N}^k$, *there is a primitive recursive function* $f\colon \mathbb{N}^k \to \mathbb{N}$ *such that*

$$P^*_{f(x_1,\ldots,x_k)}(X) \equiv K(X, x_1, \ldots, x_k).$$

(b) *Suppose* $X \in \{0,1\}^{\mathbb{N}}$. *Then* $\{e \in \mathbb{N} \mid X \notin P^*_e\} \equiv_{\mathrm{T}} \mathcal{O}^X$.

(c) $\{e \in \mathbb{N} \mid P^*_e = \varnothing\} \equiv_{\mathrm{T}} \mathcal{O}$.

Proof. Straight-forward. □

Corollary 2.3. *There exist primitive recursive functions* v, u, *and* U *such that that for all* $n, m \in \mathbb{N}$ *and* $\sigma, \tau \in N^*$ *and* $I \in \mathcal{P}_{\mathrm{fin}}(\mathbb{N})$,

$$\begin{aligned} P^*_{v(n,m)} &= P^*_n \cap P^*_m, \\ P^*_{u(e,\sigma,\tau)} &= P^*_e[\sigma,\tau] = \{X \in \{0,1\}^{\mathbb{N}} \mid \sigma \subset X \wedge \exists g\ (X \oplus g \in P_e \wedge \tau \subset g)\}, \\ P^*_{U(I,\sigma,\langle \tau_0,\ldots,\tau_{n-1}\rangle)} &= \bigcap_{k \in I \wedge k < n} P^*_k[\sigma, \tau_k]. \end{aligned}$$

Proposition 2.4. *The following partial functions are* $\mathcal{O}$*-recursive:*

(a) *The partial function* $\rho(\sigma, e) \simeq \langle \sigma_0, \sigma_1 \rangle$ *where* σ_0, σ_1 *are minimal incompatible extensions of* σ *which have extensions in* P^*_e *and* σ_0 *is lexicographically less than* σ_1, *whenever* σ *has at least two extensions in* P^*_e, *otherwise diverging.*

(b) *The partial function* $\mathrm{ext}(\langle e_1, \ldots, e_N \rangle, \sigma, \langle \tau_1, \ldots, \tau_N \rangle) \simeq (\tilde{\sigma}, \langle \tilde{\tau_1}, \ldots, \tilde{\tau_N} \rangle)$ *where* $(\tilde{\sigma}, \langle \tilde{\tau}, \ldots, \tilde{\tau_N} \rangle)$ *is the lexicographically least pair such that*

1. $\sigma \subsetneq \tilde{\sigma}$ *and* $\tau_k \subsetneq \tilde{\tau_k}$ *for* $1 \le k \le N$ *and*
2. $\bigcap_{k=1}^N P^*_{e_k}[\tilde{\sigma}, \tilde{\tau_k}] \ne \varnothing$

whenever $\bigcap_{k=1}^N P^*_{e_k}[\sigma, \tau_k] \ne \varnothing$, *otherwise diverging.*

Proof.

(a) Using $\mathcal{O}$, search for the first string ν such that $P^*_e[\sigma\frown\nu\frown\langle i \rangle, \langle\rangle] \ne \varnothing$ for $i = 0, 1$. Once such ν has been found, $\rho(\sigma, e)\!\downarrow = \langle \sigma\frown\nu\frown\langle 0 \rangle, \sigma\frown\nu\frown\langle 1 \rangle \rangle$.

(b) Using $\mathcal{O}$, search for the first of $i = 0, 1$ for which $\bigcap_{k=1}^N P^*_{e_k}[\sigma\frown\langle i \rangle, \tau_k] \ne \varnothing$, then search for the lexicographically least $\langle j_1, \ldots, j_N \rangle \in \{0,1\}^N$ such that $\bigcap_{k=1}^N P^*_{e_k}[\sigma\frown\langle i \rangle, \tau_k\frown\langle j_k \rangle] \ne \varnothing$. If no such i or $j_1, \ldots, j_N$ are found, then diverge. Otherwise, $\mathrm{ext}(\langle e_1, \ldots, e_N \rangle, \sigma, \langle \tau_1, \ldots, \tau_N \rangle)\!\downarrow = (\sigma\frown\langle i \rangle, \langle \tau_1\frown\langle j_1 \rangle, \ldots, \tau_N\frown\langle j_N \rangle \rangle)$.

□

Let ρ_0, ρ_1 be defined by

$$\rho(\sigma, e) \simeq \langle \rho_0(\sigma, e), \rho_1(\sigma, e) \rangle.$$

We use the ordinal notation description of $\mathcal{O}$ (and, more generally, $\mathcal{O}^Y$ for $Y \in \{0,1\}^{\mathbb{N}}$) described in [11] and use the following well-known lemma to describe hyperarithmetical reducibility in terms of H-sets.

Notation. For $X \in \{0,1\}^{\mathbb{N}}$ and $n \in \mathbb{N}$, define

$$(X)_n := \{x \in \mathbb{N} \mid 2^n \cdot 3^x \in X\}.$$

Lemma 2.5. *Suppose X and Y are reals in $\{0,1\}^{\mathbb{N}}$. Then $X \leq_{\mathrm{HYP}} Y$ if and only if there exists $b \in \mathcal{O}^Y$ and $n \in \mathbb{N}$ such that $X = (H^Y_b)_n$.*

Proof. Suppose $X \leq_{\mathrm{HYP}} Y$, so that there is $b \in \mathcal{O}^Y$ such that $X \leq_{\mathrm{T}} H^Y_b$. Let e be the index of such a Turing reduction, i.e., let e be such that $X = \varphi_e^{(1),H^Y_b}$. By definition [11], $2^b \in \mathcal{O}^Y$ and

$$H^Y_{2^b} \coloneqq \{2^n 3^x \mid \varphi_n^{(1),H^Y_b}(x)\downarrow\}.$$

Let f be an index such that

$$\varphi_f^{(1),H^Y_b}(x)\downarrow \iff \varphi_e^{(1),H^Y_b}(x)\downarrow = 1$$

Then

$$\begin{aligned}(H^Y_{2^b})_f &= \{x \in \mathbb{N} \mid \varphi_f^{(1),H^Y_b}(x)\downarrow\} \\ &= \{x \in \mathbb{N} \mid \varphi_e^{(1),H^Y_b}(x)\downarrow = 1\} \\ &= X\end{aligned}$$

Conversely, suppose there is $b \in \mathcal{O}^Y$ and $n \in \mathbb{N}$ such that $X = (H^Y_b)_n$. Let e be an index such that

$$\varphi_e^{(1),Z}(x) = \begin{cases} 1 & \text{if } x \in (Z)_n \\ 0 & \text{if } x \notin (Z)_n \end{cases}$$

for any $Z \in \{0,1\}^{\mathbb{N}}$. Then $\varphi_e^{(1),H^Y_b} = X$, showing that $X \leq_{\mathrm{T}} H^Y_b$. □

Proof of Theorem 2.1. By the Gandy Basis Theorem [11, Theorem III.1.4, pg. 54], assume without loss of generality that $\omega_1^Y = \omega_1^{\mathrm{CK}}$ for all $Y \in K$.

