Some fundamental issues concerning degrees of unsolvability

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Abstract

Recall that $\mathcal{R}_T$ is the upper semilattice of recursively enumerable Turing degrees. We consider two fundamental, classical, unresolved issues concerning $\mathcal{R}_T$. The first issue is to find a specific, natural, recursively enumerable Turing degree $\mathbf{a} \in \mathcal{R}_T$ which is $> \mathbf{0}$ and $< \mathbf{0}'$. The second issue is to find a "smallness property" of an infinite, co-recursively enumerable set $A \subseteq \omega$ which ensures that the Turing degree $\deg_T(A) = \mathbf{a} \in \mathcal{R}_T$ is $> \mathbf{0}$ and $< \mathbf{0}'$. In order to address these issues, we embed $\mathcal{R}_T$ into a slightly larger degree structure, $\mathcal{P}_w$, which is much better behaved. Namely, $\mathcal{P}_w$ is the lattice of weak degrees of mass problems associated with nonempty Π^0_1 subsets of 2^ω. We define a specific, natural embedding of $\mathcal{R}_T$ into $\mathcal{P}_w$, and we present some recent and new research results.

Preface

This paper is based on my talks in Evanston, Illinois, October 24, 2004, and in Singapore, August 8, 2005. The Evanston talk was part of a regional meeting of the American Mathematical Society, held at Northwestern University, October 23–24, 2004. The Singapore talk was part of a workshop entitled Computational Prospects of Infinity, held at the Institute for Mathematical Sciences, National University of Singapore, June 20 through August 15, 2005.

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Motivation

Recall that $\mathcal{D}_T$ is the upper semilattice consisting of all Turing degrees. Recall also that $\mathcal{R}_T$ is the countable sub-semilattice of $\mathcal{D}_T$ consisting of the recursively enumerable Turing degrees. Both of these semilattices have been principal objects of study in recursion theory for many decades. See for instance the monographs of Sacks [39], Rogers [38], Soare [45], and Odifreddi [35, 36].

Two fundamental, classical, unresolved issues concerning $\mathcal{R}_T$ are:

Issue 1. To find a specific, natural, recursively enumerable Turing degree $\mathbf{a} \in \mathcal{R}_T$ which is $> \mathbf{0}$ and $< \mathbf{0}'$.

Issue 2. To find a "smallness property" of an infinite, co-recursively enumerable set $A \subseteq \omega$ which ensures that the Turing degree $\deg_T(A) = \mathbf{a} \in \mathcal{R}_T$ is $> \mathbf{0}$ and $< \mathbf{0}'$.

These unresolved issues go back to Post's classical 1944 paper, *Recursively enumerable sets of positive integers and their decision problems* [37].

My recent interest in Issue 1 began in 1999 at a conference in Boulder, Colorado [10]. There I heard a talk by Shmuel Weinberger, a prominent topologist and geometer. At the time Weinberger was trying to learn something about the recursively enumerable Turing degrees, $\mathcal{R}_T$, with an eye to applying them in the study of moduli spaces in differential geometry [48], using recursion-theoretic methods pioneered by Nabutovsky [33, 34]. Weinberger was visibly frustrated by the fact that $\mathcal{R}_T$ does not appear to contain any specific, natural examples of recursively enumerable Turing degrees, beyond the two standard examples due to Turing, namely $\mathbf{0}'$ = the Turing degree of the Halting Problem, and $\mathbf{0}$ = the Turing degree of solvable problems. Weinberger expressed his frustration by lamenting the fact that there are no recursively enumerable Turing degrees with specific names such as "Bill" or "Fred".

The purpose of this paper is to show how to address Issues 1 and 2 by passing from *decision problems* to *mass problems*. Specifically, we embed $\mathcal{R}_T$ into a slightly larger degree structure, called $\mathcal{P}_w$, which is much better behaved. In the $\mathcal{P}_w$ context, we obtain satisfactory, positive answers to Issues 1 and 2. In particular, we find that $\mathcal{P}_w$ contains not only the degrees $\mathbf{0}'$ and $\mathbf{0}$ but also many other specific, natural, intermediate degrees. These degrees are assigned specific names such as "Carl", "Stanley", "Klaus", "László", "Per", "Wilhelm", and "Bjørn", as explained below.

What is this wonderful structure $\mathcal{P}_w$? Briefly, $\mathcal{P}_w$ is the lattice of weak degrees of mass problems associated with nonempty Π^0_1 subsets of 2^ω. In order to fully explain $\mathcal{P}_w$, we must first explain (1) mass problems, (2) weak degrees, and (3) nonempty Π^0_1 subsets of 2^ω.

Mass problems (informal discussion)

A "decision problem" is the problem of deciding whether a given $n \in \omega$ belongs to a fixed set $A \subseteq \omega$ or not. To compare decision problems, we use Turing reducibility. Recall that $A \leq_T B$ means that A is Turing reducible to B, i.e., A can be computed using an oracle for B.

A "mass problem" is a kind of generalized decision problem, whose solution is not necessarily unique. (By contrast, a decision problem has only one solution.)

We identify a mass problem with the set of its solutions. Here the solutions are identified with Turing oracles, i.e., elements of ω^ω. The "mass problem" associated with a set $P \subseteq \omega^\omega$ is the problem of "finding" an element of P. The "solutions" of this problem are the elements of P.

A mass problem is said to be "solvable" if it has a computable solution. A mass problem is said to be "reducible" to another mass problem if, given any solution of the second problem, we can use it as a Turing oracle to compute a solution of the first problem. Two mass problems are said to be "equivalent" or "of the same degree of unsolvability", if each is reducible to the other.

