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First-order theory of the degrees of recursive unsolvability[1]

By STEPHEN G. SIMPSON

Contents

1. Introduction and preliminaries

Let $\langle D, \cup \rangle$ be the semilattice of degrees of recursive unsolvability. The main result of this paper is that the first-order theory of $\langle D, \cup \rangle$ is recursively isomorphic to the truth set of second-order arithmetic (Corollary 5.6). We also obtain a strong result concerning first-order definability in $\langle D, \cup, j \rangle$ where j is the jump operator (Theorem 3.12).

The structure of $\langle D, \cup \rangle$ has been investigated strenuously by Kleene and Post [12], Spector [29], Sacks [20], Lerman [15] and a host of others. The first-order theory of $\langle D, \cup \rangle$ has been commented upon from time to time by various authors including Jockusch and Soare [10], Miller and Martin [17], Rogers [19], Shoenfield [24], [26] and Stillwell [30]. Our main result can be regarded as a refinement of the theorem of Lachlan [13] that the first-order theory of $\langle D, \cup \rangle$ is undecidable. Like Lachlan we use initial segments, but we combine them with the jump operator (Theorem 2.1). Our curiosity about the subject of this paper was first awakened in 1969 by Gerald E. Sacks who asked whether the first-order theory of $\langle D, \cup \rangle$ is hyperarithmetical (see also Problem 70 in [5]). We are also grateful to Carl G. Jockusch, Jr. for timely expressions of interest in this work.

We use ω to denote the set of nonnegative integers $\{0, 1, 2, \cdots\}$. Letters such as i, j, k, m, n denote elements of ω. We write 2^ω for the set of totally defined, $\{0, 1\}$-valued functions on ω. Letters such as f, g, h denote elements of 2^ω. We write $f \oplus g$ for the unique function h such that $h(2n) = f(n)$ and $h(2n + 1) = g(n)$ for all $n \in \omega$. The jump of $f \in 2^\omega$ is f^* which is again an element of 2^ω. Finite iterates of $*$ are defined by $f^{(0)} = f$ and $f^{(n+1)} = (f^{(n)})^*$

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for $n \in \omega$. The ωth iterate of $*$ is defined by

$$f^{(\omega)}\big(2^m(2n+1)-1\big) = f^{(m)}(n) .$$

Boldface letters such as $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}$ denote *degrees*, i.e., degrees of recursive unsolvability. The degree of f is written $\deg(f)$. The degree of recursive functions is denoted $\mathbf{0}$. If $\boldsymbol{a} = \deg(f)$ then we write $\boldsymbol{a}' = \deg(f^*)$, $\boldsymbol{a}^{(n)} = \deg(f^{(n)})$, and $\boldsymbol{a}^{(\omega)} = \deg(f^{(\omega)})$. The *jump operator* $j\colon D \to D$ is defined by $j(\boldsymbol{a}) = \boldsymbol{a}'$. If $\boldsymbol{a} = \deg(f)$ and $\boldsymbol{b} = \deg(g)$ then we write $\boldsymbol{a} \cup \boldsymbol{b} = \deg(f \oplus g)$, and $\boldsymbol{a} \leqq \boldsymbol{b}$ if and only if f is recursive in g.

Some of our detailed proofs will be a presented in terms of strings. A *string* is a finite sequence of elements of $\{0, 1\}$. Letters such as $\mu, \nu, \rho, \sigma, \tau$ denote strings and Σ is the set of all strings. The *length* of a string is a nonnegative integer $lh(\sigma)$. The empty string ϕ has length 0. The strings of length 1 are written 0 and 1. We write $\sigma \oplus \tau$ for the unique string ρ of length

$$\min\{2 \cdot lh(\sigma), 2 \cdot lh(\tau) + 1\}$$

defined by $\rho(2i) = \sigma(i)$, $\rho(2i+1) = \tau(i)$. We write $\sigma \subseteqq \tau$ if σ is extended by τ, i.e., $\sigma\nu = \tau$ for some string ν where as usual concatenation of strings is denoted by juxtaposition. An element of 2^ω is sometimes treated as a "string of length ω." We use the recursive relation $[m]^\sigma(n) = i$ which means that the mth algorithm with oracle information σ applied to input n halts with output $i \in \{0, 1\}$ in at most $lh(\sigma)$ steps. We write $[m]^g(n) = i$ if and only if $[m]^\sigma(n) = i$ for some $\sigma \subseteq g$. Thus f is recursive in g if and only if

$$\exists m \, \forall n [m]^g(n) = f(n) .$$

Furthermore $f^*(n) = 1$ if and only if $[n]^f(n)$ is defined.

Our main technical tool is the notion of a perfect tree (cf. Sacks [21]). A *perfect tree* is a mapping $P\colon \Sigma \to \Sigma$ such that $P(\sigma) \subseteqq P(\tau)$ if and only if $\sigma \subseteqq \tau$. Letters such as P, Q denote perfect trees. The identity mapping of Σ onto Σ is a perfect tree called the *identity tree*. If P is a perfect tree and $f \in 2^\omega$ then we write $P(f) = g$ where g is the unique element of 2^ω such that $P(\sigma) \subseteq g$ whenever $\sigma \subseteqq f$. We also write

$$[P] = \{P(f) \mid f \in 2^\omega\} .$$

Thus $[P]$ is a perfect closed subset of 2^ω in its usual topology. We write $P \subseteqq Q$ if and only if $[P] \subseteqq [Q]$. If P is a perfect tree and σ is a string, then there is a perfect tree $P^\sigma \subseteqq P$ defined by $P^\sigma(\nu) = P(\sigma\nu)$ for all $\nu \in \Sigma$. Note that $[P] = [P^0] \cup [P^1]$ and $[P^0] \cap [P^1] = \varnothing$. If D is a set of perfect trees then we write

$$[D] = \bigcup\{[P] \mid P \in D\} .$$

Strings are Gödel numbered so that we may identify perfect trees with certain number-theoretic functions. Thus it makes sense to say that P is recursive, etc. For any degree $\boldsymbol{c}$ we denote by $\mathcal{P}(\boldsymbol{c})$ the set of perfect trees recursive in $\boldsymbol{c}$.