In order to control the hyperjump $\mathcal{O}^B$, we choose B to be an element of an intersection of Σ^1_1 subsets

$$K = K_0 \supseteq K_1 \supseteq \cdots \supseteq K_n \supseteq \cdots.$$

In order for B to be an element of $K_n = P^*_{j(n)}$ for each n, there must be $g_n \in \mathbb{N}^{\mathbb{N}}$ such that $B \oplus g_n \in P_{j(n)}$, where $j(n)$ is some index of K_n. Such g_n depend on B. Thus, we additionally define sequences of strings

$$\begin{array}{ccccccccc} \sigma_0 & \subseteq & \sigma_1 & \subseteq & \cdots & \subseteq & \sigma_n & \subseteq & \cdots \\ \tau_{0,0} & \subseteq & \tau_{1,0} & \subseteq & \cdots & \subseteq & \tau_{n,0} & \subseteq & \cdots \\ \tau_{0,1} & \subseteq & \tau_{1,1} & \subseteq & \cdots & \subseteq & \tau_{n,0} & \subseteq & \cdots \\ \tau_{0,2} & \subseteq & \tau_{1,2} & \subseteq & \cdots & \subseteq & \tau_{n,0} & \subseteq & \cdots \\ \vdots & & \vdots & & \ddots & & \vdots & & \ddots \end{array}$$

so that $B = \bigcup_{n\in\omega} \sigma_n$ and $g_k = \bigcup_{n\in\omega} \tau_{n,k}$. We also define a sequence of finite subsets of $\mathbb{N}$

$$I_0 \subseteq I_1 \subseteq \cdots \subseteq I_n \subseteq \cdots$$

encoded as finite sequences $\{e_1, \ldots, e_N\} \mapsto \langle e_1, \ldots, e_N\rangle$ which keep track of the indices e of Σ^1_1 classes we have committed to intersecting, so that $K_n = \bigcap_{k\in I_n} P^*_k[\sigma_n, \tau_{n,k}]$. A function $j\colon \mathbb{N} \to \mathbb{N}$ keeps track of the index of K_n, i.e.,

$$K_n = P^*_{j(n)}.$$

In the course of the proof, we assume that j encodes all of the information from previous steps (i.e., a course-of-value computation) though we avoid making this precise to ease the burden of notation.

To ease in the notation and exposition, we set the following temporary definitions. An *intersection system* consists of the following data:

(i) a finite subset $I \subseteq \mathbb{N}$,

(ii) a string σ, and

(iii) a sequence of strings $\langle \tau_k \mid k \in I \rangle$

subject to the constraint that $\bigcap_{k\in I} P^*_k[\sigma,\tau_k]$ is nonempty. If $k \notin I$, then we assign the value $\langle\rangle$ to τ_k.

By *adding* P^*_e *to the intersection system* $I,\sigma,\langle \tau_k \mid k \in I\rangle$, we mean the following procedure, where $K = \bigcap_{k\in I} P^*_k[\sigma,\tau_k]$:

Case 1: $K \cap P^*_e = \varnothing$. Let $\tilde{I} = I$, $\tilde{K} = K$, $\tilde{\sigma} = \sigma$, and $\tilde{\tau_k} = \tau_k$ for each k.

Case 2: $K \cap P^*_e \neq \varnothing$. Let $\tilde{I} = I \cup \{e\}$, and let $\tilde{\sigma}$ and, simultaneously for all $k \in \tilde{I}$, $\tilde{\tau_k}$ be the lexicographically least proper extensions of σ and τ_k, respectively, such that $\bigcap_{k\in\tilde{I}} P^*_k[\tilde{\sigma},\tilde{\tau_k}] \neq \varnothing$.

The resulting intersection system is $\tilde{I},\tilde{\sigma},\langle\tilde{\tau_k} \mid k \in \tilde{I}\rangle$. Note that from $I,\sigma,\langle\tau_k \mid k\in I\rangle$ and e, the new intersection system $\tilde{I},\tilde{\sigma},\langle\tilde{\tau_k} \mid k \in \tilde{I}\rangle$ can be determined in a uniform way recursively in $\mathcal{O}$: representing I as $\langle e_1,\dots,e_N\rangle$ and writing $e_{N+1} = e$, then

$$\tilde{I} = \begin{cases} \langle e_1,\dots,e_N,e_{N+1}\rangle & \text{if } K\cap P^*_e \neq \varnothing,\\ I & \text{otherwise,}\end{cases}$$

$$(\tilde{\sigma},\langle\tilde{\tau_k} \mid k\in\tilde{I}\rangle) = \begin{cases} \operatorname{ext}(\tilde{I},\sigma,\langle\tau_{e_1},\dots,\tau_{e_N},\langle\rangle\rangle) & \text{if } K\cap P^*_e \neq \varnothing,\\ (\sigma,\langle\tau_k \mid k\in I\rangle) & \text{otherwise.}\end{cases}$$

In particular, the index $U(\tilde{I},\tilde{\sigma},\langle\tilde{\tau_k} \mid k < \max I\rangle)$ of $\tilde{K}$ can be determined uniformly from the intersection system $I,\sigma,\langle\tau_k \mid k\in I\rangle$ using $\mathcal{O}$ as an oracle.

Now we proceed with the construction. As K is Σ^1_1, there is e_0 such that $K = P^*_{e_0}$.

Stage $n = 0$: Define

$$K_0 := K, \qquad \sigma_0 := \langle\rangle, \qquad \tau_{0,k} := \langle\rangle, \qquad j(0) := e_0, \qquad I_0 := \{e_0\}.$$

Note that $P^*_{j(0)} = K_0 = \bigcap_{k\in I_0} P^*_k[\sigma_0,\tau_{0,k}]$.

Stage $n = 3e+1$: Let $I_n,\sigma_n,\langle\tau_{n,k} \mid k\in I_n\rangle$ be the result of adding P^*_e to the intersection system $I_{n-1},\sigma_{n-1},\langle\tau_{n-1,k} \mid k\in I_{n-1}\rangle$, and let $K_n := \bigcap_{k\in I_n} P^*_k[\sigma_n,\tau_{n,k}]$ and $j(n)$ be an index for K_n.

Stage $n = 3e+2$: At this stage we encode $A(e)$ into B.