Mass problems and weak degrees (rigorous definition)

Let P and Q be subsets of $\omega^\omega = \{f \mid f : \omega \to \omega\}$. Viewing P and Q as mass problems, we say that P is *weakly reducible* to Q if

$$(\forall g \in Q)\,(\exists f \in P)\,(f \leq_T g)\,.$$

This is abbreviated $P \leq_w Q$. Thus $P \leq_w Q$ means that, given any solution of the mass problem Q, we can use it as an oracle to compute a solution of the mass problem P.

Definition 1. We define $P, Q \subseteq \omega^\omega$ to be *weakly equivalent*, abbreviated $P \equiv_w Q$, if $P \leq_w Q$ and $Q \leq_w P$. The *weak degrees* are the equivalence classes under $\equiv_w$. There is an obvious partial ordering of the weak degrees induced by weak reducibility. Thus $\deg_w(P) \leq \deg_w(Q)$ if and only if $P \leq_w Q$. The partial ordering of all weak degrees is denoted $\mathcal{D}_w$.

It can be shown that $\mathcal{D}_w$ is a complete distributive lattice. The bottom element of $\mathcal{D}_w$ is denoted $\mathbf{0}$. This is the weak degree of solvable mass problems. Thus $\deg_w(P) = \mathbf{0}$ if and only if $P \cap \mathrm{REC} \neq \emptyset$.

Digression: weak vs. strong reducibility

Let P and Q be mass problems, i.e., subsets of ω^ω. We make the following definitions.

Definition 2.

1. As already stated, P is *weakly reducible* to Q, abbreviated $P \leq_w Q$, if for all $g \in Q$ there exists e such that $\{e\}^g \in P$.
2. P is *strongly reducible* to Q, abbreviated $P \leq_s Q$, if there exists e such that $\{e\}^g \in P$ for all $g \in Q$.

Thus strong reducibility is a uniform variant of weak reducibility.

By a result of Nerode (see Rogers [38, Chapter 9, Theorem XIX]), we have an analogy:

$$\frac{\text{weak reducibility}}{\text{Turing reducibility}} = \frac{\text{strong reducibility}}{\text{truth-table reducibility}}\,.$$

In this paper we shall deal only with weak reducibility.

As a historical note, we mention that weak reducibility goes back to Muchnik 1963 [32], while strong reducibility goes back to Medvedev 1955 [31]. Actually, as mentioned by Terwijn [46], both of these notions ultimately derive from ideas concerning the Brouwer/Heyting/Kolmogorov interpretation of intuitionistic propositional calculus.

The lattice $\mathcal{P}_w$ (rigorous definition)

Recall that $\mathcal{R}_T$, the semilattice of recursively enumerable Turing degrees, is a countable sub-semilattice of $\mathcal{D}_T$, the semilattice of all Turing degrees. Analogously we now define $\mathcal{P}_w$, a certain countable sublattice of the lattice $\mathcal{D}_w$ of all weak degrees. In defining $\mathcal{P}_w$, our guiding analogy is:

$$\frac{\mathcal{P}_w}{\mathcal{D}_w} = \frac{\mathcal{R}_T}{\mathcal{D}_T} .$$

The relevant notions are as follows. Let ω^ω be the *Baire space*, i.e., the set of all total functions $f : \omega \to \omega$. Recall that a set $P \subseteq \omega^\omega$ is said to be Π^0_1 if it is of the form $P = \{f \in \omega^\omega \mid \forall n\, R(f, n)\}$ where R is a recursive predicate. Here n ranges over ω, the set of natural numbers. It is well known that $P \subseteq \omega^\omega$ is Π^0_1 if and only if P is the set of all paths through some recursive subtree of $\omega^{<\omega}$. Here $\omega^{<\omega}$ denotes the tree of finite sequences of natural numbers. Recall also that Π^0_1 subsets of ω^ω are sometimes called *effectively closed*, because such a set is just the complement of an *effectively open* set, i.e., the union of a recursive sequence of basic open sets in ω^ω.

Additionally, $P \subseteq \omega^\omega$ is said to be *recursively bounded* if there exists a recursive function h in ω^ω such that $f(n) < h(n)$ for all $f \in P$ and all $n \in \omega$. Recall that recursively bounded Π^0_1 subsets of ω^ω are sometimes called *effectively compact*.

Definition 3. $\mathcal{P}_w$ is the set of weak degrees of nonempty, recursively bounded, Π^0_1 subsets of ω^ω. There is an obvious partial ordering of $\mathcal{P}_w$ induced by weak reducibility. Thus $\deg_w(P) \leq \deg_w(Q)$ if and only if $P \leq_w Q$.

Remark. Many authors including Jockusch/Soare [22] and Groszek/Slaman [19] have studied the Turing degrees of elements of Π^0_1 subsets of ω^ω which are nonempty and recursively bounded. This earlier research is part of the inspiration for our current study of the weak degrees of mass problems associated with such sets, i.e., $\mathcal{P}_w$.

Remark. It is well known that every recursively bounded Π^0_1 subset of ω^ω is recursively homeomorphic to a recursively bounded Π^0_1 set of a special kind, namely, a Π^0_1 subset of the *Cantor space*,

$$2^\omega = \{0, 1\}^\omega = \{X \mid X : \omega \to \{0, 1\}\}.$$

See for example [42, Theorem 4.10]. Here the recursive bounding function is the constant function 2, i.e., $h(n) = 2$ for all $n \in \omega$.