2. The Main Lemma

The purpose of this section is to prove the following theorem.

THEOREM 2.1 (The Main Lemma). *Let $\langle \boldsymbol{b}_n\rangle_{1 \leqq n \in \omega}$ be a sequence of degrees such that $\mathbf{0}'' \leqq \boldsymbol{b}_1 \leqq \cdots \leqq \boldsymbol{b}_n \leqq \cdots$. Then there exists a sequence of degrees $\langle \boldsymbol{a}_n\rangle_{1 \leqq n \in \omega}$ such that $\mathbf{0} < \boldsymbol{a}_1 < \cdots < \boldsymbol{a}_n < \cdots$ and, for each $n \geqq 1$, $\boldsymbol{a}_n'' = \boldsymbol{a}_n \cup \mathbf{0}'' = \boldsymbol{b}_n$ and $\{\mathbf{0}, \boldsymbol{a}_1, \cdots, \boldsymbol{a}_n\}$ is an initial segment of the degrees.*

The machinery which we shall develop for the proof of Theorem 2.1 will not be needed in later parts of this paper. However, we believe that the machinery has independent interest (see Yates [32] and Remark 2.14 below).

A degree $\boldsymbol{b}$ is *minimal if* $\mathbf{0} < \boldsymbol{b}$ and $\mathbf{0}$ is the unique degree less than $\boldsymbol{b}$. Let $\boldsymbol{a}$ be any degree. A *minimal cover* of $\boldsymbol{a}$ is a degree $\boldsymbol{b}$ such that $\boldsymbol{a} < \boldsymbol{b}$ and there is no degree $\boldsymbol{c}$ such that $\boldsymbol{a} < \boldsymbol{c} < \boldsymbol{b}$. A *strongly minimal cover* of $\boldsymbol{a}$ is a degree $\boldsymbol{b}$ such that $\{\boldsymbol{d} \mid \boldsymbol{d} \leqq \boldsymbol{a}\} = \{\boldsymbol{d} \mid \boldsymbol{d} < \boldsymbol{b}\}$. Clearly a strongly minimal cover of $\boldsymbol{a}$ is a minimal cover of $\boldsymbol{a}$, but the converse is false by Remark 2.13 below. Note that in Theorem 2.1, $\boldsymbol{a}_{n+1}$ is required to be strongly minimal over $\boldsymbol{a}_n$.

We assume that the reader is familiar with the arithmetical hierarchy (see §§ 7.5, 7.6 of Shoenfield [25] or Chapters 14,15 of Rogers [18]). The following lemma will be useful in verifying that certain relations are Σ_3^0 in a degree $\boldsymbol{a}$ (cf. proofs of Lemmas 3.4 and 3.8 in [9]).

LEMMA 2.2. *Let $A \subseteqq \omega \times 2^\omega \times 2^\omega$ be Π_2^0 in $\boldsymbol{a}$. Define $B \subseteqq \omega \times 2^\omega$ by*

$$B(m, f) \longleftrightarrow (\forall g \in 2^\omega) A(m, f, g) .$$

Then B is Π_2^0 in $\boldsymbol{a}$.

Proof. Immediate from the Corollary on page 187 of Shoenfield [25].

Note that $\mathcal{P}(\boldsymbol{a})$ is Σ_3^0 in $\boldsymbol{a}$. Let $\mathcal{C}$ be a nonempty subset of $\mathcal{P}(\boldsymbol{a})$ which is Σ_3^0 in $\boldsymbol{a}$. A subset D of $\mathcal{C}$ is said to be *dense* in $\mathcal{C}$ if for every $P \in \mathcal{C}$ there exists $P' \in D$ such that $P' \subseteqq P$. We shall use the following special case of a notion due to Yates [32]. A set $X \subseteqq 2^\omega$ is said to be $(\mathcal{C}, \boldsymbol{a}'')$-*comeager* if there exists a sequence of sets $\langle D_n\rangle_{n \in \omega}$ such that:

(i) each D_n is a dense subset of $\mathcal{C}$;

(ii) the relation $\{\langle n, P\rangle \mid P \in D_n\}$ is Σ_3^0 in $\boldsymbol{a}$;

(iii) $X \supseteq \bigcap_{n\in\omega}[D_n]$. It is easy to see that every $(\mathcal{C}, \boldsymbol{a}'')$-comeager set is nonempty and in fact has elements of degree $\leqq \boldsymbol{a}''$.

LEMMA 2.3. *For any degree* $\boldsymbol{a}$, *the set of* $g \in 2^\omega$ *such that* $\boldsymbol{a} \cup \deg(g)$ *is minimal over* $\boldsymbol{a}$ *is* $(\mathcal{P}(\boldsymbol{a}), \boldsymbol{a}'')$*-comeager.*

Proof. For notational simplicity was assume that $\boldsymbol{a} = \mathbf{0}$. (The easy relativization to arbitrary $\boldsymbol{a}$ is left to the reader.) Put $\mathcal{P} = \mathcal{P}(\mathbf{0})$. Let D_n be the set of $P \in \mathcal{P}$ such that (i), (ii) or (iii) holds:

(i) $\exists m\, \forall g(g \in [P] \rightarrow [n]^g(m)$ is undefined);

(ii) $\forall g, h, m\, (g, h \in [P] \rightarrow [n]^g(m) = [n]^h(m))$;

(iii) $\forall g, h\, \exists m(g, h \in [P] \wedge g \neq h \rightarrow [n]^g(m) \neq [n]^h(m))$.