By construction,

$$P^*_{j(n-1)} = K_{n-1} = \bigcap_{k\in I_{n-1}} P^*_k[\sigma_{n-1},\tau_{n-1,k}] \neq \varnothing.$$

As K_{n-1} is uncountable, there are infinitely many pairwise-incompatible extensions of σ_{n-1} which extend to elements of K_{n-1}. Thus, let

$$\sigma_n := \rho_{A(e)}(\sigma_{n-1}, j(n-1)).$$

Define

$$\begin{aligned} K_n &:= \bigcap_{k \in I_{n-1}} P^*_k[\sigma_n, \tau_{n-1,k}] = P_{U(\sigma_n, I_{n-1}, \langle \tau_{n-1,0}, \ldots, \tau_{n-1,n-1}\rangle)}, \\ \tau_{n,k} &:= \tau_{n-1,k}, \qquad \text{(for all } k) \\ I_n &:= I_{n-1}, \\ j(n) &:= U(\sigma_n, I_{n-1}, \langle \tau_{n-1,0}, \ldots, \tau_{n-1,n-1}\rangle). \end{aligned}$$

Stage $n = 3^{b+1} \cdot 5^e \cdot 7^f$: Suppose $b \in \mathcal{O}$. Let $m \in \mathbb{N}$ be the least natural number for which there are $Y_1, Y_2 \in K_{n-1}$ such that $\varphi_f^{(1),H_b^{Y_1}}(2^e \cdot 3^m)$ and $\varphi_f^{(1),H_b^{Y_2}}(2^e \cdot 3^m)$ are both defined and unequal. For $i \in \{0,1\}$, let

$$K^i_{n-1} = \{Y \in K_{n-1} \mid \varphi_f^{(1),H_b^{Y_1}}(2^e \cdot 3^m)\downarrow = i\}.$$

Because $K^0_{n-1} \cap K^1_{n-1} = \varnothing$, there is a least $k \in \mathbb{N}$ and $i \in \{0,1\}$ such that $\{Y \in K^0_{n-1} \mid Y(k) = i\}$ and $\{Y \in K^1_{n-1} \mid Y(k) \neq i\}$ are nonempty. Let $i_0 = i$ and $i_1 = 1 - i$.

Let $I_n, \sigma_n, \langle \tau_{n,k} \mid k \in I_n\rangle$ be the result of adding the (uniformly in b, e, f, m, k,and i, given $Z(m)$) Σ^1_1 class $\{Y \in \{0,1\}^{\mathbb{N}} \mid \varphi_f^{(1),H_b^Y}(2^e \cdot 3^m)\downarrow \neq Z(m) \wedge Y(k) \neq i_{Z(m)}\}$ to the intersection system $I_{n-1}, \sigma_{n-1}, \langle \tau_{n-1,k} \mid k \in I_{n-1}\rangle$, and let $K_n := \bigcap_{k \in I_n} P^*_k[\sigma_n, \tau_{n,k}]$ and $j(n)$ be an index for K_n.

If $b \notin \mathcal{O}$ or no such m exists, do nothing, i.e., let

$$K_n := K_{n-1}, \qquad \sigma_n := \sigma_{n-1}, \qquad \tau_{n,k} := \tau_{n-1,k}, \qquad j(n) := j(n-1), \qquad I_n := I_{n-1}.$$

All Other Stages n: Do nothing, i.e., let

$$K_n := K_{n-1}, \qquad \sigma_n := \sigma_{n-1}, \qquad \tau_{n,k} := \tau_{n-1,k}, \qquad j(n) := j(n-1), \qquad I_n := I_{n-1}.$$

This completes the construction.

Define

$$B := \bigcup_{n \in \mathbb{N}} \sigma_n \quad \text{and} \quad g_k := \bigcup_{n \in \mathbb{N}} \tau_{n,k}.$$

We start by claiming $B \in \bigcap_{n \in \mathbb{N}} K_n$: by construction, for $k \in \bigcap_{n \in \mathbb{N}} I_n$, we have $B \oplus g_k \in P_k$, showing $B \in P^*_k$. Additionally, by construction $B \in P^*_k[\sigma_n, \tau_{n,k}]$ for every n and every $k \in \bigcap_{n \in \mathbb{N}} I_n$, so $B \in \bigcap_{k \in I_n} P^*_k[\sigma_n, \tau_{n,k}] = K_n$. Thus, $B \in \bigcap_{n \in \mathbb{N}} K_n$. In particular, $B \in K_0 = K$, so $\omega_1^B = \omega_1^{\mathrm{CK}}$.

If $Z \leq_{\mathrm{HYP}} B$, then Lemma 2.5 shows there are $c \in \mathcal{O}^B$ and $e \in \mathbb{N}$ such that $Z = (H^B_b)_e$. Because $\omega_1^B = \omega_1^{\mathrm{CK}}$, there exists $b \in \mathcal{O}$ such that $|b| = |c|$ and hence $H^B_b \equiv_{\mathrm{T}} H^B_c$ by Spector's Uniqueness Theorem [11, Corollary II.4.6, pg. 40]. Let f be an index such that $\varphi_f^{(1),H_b^B} = H^B_c$, so that $Z = (\varphi_f^{(1),H_b^B})_e$. By construction, at Stage $n = 3^{b+1} \cdot 5^e \cdot 7^f$ it must have been the case that no m and $Y_1, Y_2 \in K_{n-1}$ existed with $\varphi_f^{(1),H_b^{Y_1}}(2^e \cdot 3^m)$ and

$\varphi_f^{(1),H_b^{Y_2}}(2^e \cdot 3^m)$ both defined and unequal. In particular, $\varphi_f^{(1),H_b^B}$ is a Σ^1_1 singleton, and so hyperarithmetical. But then $H_c^B \equiv_{\mathrm{T}} H_b^B$ is hyperarithmetical, hence $Z = (H_c^B)_e$ is hyperarithmetical, a contradiction. Thus, $Z \nleq_{\mathrm{HYP}} B$.

We now make the following observations: assuming $j(n-1)$ is known (and utilizing the implicit course-of-values procedure to yield $I_{n-1}, \sigma_{n-1}, \langle \tau_{n-1,k} \rangle_{k \in \mathbb{N}}$), then...

...in Stage $n = 3e+1$, the determination of $I_n, \sigma_n, \langle \tau_{n,k} \rangle_{k\in\mathbb{N}}$ (and hence also $j(n)$) is recursive in $\mathcal{O}$ by Proposition 2.4.

...in Stage $n = 3e+2$, the determination of $I_n, \sigma_n, \langle \tau_{n,k} \rangle_{k\in\mathbb{N}}$ (and hence also $j(n)$) is recursive in A (by construction) or $B \oplus \mathcal{O}$ (by determining the unique i for which $\rho_i(\sigma_{n-1}, j(n-1)) \subset B$) by Proposition 2.4.