It follows that $\mathcal{P}_w$ may be alternatively defined as the set of weak degrees of nonempty Π^0_1 subsets of 2^ω. Note also that $\mathcal{P}_w$, as a subset of $\mathcal{D}_w$, is partially ordered by weak reducibility. See Figure 1.

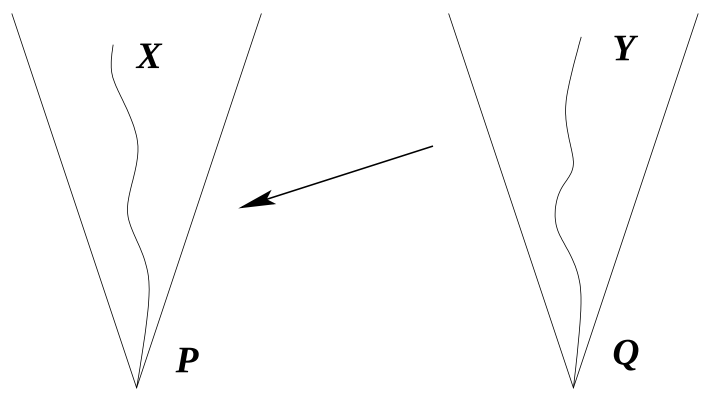

Figure 1: Weak reducibility of Π^0_1 subsets of 2^ω. In this figure, $P \leq_w Q$ means $(\forall Y \in Q)\,(\exists X \in P)\,(X \leq_T Y)$. Here P and Q are given by infinite recursive subtrees of the full binary tree $\{0,1\}^{<\omega}$ of all finite sequences of 0's and 1's. Also, X and Y are infinite paths through P and Q respectively.

Remark. Some basic facts about $\mathcal{P}_w$ are as follows.

1. $\mathcal{P}_w$ is a countable sublattice of $\mathcal{D}_w$.
2. The bottom element of $\mathcal{P}_w$ is $\mathbf{0}$, the same as the bottom element of $\mathcal{D}_w$.
3. The top element of $\mathcal{P}_w$ is the weak degree of

   $$\text{PA} = \{\text{completions of Peano Arithmetic}\}.$$

   This goes back to Scott/Tennenbaum [40]. See also Jockusch/Soare [22].

Remark. $\mathcal{R}_T$ is usually regarded as the smallest or simplest natural subsemilattice of $\mathcal{D}_T$. Similarly, $\mathcal{P}_w$ may be regarded as the smallest or simplest natural sublattice of $\mathcal{D}_w$. In addition, $\mathcal{P}_w$ resembles $\mathcal{R}_T$ in other respects, as we shall see below. In particular, $\mathcal{P}_w$ includes a copy of $\mathcal{R}_T$, as we now show.

Embedding $\mathcal{R}_T$ into $\mathcal{P}_w$

Recall that $\mathcal{R}_T$ = the semilattice of Turing degrees of recursively enumerable subsets of ω, and $\mathcal{P}_w$ = the lattice of weak degrees of nonempty Π^0_1 subsets of 2^ω. The following embedding theorem was obtained by Simpson in 2002 [44].

Theorem 1. *There is a specific, natural embedding*

$$\phi : \mathcal{R}_T \hookrightarrow \mathcal{P}_w \ .$$

The embedding ϕ is given by

$$\phi : \deg_T(A) \mapsto \deg_w(\mathrm{PA} \cup \{A\}) \ .$$

Here ϕ is one-to-one and preserves the partial ordering $\leq$, the least upper bound operation sup*, and the top and bottom elements.*

Remark. In Theorem 1, the fact that $\deg_w(\mathrm{PA} \cup \{A\})$ belongs to $\mathcal{P}_w$ is not obvious, because $\mathrm{PA} \cup \{A\}$ is usually not a Π^0_1 set. However, it turns out that $\mathrm{PA} \cup \{A\}$ is always *of the same weak degree as* a Π^0_1 subset of 2^ω. This fact is a consequence of our Embedding Lemma [44, Lemma 3.3], Lemma 4 below.

Likewise, it may not be obvious that the embedding $\phi : \mathcal{R}_T \to \mathcal{P}_w$ is one-to-one. However, the one-to-oneness of ϕ can be shown as a consequence of a famous theorem known as the Arslanov Completeness Criterion. This theorem can be found in textbooks, e.g., Soare [45, Theorem V.5.1].

Convention. *Throughout this paper, we shall identify $\mathcal{R}_T$ with its image in $\mathcal{P}_w$ under the embedding ϕ.* We shall also identify each recursively enumerable Turing degree with the weak degree which is its image in $\mathcal{P}_w$ under the embedding ϕ. In particular, we identify $\mathbf{0}', \mathbf{0} \in \mathcal{R}_T$ with the top and bottom elements of $\mathcal{P}_w$. See Figure 2.

Structural properties of $\mathcal{P}_w$

It can be shown that $\mathcal{P}_w$ has many structural features which are similar to those of $\mathcal{R}_T$. Some of the similar features are as follows.

1. $\mathcal{P}_w$ is a countable distributive lattice. Moreover, every countable distributive lattice is lattice embeddable in every initial segment of $\mathcal{P}_w$. This result is due to Binns/Simpson [4, 8].

2. The $\mathcal{P}_w$ analog of the Sacks Splitttting Theorem holds. In other words, for all $\mathbf{a}, \mathbf{c} > \mathbf{0}$ in $\mathcal{P}_w$ we can find $\mathbf{b}_1, \mathbf{b}_2 \in \mathcal{P}_w$ such that $\mathbf{a} = \sup(\mathbf{b}_1, \mathbf{b}_2)$ and $\mathbf{b}_1 \not\geq \mathbf{c}$ and $\mathbf{b}_2 \not\geq \mathbf{c}$. This result is due to Binns [4, 5].