Clearly if $g \in [D_n]$ then either (i) $[n]^g$ is not totally defined, or (ii) $[n]^g$ is recursive, or (iii) g is recursive in $[n]^g$. Hence if $g \in \bigcap_n [D_n]$ we see that $\deg(g)$ is minimal. By Lemma 2.2 the D_n are uniformly Σ^0_3, so it remains only to show that D_n is dense in $\mathcal{P}$.

Let $P \in \mathcal{P}$ be given. Case I: there exist m and σ such that $[n]^{P(\tau)}(m)$ is undefined for all $\tau \supseteq \sigma$. Then we choose such a σ and define $P' \subseteq P$ by $P'(\nu) = P(\sigma\nu)$. Then clearly (i) holds for P'. Case II: otherwise. Then we define $P' \subseteq P$ recursively so that $[n]^{P'(\sigma)}(lh(\sigma))$ is defined for all σ. Let us say that σ and τ *disagree if* $[n]^\sigma(m)$ and $[n]^\tau(m)$ are defined and unequal for some m. Case II(a): there exists σ such that for no $\tau_1, \tau_2 \supseteq \sigma$ do $P'(\tau_1)$ and $P'(\tau_2)$ disagree. Then we choose such a σ and define $P'' \subseteq P'$ by $P''(\nu) = P'(\sigma\nu)$. Then (ii) holds for P''. Case II(b): otherwise. Then we define $P'' \subseteq P'$ so that for all σ, $P''(\sigma 0)$ and $P''(\sigma 1)$ disagree. Then (iii) holds for P''. Thus D_n is dense in $\mathcal{P}$ and Lemma 2.3 is proved.

LEMMA 2.4. *Let* $X \subseteq 2^\omega$ *be* $(\mathcal{P}(\boldsymbol{a}), \boldsymbol{a}'')$*-comeager. Then for all* $\boldsymbol{c} \geqq \boldsymbol{a}''$ *there exists* $\boldsymbol{b} = \boldsymbol{a} \cup \deg(g)$, $g \in X$ *such that* $\boldsymbol{b}'' = \boldsymbol{b} \cup \boldsymbol{a}'' = \boldsymbol{c}$.

Proof. Again we assume $\boldsymbol{a} = \mathbf{0}$ and write $\mathcal{P} = \mathcal{P}(\mathbf{0})$. Let D^0_n be the set of $P \in \mathcal{P}$ such that $\forall f, m(f \in [P] \rightarrow [n]^f(m)$ is defined). Let D^1_n be the set of $P \in \mathcal{P}$ such that $\exists m\, \forall f(f \in [P] \rightarrow [n]^f(m)$ is undefined). The proof of Lemma 2.3 shows that $D^0_n \cup D^1_n$ is dense in $\mathcal{P}$. By Lemma 2.2 the D^i_n are uniformly Σ^0_3. Since X is $(\mathcal{P}, \mathbf{0}'')$-comeager, we also have $X \supseteq \bigcap_n [E_n]$ where the E_n are dense in $\mathcal{P}$ and uniformly Σ^0_3. Let $\boldsymbol{c} \geqq \mathbf{0}''$, $\boldsymbol{c} = \deg(h)$ be given. We shall construct a nested sequence of perfect trees $\langle P_n \rangle_{n\in\omega}$, $P_n \in \mathcal{P}$. Let P_0 be the identity tree. Choose $P_{2n+1} \subseteq P_{2n}$ so that $P_{2n+1} \in D^0_n \cup D^1_n$. Choose $P_{2n+2} \subseteq P^{h(n)}_{2n+1}$ so that $P_{2n+2} \in E_n$. Finally let $\boldsymbol{b} = \deg(g)$, $g \in \bigcap_n [P_n]$. The entire construction is recursive in $\mathbf{0}''$ using $h(n)$ only at stage $2n + 2$. Thus $\boldsymbol{b}'' \leqq \boldsymbol{c}$ since $\boldsymbol{b}''$ is the degree of the set

$$\{n \mid [n]^g \text{ is totally defined}\} = \{n \mid g \in [D_n^0]\}\ .$$

On the other hand $h(n) = m$ if and only if $g \in [P_{2n+1}^m]$, so $\boldsymbol{c} \leqq \boldsymbol{b} \cup \boldsymbol{0}''$. Clearly $g \in X$ so Lemma 2.4 is proved.

COROLLARY 2.5. *If* $\boldsymbol{c} \geqq \boldsymbol{a}''$ *then there exists* $\boldsymbol{b}$ *such that* $\boldsymbol{b}'' = \boldsymbol{b} \cup \boldsymbol{a}'' = \boldsymbol{c}$ *and* $\boldsymbol{b}$ *is minimal over* $\boldsymbol{a}$.

Proof. Immediate from Lemmas 2.3 and 2.4.

Remark 2.6. Lemma 2.3 is essentially well-known (cf. Sacks [21] and Yates [32]). Cooper [1] has shown that Corollary 2.5 remains true if double jump (″) is replaced throughout by jump (′). Cooper's proof is an infinite injury priority argument and so is much more difficult than the proof of 2.5. Nevertheless we conjecture that Theorem 2.1 also remains true if double jump is replaced by jump.