...in Stage $n = 3^{b+1} \cdot 5^e \cdot 7^f$, the determination of $I_n, \sigma_n, \langle \tau_{n,k} \rangle_{k\in\mathbb{N}}$ (and hence also $j(n)$) is recursive in $B \oplus \mathcal{O}$ (the determination of whether $b \in \mathcal{O}$ and whether there exists an m and $Y_1, Y_2 \in K_{n-1}$ for which $\varphi_f^{(1),H_b^{Y_1}}(2^e \cdot 3^m)$ and $\varphi_f^{(1),H_b^{Y_2}}(2^e \cdot 3^m)$ are both defined and unequal may be performed recursively in $\mathcal{O}$ since it corresponds to checking whether a particular Σ^1_1 class is nonempty, and once the least such m is found, we may determine the least k and $i \in \{0,1\}$ for which $\{Y \in K^0_{n-1} \mid Y(k) = i\}$ and $\{Y \in K^1_{n-1} \mid Y(k) = 1-i\}$ are nonempty; finally, checking whether $B(k) = i$ or $B(k) = 1-i$ determines whether we intersected $\{Y \in \{0,1\}^{\mathbb{N}} \mid \varphi_f^{(1),H_b^Y}(2^e \cdot 3^m) \downarrow= 0 \wedge Y(k) = i\}$ or $\{Y \in \{0,1\}^{\mathbb{N}} \mid \varphi_f^{(1),H_b^Y}(2^e \cdot 3^m) \downarrow= 1 \wedge Y(k) = 1-i\}$, respectively) or A (as before, the determination of whether $b \in \mathcal{O}$ and of the existence of such an m may be done recursively in $\mathcal{O} \leq_{\mathrm{T}} A$, and $Z \leq_{\mathrm{T}} A$).

...in all other Stages n, the determination of $I_n, \sigma_n, \langle \tau_{n,k} \rangle_{k\in\mathbb{N}}$ (and hence also $j(n)$) is recursive.

In particular, $j \leq_{\mathrm{T}} A$ and $j \leq_{\mathrm{T}} B \oplus \mathcal{O}$.

We make the following final observations:

- $A \leq_{\mathrm{T}} j \oplus \mathcal{O}$ as $A(e) = i$ if and only if $j(n) = U(\rho_i(\sigma_{n-1}, j(n-1)), I_{n-1}, \langle \tau_{n-1,0}, \dots, \tau_{n-1,n-1} \rangle)$, where $n = 3e+2$.
- $\mathcal{O}^B \leq_{\mathrm{T}} j \oplus \mathcal{O}$ as $B \in P_e^*$ if and only if $v(j(n-1), e) \notin \{i \mid P_i^* = \varnothing\} \equiv_{\mathrm{T}} \mathcal{O}$. The determination $v(j(n-1), e) \notin \{i \mid P_i^* = \varnothing\} \equiv_{\mathrm{T}} \mathcal{O}$ can be made recursively in $j \oplus \mathcal{O}$.

Thus, we find that

$$A \leq_{\mathrm{T}} j \oplus \mathcal{O} \leq_{\mathrm{T}} B \oplus \mathcal{O} \leq_{\mathrm{T}} \mathcal{O}^B \leq_{\mathrm{T}} j \oplus \mathcal{O} \leq_{\mathrm{T}} A$$

so we have Turing equivalence throughout. □

The following corollary is originally due to Macintyre [7, Theorem 3, pg. 9].

Corollary 2.6. *Suppose A is a real such that $\mathcal{O} \leq_{\mathrm{T}} A$. Then there exists B such that*

$$A \equiv_{\mathrm{T}} \mathcal{O}^B \equiv_{\mathrm{T}} B \oplus \mathcal{O}.$$

The following corollary is "folklore", being unpublished but known to researchers and stated in [1, Exercise 2.5.6, pg. 40] without proof or references. Other than [1, Exercise 2.5.6, pg. 40] we have not seen any statement of Corollary 2.7 in the literature.

Corollary 2.7. *Suppose K is a nonempty Σ^1_1 class. Then there exists $B \in K$ such that $\mathcal{O} \equiv_{\mathrm{T}} \mathcal{O}^B \equiv_{\mathrm{T}} B \oplus \mathcal{O}$.*

Proof. If K is uncountable, then we apply Theorem 2.1 with $Z = A = \mathcal{O}$.

If K is countable, then its elements are hyperarithmetical [11, Theorem III.6.2, pg. 72] and so any $B \in K$ satisfies $\mathcal{O} \equiv_{\mathrm{T}} \mathcal{O}^B \equiv_{\mathrm{T}} B \oplus \mathcal{O}$. □

We can generalize Theorem 2.1, replacing the real Z by a sequence of reals, as follows.

Theorem 2.8. *Suppose K is an uncountable Σ^1_1 class and Z and A are reals such that $Z \oplus \mathcal{O} \le_{\mathrm{T}} A$ and $0 <_{\mathrm{HYP}} (Z)_k$ for each $k \in \mathbb{N}$. Then there exists $B \in K$ such that*

$$A \equiv_{\mathrm{T}} \mathcal{O}^B \equiv_{\mathrm{T}} B \oplus \mathcal{O}$$

and $(Z)_k \not\le_{\mathrm{HYP}} B$ for all k.

Proof. The proof of Theorem 2.1 may be adapted by replacing Stage $n = 3^{b+1} \cdot 5^e \cdot 7^f$ with $n = 3^{b+1} \cdot 5^e \cdot 7^f \cdot 11^k$ and replacing therein Z with $(Z)_k$.

□

3 Posner-Robinson for Turing Degrees of Hyperjumps

Theorem 3.1 (Posner-Robinson for Turing Degrees of Hyperjumps)**.** *Suppose Z and A are reals such that $Z \oplus \mathcal{O} \le_{\mathrm{T}} A$ and $0 <_{\mathrm{HYP}} Z$. Then there exists B such that*

$$A \equiv_{\mathrm{T}} \mathcal{O}^B \equiv_{\mathrm{T}} B \oplus Z \equiv_{\mathrm{T}} B \oplus \mathcal{O}.$$

Theorem 3.1 is essentially due to Slaman [13]. The rest of this section is devoted to a proof of Theorem 3.1. The key to the proof is a forcing notion known as Kumabe-Slaman forcing, which was originally introduced in [12].

3.1 Kumabe-Slaman Forcing

In order to prove Theorem 3.1, we will use Turing functionals and an associated notion of forcing to construct the desired B.

Definition 3.2 (Turing Functionals)**.** [12, 9] A **Turing functional** Φ is a set of triples $(x, y, \sigma) \in \mathbb{N} \times \{0,1\} \times \{0,1\}^*$ (called **computations in** Φ) such that if $(x, y_1, \sigma_1), (x, y_2, \sigma_2) \in \Phi$ and σ_1 and σ_2 are compatible, then $y_1 = y_2$ and $\sigma_1 = \sigma_2$.

A Turing functional Φ is **use-monotone** if:

(i) For all (x_1, y_1, σ_1) and (x_2, y_2, σ_2) are elements of Φ and $\sigma_1 \subset \sigma_2$, then $x_1 < x_2$.

(ii) For all x_1 and $(x_2, y_2, \sigma_2) \in \Phi$ where $x_2 > x_1$, then there are y_1 and σ_1 such that $\sigma_1 \subseteq \sigma_2$ and $(x_1, y_1, \sigma_1) \in \Phi$.