3. We conjecture that the $\mathcal{P}_w$ analog of the Sacks Density Theorem holds. This would mean that for all $\mathbf{a}, \mathbf{b} \in \mathcal{P}_w$ with $\mathbf{a} < \mathbf{b}$ there exists $\mathbf{c} \in \mathcal{P}_w$ such that $\mathbf{a} < \mathbf{c} < \mathbf{b}$.

4. There are some degrees in $\mathcal{P}_w$ with interesting lattice-theoretic properties, such as being meet-reducible or not, and joining to $\mathbf{0}'$ or not. See Theorem 3 below. See also Simpson [44, 42].

Note that these structural properties of $\mathcal{P}_w$ are proved by means of priority arguments, just as for $\mathcal{R}_T$. On the other hand, there are some structural differences between $\mathcal{P}_w$ and $\mathcal{R}_T$. For example:

5. Within $\mathcal{P}_w$, the degree $\mathbf{0}$ is meet-irreducible. (This is trivial.)

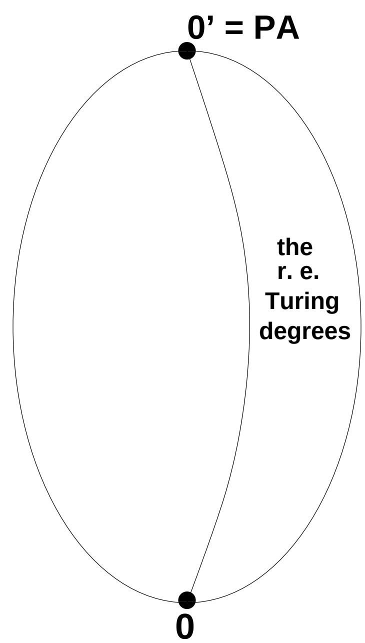

Figure 2: A picture of $\mathcal{P}_w$, the lattice of weak degrees of nonempty Π^0_1 subsets of 2^ω. Note that $\mathcal{R}_T$, the upper semilattice of recursively enumerable Turing degrees, is embedded in $\mathcal{P}_w$. Moreover, $\mathbf{0}'$ and $\mathbf{0}$ are the top and bottom elements of both $\mathcal{R}_T$ and $\mathcal{P}_w$.

Response to Issue 1

Recall that Issue 1 posed the problem of finding a specific, natural example of a recursively enumerable Turing degree which is $> \mathbf{0}$ and $< \mathbf{0}'$. We do not know how to solve this problem.

However, in the slightly broader $\mathcal{P}_w$ context, we [44, 42] have discovered many specific, natural degrees which are $> \mathbf{0}$ and $< \mathbf{0}'$. See Figure 3.

Moreover, as noted in [44, 42], several of the specific, natural degrees in $\mathcal{P}_w$ which we have discovered are related to foundationally interesting topics:

- reverse mathematics,
- algorithmic randomness,
- subrecursive hierarchies,
- computational complexity,
- diagonal nonrecursiveness.

See also the additional explanations below.

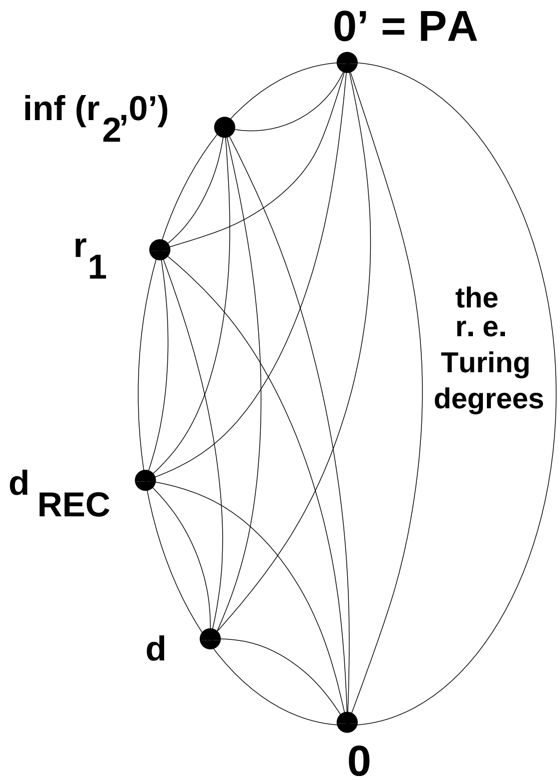

Figure 3: Some specific, natural degrees in $\mathcal{P}_w$. Note that each of these specific, natural degrees in $\mathcal{P}_w$ is incomparable with all of the recursively enumerable Turing degrees, except $\mathbf{0}'$ and $\mathbf{0}$.

Some specific, natural degrees in $\mathcal{P}_w$

Conside the following specific, natural, weak degrees. Let $\mathbf{r}_n$ be the weak degree of the set of n-random reals [25]. Let $\mathbf{d}$ be the weak degree of the set of diagonally nonrecursive functions [21]. Let $\mathbf{d}_{\mathrm{REC}}$ be the weak degree of the set of diagonally nonrecursive functions which are recursively bounded.

Not all of these weak degrees belong to $\mathcal{P}_w$. However, we have the following theorem.

Theorem 2. *In $\mathcal{P}_w$ we have*

$$\mathbf{0} < \mathbf{d} < \mathbf{d}_{\mathrm{REC}} < \mathbf{r}_1 < \inf(\mathbf{r}_2, \mathbf{0}') < \mathbf{0}'.$$

Moreover, all of these specific, natural degrees in $\mathcal{P}_w$ are incomparable with all of the recursively enumerable Turing degrees, except $\mathbf{0}'$ and $\mathbf{0}$.