Let $\mathcal{P} = \mathcal{P}(\boldsymbol{0})$, the set of recursive perfect trees. Note that $\mathcal{P}$ is Σ_3^0. If $P \in \mathcal{P}$, a subset S of the range of P is said to be *dense open* in P if each string in the range of P is extended by an element of S, and every string in the range of P which extends an element of S belongs to S. A useful fact is that the intersection of finitely many dense open subsets of P is again dense open in P.

DEFINITION 2.7. A subset $\mathcal{C}$ of $\mathcal{P}$ is said to be *adequate* if it is nonempty and Σ_3^0 and has the following properties:

(i) Let $P \in \mathcal{C}$ and let σ be a string. Then there exists $P' \subseteqq P$ such that $P' \in \mathcal{C}$ and $P'(\phi) \supseteqq P(\sigma)$.

(ii) Let $P \in \mathcal{C}$ and let $\langle S_n \rangle_{n \in \omega}$ be a recursive sequence of recursively enumerable, dense open subsets of P. Then there exists $P' \subseteq P$ such that $P' \in \mathcal{C}$ and $P'(\sigma) \in S_n$ for all σ and $n \leqq lh(\sigma)$.

(iii) Let $P \in \mathcal{C}$. Then there exists a recursive function $p\colon \omega \rightarrow \omega$ such that if i is an index for a recursive sequence $\langle S_n \rangle_{n \in \omega}$ of recursively enumerable subsets of the range of P, then $p(i)$ is an index for a partial recursive function P' from Σ into Σ, such that: (a) $P'(\sigma) \in S_n$ provided S_n is dense open in P for all $n \leqq lh(\sigma)$; (b) $P' \subseteqq P$ and $P' \in \mathcal{C}$ provided S_n is dense open in P for all $n \in \omega$.

The prime example of an adequate set is of course $\mathcal{P}$ itself. Further examples occur in Lemma 2.10 and Remark 2.14 below. The definition of adequacy is an attempt to isolate properties which are useful in "local forcing" and "splitting" arguments (cf. pp. 350–351 of Sacks [21]). The basic idea of adequacy is embodied in clauses 2.7 (i) and (ii). Clause (iii) is merely an elaboration of (ii) which we seem to need for the proofs of Lemmas 2.10 and 2.11.

LEMMA 2.8. *Let $\mathcal{C}$ be adequate and let $A \subseteq \omega \times 2^\omega$ be Π^0_2. Let D be the set of $P \in \mathcal{C}$ such that either* (i) *or* (ii) *holds*:

(i) $\exists n \, \forall f(f \in [P] \rightarrow \neg A(n, f))$;

(ii) $\forall n \, \forall f(f \in [P] \rightarrow A(n, f))$.

Then D is Σ^0_3 and dense in $\mathcal{C}$.

Proof. That D is Σ^0_3 is immediate from Lemma 2.2. It suffices to prove density in the special case when A is Σ^0_1. Say

$$A(n, f) \longleftrightarrow (\exists \sigma \subseteq f) B(n, \sigma)$$

where B is recursive and $B(n, \tau)$ if $B(n, \sigma)$, $\sigma \subseteq \tau$. Let $P \in \mathcal{C}$ be given. We seek $P' \subseteq P$ such that $P' \in D$. Case I: there exist n and σ such that $B(n, P(\tau))$ for no $\tau \supseteq \sigma$. By 2.7 (i) let $P' \subseteq P$ be such that $P' \in \mathcal{C}$ and $P'(\phi) \supseteq P(\sigma)$. Clearly 2.8 (i) holds for P'. Case II: otherwise. Then for each n the set of $P(\mu)$ such that $B(n, P(\mu))$ holds is dense open in P. Thus by 2.7 (ii) there is $P' \subseteq P$ such that $P' \in \mathcal{C}$ and $B(n, P'(\sigma))$ for all σ and $n \leqq lh(\sigma)$. Thus 2.8 (ii) holds for P' and the lemma is proved.

If $P, Q \in \mathcal{P}$ we say that Q is *P-based* if there exists an integer i such that

$$[P] \subseteq \{f \mid [i]^f \colon \Sigma \longrightarrow \Sigma \text{ is a perfect tree}\}$$

and

$$[Q] = [P, i] = \{f \oplus g \mid f \in [P] \text{ and } g \in [[i]^f]\} .$$

If $\mathcal{C}$ is adequate, let $\mathcal{C}^+$ be the set of all $Q \in \mathcal{P}$ such that Q is P-based for some $P \in \mathcal{C}$. By Lemma 2.2 $\mathcal{C}^+$ is again a Σ^0_3 subset of $\mathcal{P}$. If $Y \subseteq 2^\omega$ and $f \in 2^\omega$ we write $Y_f = \{g \mid f \oplus g \in Y\}$ and

$$Y^- = \{f \mid Y_f \text{ is } (\mathcal{P}(\boldsymbol{a}), \boldsymbol{a}'')\text{-comeager where } \boldsymbol{a} = \deg(f)\} .$$

The next lemma embodies an "iterated forcing" argument which links the definitions of $\mathcal{C}^+$ and Y^-. The method of "iterated forcing" has been studied in the context of transitive models of ZF set theory by Solovay and Tennenbaum [28] and in the context of admissible sets by Sacks [22].