Remark 3.3. Despite the terminology, a Turing functional Φ is not assumed to be recursive or even recursively enumerable.

Definition 3.4 (Computations along a Real)**.** [12, 9] Suppose Φ is a Turing functional and $X \in \{0,1\}^{\mathbb{N}}$. Then $(x, y, \sigma) \in \Phi$ is a **computation along** X if $\sigma \subset X$, in which case we write $\Phi(X)(x) = y$. If for every $x \in \mathbb{N}$ there exists $y \in \{0,1\}$ and $\sigma \subset X$ such that $(x, y, \sigma) \in \Phi$, then $\Phi(X)$ defines an element of $\{0,1\}^{\mathbb{N}}$ (otherwise it is a partial function).

Lemma 3.5. *Suppose Φ is a Turing functional, $X \in \{0,1\}^{\mathbb{N}}$, and $\Phi(X) \in \{0,1\}^{\mathbb{N}}$. Then*

$$\Phi(X) \leq_{\mathrm{T}} \Phi \oplus X.$$

Proof. Obvious from the definition of $\Phi(X)$. □

Definition 3.6 (Kumabe-Slaman Forcing)**.** [12, 9] Define the poset $(\mathbb{P}, \leq)$ as follows:

(i) Elements of $\mathbb{P}$ are pairs $(\Phi, \mathbf{X})$ where Φ is a finite use-monotone Turing functional and $\mathbf{X}$ is a finite subset of $\{0,1\}^{\mathbb{N}}$.

(ii) If $p = (\Phi_p, \mathbf{X}_p)$ and $q = (\Phi_q, \mathbf{X}_q)$ are in $\mathbb{P}$, then $p \leq q$ if

   (a) $\Phi_p \subseteq \Phi_q$ and for all $(x_q, y_q, \sigma_q) \in \Phi_q \setminus \Phi_p$ and all $(\mathbf{X}_p, y_p, \sigma_p) \in \Phi_p$, the length of σ_q is greater than the length of σ_p.

   (b) $\mathbf{X}_p \subseteq \mathbf{X}_q$.

   (c) For every x, y, and $X \in \mathbf{X}$, if $\Phi_q(X)(x) = y$, then $\Phi_p(X)(x) = y$.

   In other words, a stronger condition than p can add longer computations to Φ_p, provided they don't apply to any element of $\mathbf{X}_p$.

In the remainder of §3, we will be discussing Kumabe-Slaman forcing over countable ω-models of ZFC[1]. Unlike in the forcing constructions in axiomatic set theory, it will be important here that the countable ground model M is *not* well-founded. We now introduce some conventions for discussing such models.

Let M be a countable non-well-founded ω-model of ZFC. Let $\theta(x_1, \dots, x_n)$ be a sentence in the language of ZFC with parameters $x_1, \dots, x_n$ from M. We write $\theta^M(x_1, \dots, x_m)$ or $M \vDash \theta(x_1, \dots, x_n)$ to mean that $\theta(x_1, \dots, x_n)$ holds in M. In particular, $x_1 \in^M x_2$ means that $M \vDash x_1 \in x_2$, etc. We tacitly identity the natural number system of M with the standard natural number system, the reals of M with standard reals, etc. In particular, let $\mathbb{P}^M$ be the set of pairs (Φ, X) such that $M \vDash$ "(Φ, X) is a Kumabe-Slaman forcing condition". In this case, Φ is identified with a finite Turing functional, X is identified with a finite set of reals belonging to M, etc., so (Φ, X) actually *is* a Kumabe-Slaman forcing condition.

The key property of Kumabe-Slaman Forcing is the following:

Lemma 3.7. *[9, based on Lemma 3.10, pg. 23] Suppose M is an ω-model of* ZFC*, $D \in M$ is dense in $\mathbb{P}^M$, and $X_1, \dots, X_n \in \{0,1\}^{\mathbb{N}}$. Then for any $p \in \mathbb{P}^M$, there exists $q \geq p$ such that $q \in D$ and Φ_q does not add any new computations along any X_k.*

Proof. Suppose $p = (\Phi_p, \mathbf{X}_p) \in \mathbb{P}^M$. Say that an n-tuple of strings $\vec{\tau}$ is *essential* for (p, D) if $q > p$ and $q \in D$ implies the existence of $(x, y, \sigma) \in \Phi_q \setminus \Phi_p$ such that σ is compatible with some component of $\vec{\tau}$, i.e., any extension of p in D adds a computation along a string compatible with a component of $\vec{\tau}$. Being essential for (p, D) is definable in M.

Define

$$T_n(p, D) \coloneqq \{\vec{\tau} \in (\{0,1\}^*)^n \mid \vec{\tau} \text{ is essential for } (p, D) \text{ and } |\tau_1| = \cdots = |\tau_n|\}.$$

Being essential for (p, D) is closed under taking (component-wise) initial segments, so $T_n(p, D)$ is a finitely branching tree in M.

[1] Here ZFC denotes Zermelo-Fraenkel Set Theory with the Axiom of Choice. However, for the purposes of this paper, our ω-models need not satisfy ZFC but only a small subsystem of ZFC or actually of second-order arithmetic.

Suppose for the sake of a contradiction that for every $q > p$, either $q \notin D$ or else q adds a new computation along some X_k. We claim that $\langle X_1 \upharpoonright m, \dots, X_n \upharpoonright m\rangle$ is essential for (p, D) for all $m \in \mathbb{N}$. Given $q > p$ with $q \in D$, by hypothesis there is some computation $(x, y, \sigma) \in \Phi_q \setminus \Phi_p$ along some X_k. This means that $\sigma \subset X_k$ (outside of M), so σ is compatible with $X_k \upharpoonright m$.

This shows that $T_n(p, D)$ is infinite. As M is a model of ZFC, it follows that $T_n(p, D)$ has a path through it. The requirement that the components of any element of $T_n(p, D)$ are of the same length implies that such a path is of the form $(Y_1, \dots, Y_n)$ for $Y_1, \dots, Y_n \in M \cap \{0,1\}^{\mathbb{N}}$.

Consider $p_1 = (\Phi_p, \mathbf{X}_p \cup \{Y_1, \dots, Y_n\})$. Suppose $q \geq p_1$ and $q \in D$. Each n-tuple $\langle Y_1 \upharpoonright m, \dots, Y_n \upharpoonright m\rangle$ is essential for (p, D) for each m, so there exists $(x_m, y_m, \sigma_m) \in \Phi_q \setminus \Phi_p$ such that σ_m is compatible with $Y_k \upharpoonright m$ for some k. As Φ_q is finite, letting m be sufficiently large shows that there is $(x, y, \sigma) \in \Phi_q \setminus \Phi_p$ for which σ is an initial segment of Y_k for some k. However, this is not possible since $q \leq p_1$ implies $Y_k \in \mathbf{X}_q$. This provides the needed contradiction. □

Suppose G is an M-generic filter for $\mathbb{P}^M$. Then for every X

$$X \subseteq^M \mathbb{N} \iff \text{there is } p \in G \text{ with } X \in \mathbf{X}_p$$

since for any $X \subseteq^M \mathbb{N}$, the set $\{p \in \mathbb{P}^M \mid (\varnothing, \{X\}) \leq p\}^M$ is a dense open subset of $\mathbb{P}^M$ in M. Thus, the essential parts of an M-generic filter G are the Turing functionals Φ_p for $p \in G$.