Proof. See Simpson [44, 42]. The strict inequalities $\mathbf{d} < \mathbf{d}_{\mathrm{REC}} < \mathbf{r}_1$ follow from Kumabe [28] and Ambos-Spies et al [3]. □

We also have:

Theorem 3.

1. *We may characterize* $\mathbf{r}_1$ *as the maximum weak degree of a* Π^0_1 *subset of* 2^ω *which is of positive measure.*

2. *We may characterize* $\inf(\mathbf{r}_2, \mathbf{0}')$ *as the maximum weak degree of a* Π^0_1 *subset of* 2^ω *whose Turing upward closure is of positive measure.*

3. *Each of the degrees* $\mathbf{r}_1$ *and* $\inf(\mathbf{r}_2, \mathbf{0}')$ *is meet-irreducible and does not join to* $\mathbf{0}'$*.*

Proof. See Simpson [44, 42]. □

Remark. The weak degrees $\mathbf{r}_1$ and $\mathbf{d}$ have arisen in the reverse mathematics of measure theory and of the Tietze Extension Theorem, respectively [9, 18]. See also [41, Chapter X].

Remark. We hereby assign the names "Carl", "Klaus", and "Per" [21, 3, 30] to the respective weak degrees $\mathbf{d}$, $\mathbf{d}_{\mathrm{REC}}$, and $\mathbf{r}_1$ in $\mathcal{P}_w$.

The Embedding Lemma and some of its consequences

Several of the results stated above are consequences of the following lemma, due to Simpson [44, Lemma 3.3], which we call the Embedding Lemma:

If $S \subseteq \omega^\omega$ *is* Σ^0_3 *and if* $P \subseteq 2^\omega$ *is nonempty* Π^0_1*, then* $\deg_w(S \cup P) \in \mathcal{P}_w$.

Using the Embedding Lemma we can show that the weak degrees of many specific, natural Σ^0_3 subsets of ω^ω belong to $\mathcal{P}_w$.

Examples.

1. Let $R_1 = \{X \in 2^\omega \mid X \text{ is 1-random}\}$. Since R_1 is Σ^0_2, it follows by the Embedding Lemma that $\mathbf{r}_1 = \deg_w(R_1) \in \mathcal{P}_w$.

2. Let $R_2 = \{X \in 2^\omega \mid X \text{ is 2-random}\}$. Since R_2 is Σ^0_3, it follows by the Embedding Lemma that $\inf(\mathbf{r}_2, \mathbf{0}') = \deg_w(R_2 \cup \mathrm{PA}) \in \mathcal{P}_w$.

3. Let $D = \{f \in \omega^\omega \mid f \text{ is diagonally nonrecursive}\}$. Since D is Π^0_1, it follows by the Embedding Lemma that $\mathbf{d} = \deg_w(D) \in \mathcal{P}_w$.

4. Let $D_{\mathrm{REC}} = \{f \in D \mid f \text{ is recursively bounded}\}$. Since D_{REC} is Σ^0_3, it follows by the Embedding Lemma that $\mathbf{d}_{\mathrm{REC}} = \deg_w(D_{\mathrm{REC}}) \in \mathcal{P}_w$.

5. Let $A \subseteq \omega$ be recursively enumerable. Since $\{A\}$ is Π^0_2, it follows by the Embedding Lemma that $\deg_w(\{A\} \cup \mathrm{PA}) \in \mathcal{P}_w$. This gives our embedding of $\mathcal{R}_T$ into $\mathcal{P}_w$.

Proof of the Embedding Lemma

We restate the Embedding Lemma as follows.

Lemma 4 (The Embedding Lemma)**.** *Let $S \subseteq \omega^\omega$ be Σ^0_3. Let $P \subseteq 2^\omega$ be nonempty Π^0_1. Then we can find $Q \subseteq 2^\omega$ nonempty Π^0_1 such that $Q \equiv_w S \cup P$.*

Proof. First use a Skolem function technique to reduce to the case when S is Π^0_1. Namely, letting $S = \{f \in \omega^\omega \mid \exists k \, \forall n \, \exists m \, R(f,k,n,m)\}$ where R is recursive, replace S by the set of all $\langle k\rangle^\frown(f \oplus g) \in \omega^\omega$ such that $\forall n \, R(f,k,n,g(n))$ holds. Clearly the latter set is $\equiv_w S$ and Π^0_1.

Assuming now that S is a Π^0_1 subset of ω^ω, let T_S be a recursive subtree of $\omega^{<\omega}$ such that S is the set of paths through T_S. We may safely assume that, for all $\tau \in T_S$ and all $n <$ the length of τ, $\tau(n) \geq 2$. Let T_P be a recursive subtree of $\{0,1\}^{<\omega}$ such that P is the set of paths through T_P. Define T_Q to be the set of finite sequences $\rho \in \omega^{<\omega}$ of the form

$$\rho = \sigma_0{}^\frown\langle m_0\rangle^\frown\sigma_1{}^\frown\langle m_1\rangle^\frown\cdots^\frown\langle m_{k-1}\rangle^\frown\sigma_k$$

such that $\langle m_0, m_1, \ldots, m_{k-1}\rangle \in T_S$, $\sigma_0, \sigma_1, \ldots, \sigma_k \in T_P$, and $\rho(n) \leq \max(n,2)$ for all $n <$ the length of ρ. Thus T_Q is a recursive subtree of $\omega^{<\omega}$. Let $Q \subseteq \omega^\omega$ be the set of paths through T_Q.