LEMMA 2.9. *Let $\mathcal{C}$ be adequate. Suppose that $Y \subseteq 2^\omega$ is $(\mathcal{C}^+, \mathbf{0}'')$-comeager. Then Y^- is $(\mathcal{C}, \mathbf{0}'')$-comeager.*

Proof. Let $Y \supseteq \bigcap_n [D_n]$ where the D_n are dense in $\mathcal{C}^+$ and uniformly Σ^0_3. For each n and i, using Lemma 2.8 and the density of D_n in $\mathcal{C}^+$, we can effectively find D^-_{ni} a dense Σ^0_3 subset of $\mathcal{C}$, such that for each $P \in D^-_{ni}$ either (i) or (ii) holds:

(i) $\forall f(f \in [P] \rightarrow [i]^f$ is not a perfect tree);

(ii) $\forall f(f \in [P] \rightarrow [i]^f$ is a perfect tree) and there exists a P-based $Q \in D_n$ such that $[Q] \subseteq [P, i]$.

It is then easy to check that $Y^- \supseteq \bigcap_{ni} [D^-_{ni}]$ so Y^- is $(\mathcal{C}, \mathbf{0}'')$-comeager.

We define $\mathcal{P}^1 = \mathcal{P}$ and $\mathcal{P}^{n+1} = (\mathcal{P}^n)^+$ for all $n \geqq 1$.

LEMMA 2.10. *If $\mathcal{C}$ is adequate, then $\mathcal{C}^+$ is adequate. Hence, for each $n \geqq 1$, $\mathcal{P}^n$ is adequate.*

Proof. Let $Q \in \mathcal{C}^+$, $\mathcal{C}$ adequate, and let $\langle T_n \rangle_{n \in \omega}$ be a recursive sequence of recursively enumerable, dense open subsets of Q. We seek $Q' \subseteq Q$ such that $Q' \in \mathcal{C}$ and $Q'(\rho) \in T_n$ for all ρ and $n \leqq lh(\rho)$. This will prove 2.7 (ii) for $\mathcal{C}^+$. Let Q be P-based, $P \in \mathcal{C}$. Our Q' will be P'-based with $P' \subseteq P$, $P' \in \mathcal{C}$, and P' obtained as in 2.7 (ii) from a sequence $\langle S_n \rangle_{n \in \omega}$ of dense open subsets of P.

Let i be an integer such that $[Q] = [P, i]$. Let R be the partial mapping from $\Sigma \times \Sigma$ into Σ defined by $R_\sigma(\tau) = [i]^{P(\sigma)}(\tau)$. There will exist a mapping $R' \colon \Sigma \times \Sigma \to \Sigma$ such that $Q'(\sigma \oplus \tau) = P'(\sigma) \oplus R'_\sigma(\tau)$ whenever $lh(\tau) \leqq lh(\sigma)$. Using 2.7 (iii) and the recursion theorem, we may define the S_n, P', and R' simultaneously as follows. Let S_0 be the set of $P(\mu)$ such that $P(\mu) \oplus R_\mu(\nu) \in T_0 \cap T_1$ for some ν. The dense openness of S_0 in P follows from the dense openness of $T_0 \cap T_1$ in Q. Choose $P'(\phi) \in S_0$ and $R'_\phi(\phi)$ so that $P'(\phi) \oplus R'_\phi(\phi) \in T_0$. Assume inductively that S_i, $P'(\sigma)$ and $R'_\sigma(\tau)$ have been defined for $i \leqq n$, $lh(\tau) \leqq lh(\sigma) \leqq n$. Two strings are said to be *incompatible* if neither extends the other. Let S_{n+1} be the set of $P(\mu)$ such that for all σ, τ of length n, if $P(\mu) \supseteq P'(\sigma)$ then there exist ν_i, $i < 4$ such that the $P(\mu) \oplus R_\mu(\nu_i)$ extend $P'(\sigma) \oplus R'_\sigma(\tau)$ and are pairwise incompatible and belong to $T_{2n+2} \cap T_{2n+3}$. The dense openness of S_{n+1} in P follows from the dense openness of $T_{2n+2} \cap T_{2n+3}$ in Q. Now for σ, τ of length n and $i, j \in \{0, 1\}$ define P' and R' so that the $P'(\sigma i) \oplus R'_{\sigma i}(\tau j)$ are pairwise incompatible and belong to $T_{2n+2} \cap T_{2n+3}$. This completes the proof of 2.7 (ii) for $\mathcal{C}^+$. The proof of the other clauses is left to the reader.

The special case $\mathcal{C} = \mathcal{P}$ of the next lemma goes back to D. Titgemeyer (cf. §11 of Sacks [20]).

LEMMA 2.11. *Let $\mathcal{C}$ be adequate. Then $\{f \oplus g \mid \deg(f \oplus g)$ is strongly minimal over $\deg(f)\}$ is $(\mathcal{C}^+, \mathbf{0}'')$-comeager.*

Proof. For each n let D_n be the set of $Q \in \mathcal{C}^+$ such that either (i), (ii) or (iii) holds:

(i) $\exists m \, \forall h \, (h \in [Q] \to [n]^h(m)$ is undefined$)$;

(ii) $\forall f, g_1, g_2, m(f \oplus g_1, f \oplus g_2 \in [Q] \to [n]^{f \oplus g_1}(m) = [n]^{f \oplus g_2}(m))$;

(iii) $\forall h_1, h_2 \, \exists m(h_1, h_2 \in [Q] \wedge h_1 \neq h_2 \to [n]^{h_1}(m) \neq [n]^{h_2}(m))$.

If $f \oplus g \in [D_n]$ then either (i) $[n]^{f \oplus g}$ is not totally defined, or (ii) $[n]^{f \oplus g}$ is recursive in f, or (iii) $f \oplus g$ is recursive in $[n]^{f \oplus g}$. Hence if $f \oplus g \in \bigcap_n [D_n]$ it follows that $\deg(f \oplus g)$ is strongly minimal over $\deg(f)$. Moreover the D_n are uniformly Σ_3^0 by Lemma 2.2, so it remains only to show that D_n is dense in $\mathcal{C}^+$.