Definition 3.8. A Turing functional Φ is **M-generic for** $\mathbb{P}^M$ if and only if there exists an M-generic filter G such that

$$(x, y, \sigma) \in \Phi \iff \text{there exists } p \in G \text{ such that } (x, y, \sigma) \in^M \Phi_p.$$

Φ may be identified with an element $(\dot{\Phi})_G$ in $M[G]$, where

$$M \vDash \dot{\Phi} = \{(p, \dot{c}) \mid p \in \mathbb{P}^M \wedge c \in \Phi_p\}$$

and $\dot{c}$ is a *canonical name* for $c \in M$, defined by transfinite recursion in M to be the unique element in M for which

$$M \vDash \dot{c} = \mathbb{P}^M \times \{\dot{b} \mid b \in c\}.$$

Lemma 3.9. *The following are equivalent for a Turing functional Φ:*

(i) Φ is an M-generic Turing functional for $\mathbb{P}^M$.

(ii) For every dense open subset $D \subseteq^M \mathbb{P}^M$, there exists $p \in^M D$ such that

$$(x, y, \sigma) \in^M \Phi_p \implies (x, y, \sigma) \in \Phi.$$

Proof.

$(i) \implies (ii)$ Let G be an M-generic filter for $\mathbb{P}^M$ such that

$$(x, y, \sigma) \in \Phi \iff \text{there exists } p \in G \text{ such that } (x, y, \sigma) \in^M \Phi_p.$$

Suppose $D \subseteq^M \mathbb{P}^M$ is dense open. By definition, there exists $p \in G$ such that $p \in^M D$. Then by definition,

$$(x, y, \sigma) \in^M \Phi_p \implies (x, y, \sigma) \in \Phi.$$

$(ii) \implies (i)$ For $p \in^M \mathbb{P}^M$, temporarily write $p < \Phi$ if $(x, y, \sigma) \in^M \Phi_p$ implies $(x, y, \sigma) \in \Phi$. Define

$$G := \{q \mid \exists p\ (p < \Phi \wedge M \models (p \leq q))\}.$$

We claim that G is an M-generic filter for $\mathbb{P}^M$.

Upwards closed: Suppose $q \in G$ and $M \models (q \leq q')$. Let $p < \Phi$ be such that $M \models p \leq q$. Then $M \models p \leq q'$ since $M \models (\leq \text{ is transitive})$, so $q' \in G$.

Downwards directed: Suppose $q, q' \in G$. Let $p, p' < \Phi$ be such that $M \models (p \leq q \wedge p' \leq q')$. Then the unique p'' for which $M \models p'' = (\Phi_p \cap \Phi_{p'}, \varnothing)$ satisfies $p'' < \Phi$ and $M \models (p'' \leq q \wedge p'' \leq q')$.

M-generic: Suppose $D \subseteq^M \mathbb{P}^M$ is dense open. By hypothesis, there exists $p < \Phi$ such that $p \in^M D$. By definition of G, $p \in G$.

□

By defining an M-generic Turing functional Φ for $\mathbb{P}^M$ by means of approximations, Lemmas 3.7 and 3.10 allow us to meet dense sets without affecting $\Phi(Z)$, which can then be arranged independently.

3.2 Proof of Posner-Robinson for Hyperjumps

Now we proceed with the proof of Theorem 3.1:

Lemma 3.10. *Suppose Z and A are reals such that $Z \oplus \mathcal{O} \leq_{\mathrm{T}} A$ and $0 <_{\mathrm{HYP}} Z$. Then there exists a (code for a) countable ω-model M of* ZFC *such that $\mathcal{O}^M \equiv_{\mathrm{T}} A$ and $Z \notin M$.*

Proof. The set of codes for countable ω-models of ZFC is Σ^1_1, so the existence of a code of such an M follows from Theorem 2.1. □

Proof of Theorem 3.1: The main idea of the proof is due to Slaman [13].

We shall construct an M-generic Turing functional Φ with $B = \Phi$ the desired real. Assume without loss of generality that no initial segment of Z is an initial segment of $\mathcal{O}$. By arranging for $\Phi(Z) \in \{0,1\}^{\mathbb{N}}$ and $\Phi(Z) = \mathcal{O}^\Phi$ and $\Phi(\mathcal{O}) = A$, this will complete the proof.

By Lemma 3.10, there exists a countable ω-model M of ZFC such that $\mathcal{O}, Z \notin M$ and $\mathcal{O}^M \equiv_{\mathrm{T}} A$. Without loss of generality, $M = \langle \omega, E \rangle$.

Let $D_0, D_1, D_2, \dots$ be an enumeration, recursive in A, of the dense open subsets of $\mathbb{P}^M$ in M (M is countable and $\mathcal{O}^M \equiv_{\mathrm{T}} A$, so this is possible). To construct our M-generic Φ, we approximate it by finite initial segments

$$p_0 \leq p_1 \leq \cdots \leq p_n \leq \cdots.$$

During our construction, we alternate between meeting dense sets, arranging for $\Phi(\mathcal{O}) = A$, and arranging for $\Phi(Z) \equiv_{\mathrm{T}} \mathcal{O}^\Phi$.

Stage $n = 0$: Define $p_0 := (\varnothing, \varnothing)$.

Stage $n = 2^m$: Suppose p_{n-1} has been constructed. By Lemma 3.7, there exists $q \in D_n$ extending p_{n-1} which does not add any new computations along Z or $\mathcal{O}$. Let p_n be the least such condition.

Stage $n = 2^m \cdot 3$**:** We extend p_{n-1} to p_n by adding $(m, A(m), \sigma)$ where $\sigma \subset \mathcal{O}$ is a sufficiently long initial segment of $\mathcal{O}$ (i.e., the shortest initial segment of $\mathcal{O}$ which is longer than any existing strings in elements of $\Phi_{p_{n-1}}$).

Stage $n = 2^m \cdot 5$**:** Suppose p_{n-1} has been constructed. By construction, there is no y and $\sigma \subset Z$ such that $(m, y, \sigma) \in \Phi_{p_{n-1}}$. Now proceed as follows:

Substage 1: Consider the set D (in M) containig all $q \in \mathbb{P}^M$ such that one of the following conditions hold:

(i) $q \Vdash (m$ encodes a Φ-recursive linear order on $\omega \wedge m \in \mathcal{O}^\Phi \wedge \exists \alpha\ (\alpha \in \mathrm{Ord}^M \wedge |m| = \alpha))$,

(ii) $q \Vdash (m$ encodes a Φ-recursive linear order on $\omega \wedge m \notin \mathcal{O}^\Phi)$, or

(iii) $q \Vdash \neg(m$ encodes a Φ-recursive linear order on $\omega)$.