We claim that $Q \equiv_w S \cup P$. Let $f \in S$ be given. Since T_P is infinite, it contains a recursive sequence of finite sequences τ_n of length n for each n. Setting $g = \tau_{f(0)}{}^\frown\langle f(0)\rangle^\frown\cdots^\frown\tau_{f(n)}{}^\frown\langle f(n)\rangle^\frown\cdots$, we have $g \in Q$ and $g \leq_T f$. This shows that $Q \leq_w S$. Note also that $T_Q \supseteq T_P$, hence $Q \supseteq P$, hence $Q \leq_w P$. We now have $Q \leq_w S \cup P$. Conversely, given $g \in Q$, set

$$I = \{n \mid g(n) \geq 2\} = \{n_0 < n_1 < \cdots < n_k < \cdots\}.$$

If I is infinite, then setting $f(k) = n_k$ for all k, we have $f \in S$ and $f \leq_T g$. If I is finite, say $I = \{n_0 < n_1 < \cdots < n_{k-1}\}$, then setting $n_{-1} = -1$ and $X(i) = g(n_{k-1}+i+1)$ for all i, we have $X \in P$ and $X \leq_T g$. Thus $Q \geq_w S \cup P$ and our claim is proved.

Note that Q is Π^0_1 and recursively bounded, with bounding function $h(n) = \max(n,2)$. Therefore, we can find a Π^0_1 set $Q^* \subseteq 2^\omega$ which is recursively homeomorphic to Q, hence weakly equivalent to Q. This proves our lemma. □

Some additional, specific degrees in $\mathcal{P}_w$

By the same method as in Theorem 2, we can use the Embedding Lemma to identify some additional, specific, natural degrees in $\mathcal{P}_w$. Some of these degrees in $\mathcal{P}_w$ are associated with computational complexity classes, as follows.

Definition 4. Let C be a uniformly recursively enumerable class of total recursive functions satisfying some mild closure conditions, as explained in [42, Section 10]. Let $\mathbf{d}_C$ be the weak degree of the set of diagonally nonrecursive functions which are bounded by some $h \in C$. As a consequence of the Embedding Lemma, we have $\mathbf{d}_C \in \mathcal{P}_w$. See [42, Section 10] for a detailed justification of our claim that these degrees $\mathbf{d}_C$ are recursion-theoretically natural.

Remark. If $C^* \supseteq C$ is another such class of recursive functions, then we have $\mathbf{d}_{C^*} \leq \mathbf{d}_C$. Moreover, according to [3, Theorem 1.9], we have strict inequality $\mathbf{d}_{C^*} < \mathbf{d}_C$ provided C^* contains a function which "grows much faster than" all functions in C. There are some interesting problems here.

Examples. Let C be any of the following complexity classes:

1. PR = the class of primitive recursive functions
2. ER = the class of elementary recursive functions.
3. PTIME = the class of polynomial-time computable functions.
4. PSPACE = the class of polynomial-space computable functions.
5. EXPTIME = the class of exponential-time computable functions, etc.
6. C_α = the class of recursive functions at levels $< \omega \cdot (1+\alpha)$ of the transfinite Ackermann hierarchy due to Wainer [47]. Here α is any ordinal number $\leq \varepsilon_0$. Thus $C_0 = \text{PR}$, C_1 = the class of functions which are primitive recursive in the Ackermann function, etc.

For each of these classes C, we have a specific, natural degree $\mathbf{d}_C$ in $\mathcal{P}_w$. Thus we have

$$\mathbf{r}_1 > \mathbf{d}_{\text{PTIME}} > \mathbf{d}_{\text{PSPACE}} > \mathbf{d}_{\text{EXPTIME}} > \mathbf{d}_{\text{ER}} > \mathbf{d}_{\text{PR}} = \mathbf{d}_0$$

in $\mathcal{P}_w$, corresponding to well-known complexity classes. Also, writing $\mathbf{d}_\alpha = \mathbf{d}_{C_\alpha}$ for each $\alpha \leq \varepsilon_0$, we have

$$\mathbf{d}_0 > \mathbf{d}_1 > \cdots > \mathbf{d}_\alpha > \mathbf{d}_{\alpha+1} > \cdots > \mathbf{d}_{\varepsilon_0} > \mathbf{d}_{\text{REC}}$$

in $\mathcal{P}_w$. Moreover, if α is a limit ordinal, then by [42, Remark 10.12] we have $\mathbf{d}_\alpha = \inf_{\beta<\alpha} \mathbf{d}_\alpha$.

Remark. We hereby assign the names "Wilhelm", "László", and "Stanley" [1, 24, 47] to the respective weak degrees $\mathbf{d}_0 = \mathbf{d}_{\text{PR}}$, $\mathbf{d}_{\text{ER}}$, and $\mathbf{d}_{\varepsilon_0}$ in $\mathcal{P}_w$.

In addition, let $\mathbf{d}^2$ be the weak degree of the set of $f \oplus g$ such that f is diagonally nonrecursive, and g is diagonally nonrecursive relative to f. More generally [44, Section 4], we can define $\mathbf{d}^n$ for all $n \geq 1$, and we can extend this into the transfinite.

Theorem 5. *In $\mathcal{P}_w$ we have*

$$\mathbf{d} = \mathbf{d}^1 < \mathbf{d}^2 < \cdots < \mathbf{d}^n < \cdots < \mathbf{r}_1 \,.$$

Proof. This is a consequence of Kumabe [28]. See Simpson [44, Section 4]. □

Remark. We conjecture that all of the $\mathbf{d}^n$'s, $n \geq 2$ are incomparable with all of the $\mathbf{d}_\alpha$'s, $\alpha \leq \varepsilon_0$, and with $\mathbf{d}_{\text{REC}}$.