Let $Q \in \mathcal{C}^+$ be given. We seek $Q' \subseteqq Q$ such that $Q' \in D_n$. If 2.11 (i) holds for some $Q' \subseteqq Q$, $Q' \in \mathcal{C}^+$ then we are done. If not, then by 2.10 and 2.8 we may assume that $\forall h$, $m(h \in [Q] \rightarrow [n]^h(m)$ is defined$)$. Let Q be P-based, $P \in \mathcal{C}$, and proceed as in the proof of Lemma 2.10. We say that strings σ and τ *disagree* if $[n]^\sigma(m)$ and $[n]^\tau(m)$ are defined and unequal for some m. Case I: there exist σ, τ such that $lh(\sigma) = lh(\tau)$ and for no $\sigma' \supseteqq \sigma$, $\tau_1' \supseteqq \tau$, $\tau_2' \supseteqq \tau$ do $P(\sigma') \oplus R_{\sigma'}(\tau_1')$ and $P(\sigma') \oplus R_{\sigma'}(\tau_2')$ disagree. Then we choose such σ, τ and by 2.10 and 2.7 (i) we can find $Q' \subseteqq Q$ such that $Q' \in \mathcal{C}^+$ and $Q'(\phi) \supseteqq P(\sigma) \oplus R_\sigma(\tau)$. Then 2.11 (ii) holds for this Q'. Case II: otherwise. In this case we shall imitate the proof of Lemma 2.10 to obtain $Q' \subseteqq Q$ such that $Q' \in \mathcal{C}^+$ and 2.11 (iii) holds for Q'. Define S_0, $P'(\phi)$ and $R_\phi'(\phi)$ arbitrarily. Assume inductively that S_i, $P'(\sigma)$, $R_\sigma'(\tau)$ have been defined for $i \leqq n$, $lh(\tau) \leqq lh(\sigma) \leqq n$. Let S_{n+1} be the set of $P(\mu)$ such that for all σ, τ of length n, if $P(\mu) \supseteqq P'(\sigma)$ then there exist ν_i, $i < 4$, such that the $P(\mu) \oplus R_\mu(\nu_i)$ extend $P'(\sigma) \oplus R_\sigma'(\tau)$ and disagree pairwise. The dense openness of S_{n+1} in P follows from the fact that we are not in Case I. For σ, τ of length n and $i, j \in \{0, 1\}$ define P', R' so that $P'(\sigma i) \oplus R_{\sigma i}'(\tau j)$ disagree pairwise for $i, j \in \{0, 1\}$. This completes the proof of Lemma 2.11.

COROLLARY 2.12. *Let $\mathcal{C}$ be adequate. Then* $\{f \mid \deg(f)$ *has a strongly minimal cover*$\}$ *is* $(\mathcal{C}, \mathbf{0}'')$*-comeager*.

Proof. Immediate from Lemmas 2.11 and 2.9.

Remark 2.13. The theorem that every degree has a minimal cover is due to Spector [29] who in the same paper raised the following problem: Which degrees have strongly minimal covers? Corollary 2.12 is a new, positive result on Spector's problem. For interest's sake we list here the other known results on Spector's problem. First, the theorem of Lachlan and Lebeuf [14] implies that for any countable upper semilattice L with greatest and least elements, there exists a degree $\boldsymbol{a}$ such that $\{\boldsymbol{d} \mid \boldsymbol{d} \leqq \boldsymbol{a}\}$ is isomorphic to L and $\boldsymbol{a}$ has a strongly minimal cover. There is also a surprising result of Cooper [2] which says that there exists a nonzero recursively enumerable degree which has a strongly minimal cover. On the negative side, there is the well-known theorem of Friedberg [4] which immediately implies that no degree $\geqq \mathbf{0}'$ is a strongly minimal cover. A related theorem of Jockusch [8] implies that $\{f \mid$no degree $\geqq \deg(f)$ is a strongly minimal cover$\}$ is comeager in 2^ω. An open question of long standing is whether every minimal degree has a strongly minimal cover.

Remark 2.14. The notion of adequacy (Definition 2.7) appears to be convenient in that it permits formulation of strong results. For instance, the construction of Yates [31, §3] can be adapted to show that, for any finite distributive lattice L, there exists an adequate set $\mathcal{C}_L$ such that

$$\{f \mid L \text{ is isomorphic to } \{\boldsymbol{d} \mid \boldsymbol{d} \leqq \deg(f)\}\}$$

is $(\mathcal{C}_L, \mathbf{0}'')$-comeager. (The set $\mathcal{P}^n$ of 2.10 is essentially $\mathcal{C}_L$ where L is a linear ordering of size $n + 1$.) In another direction, ideas of Jockusch [7], Miller and Martin [17], and Sasso [23] can be combined with the proof of Lemma 2.8 to show that if $\mathcal{C}$ is adequate, then $\{f \mid \boldsymbol{a} = \deg(f)$ is bi-immune free and hyperimmune free and $\boldsymbol{a}' > \boldsymbol{a} \cup \mathbf{0}'\}$ is $(\mathcal{C}, \mathbf{0}'')$-comeager. Finally, the proof of Lemma 2.4 can be combined with Lemma 2.8 to yield the following result. Let $\mathcal{C}$ be adequate, and let $X \subseteqq 2^\omega$ be $(\mathcal{C}, \mathbf{0}'')$-comeager. Then for all $\boldsymbol{b} \geqq \mathbf{0}''$ there exists $\boldsymbol{a} = \deg(f)$, $f \in X$ such that $\boldsymbol{a}'' = \boldsymbol{a} \cup \mathbf{0}'' = \boldsymbol{b}$. We shall not prove these results here.