D is dense. By Lemma 3.7, there exists $q \in D$ extending p_{n-1} which does not add any new computations along Z or $\mathcal{O}$. Let q be minimal with that property.

Substage 2: Extend q to p_n by adding (m, y, σ), where $\sigma \subset Z$ is a sufficiently long initial segment of Z (i.e., the shortest initial segment of Z which is longer than any existing strings in elements of Φ_q) and y depends on the following cases:

Case 1: If $q \Vdash (m$ encodes a Φ-recursive linear order on $\omega \wedge m \in \mathcal{O}^\Phi \wedge \exists \alpha\ (\alpha \in \mathrm{Ord}^M \wedge |m| = \alpha))$, then we break into two subcases:

Case 1a: If α is in the standard part of Ord^M, then α is actually an ordinal and m *does* encode a Φ-recursive linear order on ω. Thus, set $y := 1$.

Case 1b: If α is not in the standard part of Ord^M, then α is not actually well-ordered (it is only well-ordered when viewed in M) so m *does not* encode a Φ-recursive linear order on ω. Thus, set $y := 0$.

Case 2: If $q \Vdash (m$ encodes a Φ-recursive linear order on $\omega \wedge m \notin \mathcal{O}^\Phi)$, then m cannot encode a Φ-recursive well-ordering of ω. Thus, set $y := 0$.

Case 3: If $q \Vdash \neg(m$ encodes a Φ-recursive linear order on $\omega)$, then set $y := 0$.

All Other Stages n**:** Let $p_n = p_{n-1}$.

Define Φ to be the unique set such that

$$(x, y, \sigma) \in \Phi \iff \text{there exists } n \in \mathbb{N} \text{ such that } (x, y, \sigma) \in^M \Phi_{p_n}.$$

Thanks to Stages $n = 2^m$ and Lemma 3.9, Φ is an M-generic Turing functional. Thanks to Stages $n = 2^m \cdot 3$, $\Phi(\mathcal{O}) = A$. Thanks to Stages $n = 2^m \cdot 5$, $\Phi(Z) = \mathcal{O}^\Phi$.

We also note that in the above construction of Φ, (assuming p_{n-1} is given)...

...Stage $n = 2^m$ is recursive in $\mathcal{O}^M \oplus Z \oplus \mathcal{O} \leq_{\mathrm{T}} A$,

...Stage $n = 2^m \cdot 3$ is recursive in $\mathcal{O} \leq_{\mathrm{T}} A$,

...Stage $n = 2^m \cdot 5$ (Substage 1) is recursive in $\mathcal{O} \oplus \mathcal{O}^M \oplus Z \leq_{\mathrm{T}} A$,

...Stage $n = 2^m \cdot 5$ (Substage 2) is recursive in $Z \leq_{\mathrm{T}} A$, and

...Stage n (for all other n) is recursive.

Thus,

$$\Phi \leq_{\mathrm{T}} M \oplus (\mathcal{O}^M \oplus Z) \oplus A \leq_{\mathrm{T}} A.$$

Applying Lemma 3.5 we find

$$A = \Phi(\mathcal{O}) \leq_{\mathrm{T}} \mathcal{O} \oplus \Phi \leq_{\mathrm{T}} \mathcal{O}^\Phi \equiv_{\mathrm{T}} \Phi(Z) \leq_{\mathrm{T}} Z \oplus \Phi \leq_{\mathrm{T}} Z \oplus A \equiv_{\mathrm{T}} A$$

so we have Turing equivalence throughout. $B = \Phi$ is hence the desired real. □

Theorem 3.1 can be generalized, replacing the real Z by a sequence of reals.

Theorem 3.11. *Suppose Z and A are reals such that $Z \oplus \mathcal{O} \leq_{\mathrm{T}} A$ and $0 <_{\mathrm{HYP}} (Z)_k$ for every $k \in \mathbb{N}$. Then there exists B such that for every $k \in \mathbb{N}$*

$$A \equiv_{\mathrm{T}} \mathcal{O}^B \equiv_{\mathrm{T}} B \oplus (Z)_k \equiv_{\mathrm{T}} B \oplus \mathcal{O}.$$

Proof. The proof of Theorem 3.1 may be adapted by making the following adjustments. First, we replace the use of Lemma 3.10 with the following lemma:

Lemma 3.12. *Suppose Z and A are reals such that $Z \oplus \mathcal{O} \leq_{\mathrm{T}} A$ and $0 <_{\mathrm{HYP}} (Z)_k$ for every $k \in \mathbb{N}$. Then there exists a (code for a) countable ω-model M of* ZFC *such that $\mathcal{O}^M \equiv_{\mathrm{T}} A$ and $(Z)_k \notin M$ for every $k \in \mathbb{N}$.*

Proof. Replace the usage of Theorem 2.1 in the proof of Lemma 3.10 with Theorem 2.8. □

This yields a (code for a) countable ω-model M of ZFC such that $\mathcal{O}, (Z)_0, (Z)_1, \ldots \notin M$ and $\mathcal{O}^M \equiv_{\mathrm{T}} A$. We assume without loss of generality that $\mathcal{O} \neq (Z)_k$ for each k.

The adjustments to the construction are the following:

- In Stages $n = 2^m$ and $n = 2^m{\cdot}3$, we avoid adding new computations along $(Z)_0, \ldots, (Z)_n$ and $\mathcal{O}$.
- Replace Stage $n = 2^m \cdot 5$ with Stages $n = 2^m \cdot 5^{k+1}$, and at the beginning of Stage $n = 2^m \cdot 5^{k+1}$, first check if there exists y and $\sigma \subset (Z)_k$ such that $(m, y, \sigma) \in \Phi_{p_{n-1}}$. If such a y and σ are found, do nothing and proceed to the next stage. Otherwise, proceed as in Stage $n = 2^m \cdot 5$ of the proof of Theorem 3.1, with the same adjustment of avoiding adding new computations along $(Z)_0, \ldots, (Z)_n$ and $\mathcal{O}$ as above.

Note that it is no longer necessarily the case that $\Phi((Z)_k) = \mathcal{O}^\Phi$ for every $k \in \mathbb{N}$, as early stages may have added computations to Φ which make $\Phi((Z)_k)$ disagree with $\mathcal{O}^\Phi$. However, after Stage k, no other stages add new computations along $(Z)_k$ except for those purposely added (i.e., in Stages $n = 2^m \cdot 5^{k+1}$). It follows that $\Phi((Z)_k)$ and $\mathcal{O}^\Phi$ differ only on a finite set of indices, so $\Phi((Z)_k) \equiv_{\mathrm{T}} \mathcal{O}^\Phi$.