Positive-measure domination

Up until this point, all of our examples of specific, natural degrees in $\mathcal{P}_w$ have turned out to be incomparable with all of the recursively enumerable Turing degrees, except $\mathbf{0}'$ and $\mathbf{0}$. We now present an example which behaves differently in this respect.

Starting with Dobrinen/Simpson [14] and continuing with Cholak/Greenberg/Miller [12], Binns et al [7], and Kjos-Hanssen [26], there has been a recent upsurge of interest in domination properties related to the reverse mathematics of measure theory. We consider one such property.

Definition 5. $A \in 2^\omega$ is said to be *positive-measure dominating* if every Π^0_2 subset of 2^ω of positive measure includes a $\Pi^{0,A}_1$ set of positive measure. This notion was implicitly introduced in [14, Conjecture 3.1] and has been developed in [26].

Remark. Positive-measure domination is related to the reverse mathematics of the following measure-theoretic regularity statement:

> Every G_δ set of positive measure includes a closed set of positive measure.

For more on the reverse mathematics of measure-theoretic regularity, see [14].

Remark. Kjos-Hanssen [26] has shown that the set

$$\mathrm{PMD} = \{A \in 2^\omega \mid A \text{ is positive-measure dominating}\}$$

is Σ^0_3. Setting $\mathbf{m} = \deg_w(\mathrm{PMD})$, we may apply the Embedding Lemma to conclude that $\inf(\mathbf{m}, \mathbf{0}') \in \mathcal{P}_w$. It follows from the results of [12, 7] that $\inf(\mathbf{m}, \mathbf{0}')$ is incomparable with $\mathbf{d}$ and that there exist recursively enumerable Turing degrees $\mathbf{a}$ such that $\mathbf{0} < \inf(\mathbf{m}, \mathbf{0}') < \mathbf{a} < \mathbf{0}'$ in $\mathcal{P}_w$. See Figure 4.

Remark. We hereby assign the name "Bjørn" [26] to the specific, natural degree $\inf(\mathbf{m}, \mathbf{0}')$ in $\mathcal{P}_w$.

Note added June 30, 2006: Recently Kjos-Hanssen, Miller and Solomon [27] have shown that $A \in 2^\omega$ is positive-measure dominating if and only if A is almost everywhere dominating, if and only if A is uniformly almost everywhere dominating. In addition, Simpson [43] has shown that any such A is *superhigh*, i.e., $A' \geq_{tt} 0''$, i.e., $0''$ is truth-table reducible to A'.

Some further specific, natural degrees in $\mathcal{P}_w$

We now mention some further examples of specific, natural, Σ^0_3 subsets of ω^ω which, via the Embedding Lemma, give rise to specific, natural degrees in $\mathcal{P}_w$. These degrees are earmarked for future research.

Examples.

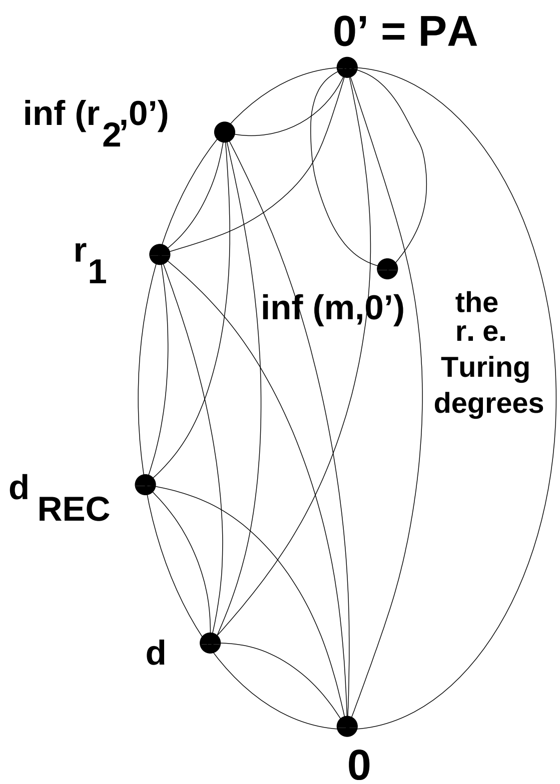

Figure 4: Another specific, natural degree in $\mathcal{P}_w$. Note that $\inf(\mathbf{m}, \mathbf{0}')$, unlike $\mathbf{d}$, $\mathbf{d}_{\mathrm{REC}}$, $\mathbf{r}_1$, and $\inf(\mathbf{r}_2, \mathbf{0}')$, is bounded above by some recursively enumerable Turing degrees which are strictly less than $\mathbf{0}'$. In addition, $\inf(\mathbf{m}, \mathbf{0}')$ is incomparable with $\mathbf{d}$.

1. Let $\mathbf{d}^*$ be the weak degree of the set of functions which are diagonally nonrecursive relative to the Halting Problem. This set of functions has arisen in the reverse mathematics of Ramsey's Theorem [20]. The Embedding Lemma tells us that $\inf(\mathbf{d}^*, \mathbf{0}') \in \mathcal{P}_w$.

2. Let $\mathbf{d}^*_{\mathrm{REC}}$ = the weak degree of the set of functions which are (a) diagonally nonrecursive relative to the Halting Problem, and (b) recursively bounded. The Embedding Lemma tells us that $\inf(\mathbf{d}^*_{\mathrm{REC}}, \mathbf{0}') \in \mathcal{P}_w$.