LEMMA 2.15. *There exists a sequence of sets* $\langle Y_n \rangle_{1 \leqq n \in \omega}$ *such that*

(i) Y_1 *is* $(\mathcal{P}, \mathbf{0}'')$-*comeager*;

(ii) $Y_1 \subseteqq \{f \mid \deg(f)$ *is minimal*$\}$;

(iii) *for each* $n \geqq 1$, $Y_n \subseteqq Y_{n+1}^-$;

(iv) *for each* $n \geqq 1$, $Y_{n+1} \subseteqq \{f \oplus g \mid \deg(f \oplus g)$ *is strongly minimal over* $\deg(f)\}$.

Proof. Let X_1 be the set of f such that $\deg(f)$ is minimal. For $n \geqq 2$ let X_n be the set of $f \oplus g$ such that $\deg(f \oplus g)$ is strongly minimal over $\deg(f)$. Lemmas 2.4 and 2.11 imply that X_n is $(\mathcal{P}^n, \mathbf{0}'')$-comeager. We put $Y_n = \bigcap_i X_n^i$ where $X_n^0 = X_n$, $X_n^{i+1} = (X_{n+1}^i)^-$. By Lemma 2.9 each X_n^i is $(\mathcal{P}^n, \mathbf{0}'')$-comeager. Furthermore, the uniformity of the proof of Lemma 2.9 implies that the X_n^i are uniformly $(\mathcal{P}^n, \mathbf{0}'')$-comeager, i.e., $X_n^i \supseteq \bigcap_j [D_{nij}]$ where D_{nij} is dense in $\mathcal{P}^n$ and $\{\langle P, n, i, j \rangle \mid P \in D_{nij}\}$ is Σ_3^0. Properties (i)–(iv) are immediate.

Proof of Theorem 2.1. Let Y_n, $1 \leqq n \in \omega$ be as in Lemma 2.15. By 2.15 (i) and 2.4 we can find $\boldsymbol{a}_1 = \deg(f_1)$, $f_1 \in Y_1$ with $\boldsymbol{a}_1'' = \boldsymbol{a}_1 \cup \mathbf{0}'' = \boldsymbol{b}_1$. By 2.15 (ii) $\boldsymbol{a}_1$ is minimal. Assume inductively that $\boldsymbol{a}_n = \deg(f_n)$ has been defined, $f_n \in Y_n$, $\boldsymbol{a}_n'' = \boldsymbol{a}_n \cup \mathbf{0}'' = \boldsymbol{b}_n$. Then of course $\boldsymbol{a}_n'' \leqq \boldsymbol{b}_{n+1}$ so by 2.15 (iii) and 2.4 we can find $\boldsymbol{a}_{n+1} = \deg(f_{n+1})$, $f_{n+1} = f_n \oplus g_n \in Y_{n+1}$ such that $\boldsymbol{a}_{n+1}'' = \boldsymbol{a}_{n+1} \cup \mathbf{0}'' = \boldsymbol{b}_{n+1}$. By 2.15 (iv) $\boldsymbol{a}_{n+1}$ is strongly minimal over $\boldsymbol{a}_n$. This completes the proof.

3. First-order definability with jump

By *analysis* we mean second-order arithmetic, i.e., the second-order theory of $\langle \omega, +, \cdot \rangle$ or equivalently the first-order theory of the 2-sorted structure

$$\mathfrak{A} = \langle 2^\omega, \omega, +, \cdot, E \rangle$$

where $E: 2^\omega \times \omega \rightarrow \omega$ is defined by $E(f, n) = f(n)$. General information on analysis can be found in the textbooks of Rogers [18, §16.2] and Shoenfield [25, §8.5]. Let

$$\mathfrak{D} = \langle D, \cup, j \rangle$$

where $\langle D, \cup \rangle$ is the semilattice of degrees and j is the jump operator. The purpose of this section is to prove the following theorem which says that the first-order languages associated with $\mathfrak{A}$ and $\mathfrak{D}$ have roughly the same expressive power.

MAIN THEOREM. *Let S be a set of degrees such that $S \subseteq \{\boldsymbol{d} \mid \boldsymbol{d} \geqq \boldsymbol{0}^{(\omega)}\}$. Then S is first-order definable in $\mathfrak{D}$ if and only if*

$$\{f \mid \deg(f) \in S\} \subseteq 2^\omega$$

is first-order definable in $\mathfrak{A}$.

The proof of the Main Theorem will involve a certain translation of the language of analysis into the language of degree theory with jump. In this translation, the integer n will be interpreted as the degree $\boldsymbol{0}^{(n)}$. The relations $\{\langle m, n, k \rangle \mid m + n = k\}$ and $\{\langle m, n, k \rangle \mid m \cdot n = k\}$ on ω will be interpreted as the corresponding relations on $\Omega = \{\boldsymbol{0}^{(n)} \mid n \in \omega\}$. A function $g \in 2^\omega$ will be interpreted non-uniquely as a degree $\boldsymbol{a}$ such that for all n, $g(n) = 1$ if and only if $\boldsymbol{0}^{(n+1)} \leqq \boldsymbol{a} \cup \boldsymbol{0}^{(n)}$. Theorem 3.7 below says that there exists a faithful translation with the features just mentioned. Then Lemma 3.11 sets the stage for Theorem 3.12 which generalizes the Main Theorem to n-ary relations.