In the resulting construction of Φ, (assuming p_{n-1} is given)

…Stage $n = 2^m$ is recursive in $\mathcal{O}^M \oplus \bigoplus_{i=0}^n (Z)_i \oplus \mathcal{O} \leq_{\mathrm{T}} A$,

…Stage $n = 2^m \cdot 3$ is recursive in $\mathcal{O} \leq_{\mathrm{T}} A$,

…Stage $n = 2^m \cdot 5^{k+1}$ (Substage 1) is recursive in $\mathcal{O} \oplus \mathcal{O}^M \oplus \bigoplus_{i=0}^n (Z)_i \leq_{\mathrm{T}} A$,

…Stage $n = 2^m \cdot 5^{k+1}$ (Substage 2) is recursive in $(Z)_k \leq_{\mathrm{T}} A$, and

…Stage n (for all other n) is recursive.

Thus,

$$\Phi \leq_{\mathrm{T}} M \oplus (\mathcal{O}^M \oplus Z) \oplus A \equiv_{\mathrm{T}} A.$$

The proof concludes as in the proof of Theorem 3.1. □

4 Open Problems

In light of Theorems 2.1 and 3.1, it is natural to ask whether they can be combined into one theorem. In other words, for which uncountable Σ^1_1 classes $K \subseteq \{0,1\}^{\mathbb{N}}$ do the following properties hold?

Property 4.1. Suppose Z and A are reals such that $Z \oplus \mathcal{O} \leq_{\mathrm{T}} A$ and $0 <_{\mathrm{HYP}} (Z)_k$ for every $k \in \mathbb{N}$. Then there exists $B \in K$ such that

$$A \equiv_{\mathrm{T}} \mathcal{O}^B \equiv_{\mathrm{T}} B \oplus Z \equiv_{\mathrm{T}} B \oplus \mathcal{O}.$$

Property 4.2. Suppose Z and A are reals such that $Z \oplus \mathcal{O} \leq_{\mathrm{T}} A$ and $0 <_{\mathrm{HYP}} (Z)_k$ for every $k \in \mathbb{N}$. Then there exists $B \in K$ such that for every k

$$A \equiv_{\mathrm{T}} \mathcal{O}^B \equiv_{\mathrm{T}} B \oplus (Z)_k \equiv_{\mathrm{T}} B \oplus \mathcal{O}.$$

The following theorem answers some special cases of this problem.

Theorem 4.3. *Let $L_{\mathrm{T}} = \{X \mid \mathcal{O}^X \equiv_{\mathrm{T}} X \oplus \mathcal{O}\}$. Suppose K is an uncountable Σ^1_1 class which is Turing degree upward closed in L_{T}, i.e., whenever $X, Y \in L_{\mathrm{T}}$, $X \in K$, and $X \leq_{\mathrm{T}} Y$, then there is $Y_0 \in K$ such that $Y \equiv_{\mathrm{T}} Y_0$. Then K has Properties 4.1 and 4.2.*

Proof. This theorem is analogous to [3, Lemma 3.3]. By Theorem 2.1, let C be such that

$$A \equiv_{\mathrm{T}} \mathcal{O}^C \equiv_{\mathrm{T}} C \oplus Z \equiv_{\mathrm{T}} C \oplus \mathcal{O}. \qquad (*)$$

Theorem 2.1, relativized to C, yields $B_0 \in K$ such that

$$\mathcal{O}^C \equiv_{\mathrm{T}} \mathcal{O}^{B_0 \oplus C} \equiv_{\mathrm{T}} B_0 \oplus \mathcal{O}^C. \qquad (\dagger)$$

Combining (†) and (∗) shows that $\mathcal{O}^{B_0 \oplus C} \equiv_{\mathrm{T}} B_0 \oplus C \oplus \mathcal{O}$. As $B_0 \leq_{\mathrm{T}} B_0 \oplus C$, there is $B \in K$ such that $B \equiv_{\mathrm{T}} B_0 \oplus C$ by hypothesis. In particular,

$$\mathcal{O}^C \equiv_{\mathrm{T}} \mathcal{O}^B \equiv_{\mathrm{T}} B \oplus \mathcal{O}.$$

Moreover, in combination with (∗),

$$A \equiv_{\mathrm{T}} \mathcal{O}^B \equiv_{\mathrm{T}} B \oplus \mathcal{O} \equiv_{\mathrm{T}} B \oplus Z.$$

This shows that K has Property 4.1.

To show that K has Property 4.2, repeat the above argument using Theorem 2.8 instead of Theorem 2.1. □

Remark 4.4. The proof of Theorem 4.3 is easily adapted to prove the same result with $L_{\mathrm{T}} = \{X \mid \mathcal{O}^X \equiv_{\mathrm{T}} X \oplus \mathcal{O}\}$ replaced by $L_{\mathrm{HYP}} = \{X \mid \mathcal{O}^X \equiv_{\mathrm{HYP}} X \oplus \mathcal{O}\}$.

The hyperarithmetical analog of the Pseudojump Inversion Theorem [4, Theorem 2.1, pg. 601] also remains open. Namely, suppose V^X_e is an effective enumeration of the $\Pi^{1,X}_1$ predicates, uniformly in X. Define the **e-th pseudo-hyperjump** by

$$\mathrm{HJ}_e(X) \coloneqq X \oplus V^X_e.$$

Does the following result hold?

Conjecture 4.5. *Suppose* $e \in \mathbb{N}$ *and* A *is a real such that* $\mathcal{O} \leq_{\mathrm{T}} A$. *Then there exists* B *such that*

$$A \equiv_{\mathrm{T}} \mathrm{HJ}_e(B) \equiv_{\mathrm{T}} B \oplus \mathcal{O}. \tag{1}$$

Even if Conjecture 4.5 holds, this leaves open the question of characterizing the Σ^1_1 classes $K \subseteq \{0,1\}^{\mathbb{N}}$ with the following properties:

Property 4.6. Suppose $e \in \mathbb{N}$ and A is a real such that $\mathcal{O} \leq_{\mathrm{T}} A$. Then there exists $B \in K$ such that Equation (1) holds.

Property 4.7. Suppose $e \in \mathbb{N}$ and Z and A are reals such that $Z \oplus \mathcal{O} \leq_{\mathrm{T}} A$ and $0 <_{\mathrm{HYP}} (Z)_k$ for each $k \in \mathbb{N}$. Then there exists $B \in K$ such that Equation (1) holds and $(Z)_k \nleq_{\mathrm{HYP}} B$ for every $k \in \mathbb{N}$.

Property 4.8. Suppose $e \in \mathbb{N}$ and Z and A are reals such that $Z \oplus \mathcal{O} \leq_{\mathrm{T}} A$ and $0 <_{\mathrm{HYP}} Z$. Then there exists $B \in K$ such that

$$A \equiv_{\mathrm{T}} \mathrm{HJ}_e(B) \equiv_{\mathrm{T}} B \oplus Z \equiv_{\mathrm{T}} B \oplus \mathcal{O}.$$

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