3. For each computational complexity class C as considered in Definition 4, let $\mathbf{d}^*_C$ = the weak degree of the set of functions which are (a) diagonally nonrecursive relative to the Halting Problem, and (b) bounded by a function in C. The Embedding Lemma tells us that $\inf(\mathbf{d}^*_C, \mathbf{0}') \in \mathcal{P}_w$.

Remark. It should be interesting to explore the relationships among these newly identified degrees $\inf(\mathbf{d}^*, \mathbf{0}')$, $\inf(\mathbf{d}^*_{\mathrm{REC}}, \mathbf{0}')$, $\inf(\mathbf{d}^*_C, \mathbf{0}')$ in $\mathcal{P}_w$, as well as their relationships with the previously identified degrees $\mathbf{d}$, $\mathbf{d}_{\mathrm{REC}}$, $\mathbf{d}_C$ in $\mathcal{P}_w$, and with reverse mathematics.

Remark. Another promising source of examples is as follows. Let $Q \subseteq \omega^\omega$ be Σ^0_2 relative to the Halting Problem. Let $\mathbf{s}$ be the weak degree of S, where S is either

$$\{f \in \omega^\omega \mid \exists g\,(g \in Q \text{ and } g \leq_{tt} f')\}$$

or

$$\{f \in \omega^\omega \mid \exists g\,(g \in Q \text{ and } g \leq_{tt} f \oplus 0')\}.$$

Here $\leq_{tt}$ denotes truth-table reducibility, and f' denotes the Turing jump of f. Since S is Σ^0_3, the Embedding Lemma applies. Moreover, if Q is specific and natural, then so is $\mathbf{s}$, hence so is $\inf(\mathbf{s}, \mathbf{0}') \in \mathcal{P}_w$. It should be interesting to explore the relationships among these degrees and others in $\mathcal{P}_w$. The ideas of Jockusch/Stephan [23] and Kjos-Hanssen [7, 26] concerning cohesiveness and superhighness may be relevant.

Response to Issue 2

Issue 2 was the problem of finding a "smallness property" of infinite Π^0_1 (i.e., co-recursively enumerable) sets $A \subseteq \omega$ which ensures that the Turing degree of A is $> \mathbf{0}$ and $< \mathbf{0}'$. We do not know how to do this.

However, in the $\mathcal{P}_w$ context, we have identified several "smallness properties" of Π^0_1 sets $P \subseteq 2^\omega$ which ensure that the weak degree of P is $> \mathbf{0}$ and $< \mathbf{0}'$.

Here is one result of this type.

Definition 6. A Π^0_1 set $P \subseteq 2^\omega$ is said to be *thin* if, for all Π^0_1 sets $Q \subseteq P$, $P \setminus Q$ is Π^0_1. Equivalently, all Π^0_1 sets $Q \subseteq P$ are of the form $P \cap D$ where D is clopen. A set $P \subseteq 2^\omega$ is said to be *perfect* if it has no isolated points.

Remark. Nonempty Π^0_1 subsets of 2^ω which are thin and perfect have been constructed by means of priority arguments. Much is known about them. For example, any two such sets are automorphic in the lattice of Π^0_1 subsets of 2^ω under inclusion. See Martin/Pour-El [29], Downey/Jockusch/Stob [15, 16], and Cholak et al [11].

Theorem 6. *Let $\mathbf{p} = \deg_w(P)$ where $P \subseteq 2^\omega$ is Π^0_1, nonempty, thin, and perfect. Then $\mathbf{p}$ is incomparable with $\mathbf{r}_1$. Hence $\mathbf{0} < \mathbf{p} < \mathbf{0}'$.*

Proof. See Simpson [42]. □

One may also consider other smallness properties. As above, let P be a nonempty Π^0_1 subset of 2^ω. The following definition and theorem are due to Binns [6].

Definition 7. P is *small* if there is no recursive function f such that for all n there exist n members of P which differ at level $f(n)$ in the binary tree $\{0,1\}^{<\omega}$. For example, let $A \subseteq \omega$ be hypersimple, and let $A = B_1 \cup B_2$ where B_1, B_2 are recursively enumerable. Then $P = \{X \in 2^\omega \mid X \text{ separates } B_1, B_2\}$ is small.

Theorem 7. *If P is small, then the weak degree of P is $< \mathbf{0}'$.*

Proof. See Binns [6]. ☐

The following definition and theorem are due to Simpson, unpublished.

Definition 8. P is *h-small* if there is no recursive, canonically indexed sequence of pairwise disjoint clopen sets D_n, $n \in \omega$, such that $P \cap D_n \neq \emptyset$ for all n.

Theorem 8. *If P is h-small, then the weak degree of P is $< \mathbf{0}'$.*

Summary

We summarize this paper as follows.

There are basic, unresolved issues concerning $\mathcal{R}_T$, the semilattice of recursively enumerable Turing degrees. One of the issues is the lack of specific, natural, recursively enumerable degrees.

In order to address this issue, we have embedded $\mathcal{R}_T$ into $\mathcal{P}_w$, the lattice of weak degrees of nonempty Π^0_1 subsets of 2^ω. We identify $\mathcal{R}_T$ with its image in $\mathcal{P}_w$. In the $\mathcal{P}_w$ context, some of the unresolved issues can be satisfactorily addressed. In particular, $\mathcal{P}_w$ contains many specific, natural degrees which are related to foundationally interesting topics: algorithmic randomness, reverse mathematics, computational complexity.

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