A nonempty subset I of D is called an *ideal* if
(i) $\boldsymbol{c} \leqq \boldsymbol{d}$, $\boldsymbol{d} \in I$ imply $\boldsymbol{c} \in I$;
(ii) $\boldsymbol{c}, \boldsymbol{d} \in I$ imply $\boldsymbol{c} \cup \boldsymbol{d} \in I$.

This terminology is standard (cf. Grätzer [6, §6]). Given two degrees $\boldsymbol{a}, \boldsymbol{b}$ it is easy to check that

$$I(\boldsymbol{a}, \boldsymbol{b}) = \{\boldsymbol{d} \mid \boldsymbol{d} \leqq \boldsymbol{a} \text{ and } \boldsymbol{d} \leqq \boldsymbol{b}\}$$

is a countable ideal. The next lemma says that all countable ideals are of this form.

LEMMA 3.1. *For any countable ideal I there exist degrees $\boldsymbol{a}, \boldsymbol{b}$ such that $I = I(\boldsymbol{a}, \boldsymbol{b})$.*

Proof. This follows from Theorem 3 of Spector [29]. See also Sacks [20, §2].

Lemma 3.1 is very useful because it tells us that the first-order language of $\langle D, \cup \rangle$ is strong enough to express quantification over all countable ideals. This idea will be exploited in the proof of the next lemma, where mention of a countable ideal I is to be tacitly replaced by mention of degrees $\boldsymbol{a}, \boldsymbol{b}$ such that $I(\boldsymbol{a}, \boldsymbol{b}) = I$. Further exploitation of Lemma 3.1 has occurred in [9].

LEMMA 3.2. *The set of degrees*

$$\Omega = \{\mathbf{0}^{(n)} \mid n \in \omega\}$$

is first-order definable in $\mathfrak{D}$.

Proof. By the Main Lemma (Theorem 2.1) there exists a countable ideal $I = \{\boldsymbol{a}_n \mid n \in \omega\}$ of order type ω, such that $\boldsymbol{a}_n \cup \mathbf{0}^{(2)} = \mathbf{0}^{(n+2)}$ for all n. If I is any such ideal then we have $\boldsymbol{b} \in \Omega$ if and only if $\boldsymbol{b} = \mathbf{0}$ or $\boldsymbol{b} = \mathbf{0}^{(1)}$ or $\boldsymbol{b} = \boldsymbol{a} \cup \mathbf{0}^{(2)}$ for some $\boldsymbol{a} \in I$. Therefore by Lemma 3.1 it suffices to show that the set of all such I is first-order definable in $\mathfrak{D}$. Well, I is such an ideal if and only if I is linearly ordered, every element of I has a minimal cover in I, every proper subideal of I has top element, and $(\boldsymbol{c} \cup \mathbf{0}^{(2)})' = \boldsymbol{d} \cup \mathbf{0}^{(2)}$ whenever $\boldsymbol{c}, \boldsymbol{d}$ are consecutive elements of I. By Lemma 3.1 all of these conditions on I are first-order, so Lemma 3.2 is proved.

LEMMA 3.3. *Let m be a positive integer and let* $\mathbf{0}^{(2m)} \leqq \boldsymbol{b}_1 \leqq \cdots \leqq \boldsymbol{b}_m$ *be given. Then there exists an initial segment* $\mathbf{0} < \boldsymbol{a}_1 < \cdots < \boldsymbol{a}_m$ *such that*

$$\boldsymbol{a}_i \cup \mathbf{0}^{(2m-2i)} < \boldsymbol{a}_i \cup \mathbf{0}^{(2m-2i+2)} = \boldsymbol{a}_i \cup \mathbf{0}^{(2m)} = \boldsymbol{b}_i$$

for $1 \leqq i \leqq m$.

Proof. The special case $m = 1$ is essentially just Corollary 2.5. Assume inductively that $m > 1$ and that Lemma 3.3 holds with $m - 1$ instead of m. Relativizing this statement to $\mathbf{0}^{(2)}$ we obtain degrees $\mathbf{0}^{(2)} < \boldsymbol{c}_1 < \cdots < \boldsymbol{c}_{m-1}$ such that

$$\boldsymbol{c}_i \cup \mathbf{0}^{(2m-2i)} < \boldsymbol{c}_i \cup \mathbf{0}^{(2m-2i+2)} = \boldsymbol{c}_i \cup \mathbf{0}^{(2m)} = \boldsymbol{b}_i$$

for $1 \leqq i < m$. Then the Main Lemma (Theorem 2.1) yields an initial segment $\mathbf{0} < \boldsymbol{a}_1 < \cdots < \boldsymbol{a}_{m-1} < \boldsymbol{a}_m$ such that $\boldsymbol{a}_i \cup \mathbf{0}^{(2)} = \boldsymbol{c}_i$ for $1 \leqq i < m$ and $\boldsymbol{a}_m \cup \mathbf{0}^{(2)} = \boldsymbol{b}_m$. From this it follows easily that the $\boldsymbol{a}_i$ satisfy the conclusion of Lemma 3.3.

LEMMA 3.4. *The ternary relations* $\{\langle \mathbf{0}^{(m)}, \mathbf{0}^{(n)}, \mathbf{0}^{(k)} \rangle \mid m + n = k\}$ *and* $\{\langle \mathbf{0}^{(m)}, \mathbf{0}^{(n)}, \mathbf{0}^{(\kappa)} \rangle \mid m \cdot n = k\}$ *of "addition" and "multiplication" on* Ω *are first-order definable in* $\mathfrak{D}$.

Proof. A slight modification of the proof of Lemma 3.2 shows that

$$2\Omega = \{\mathbf{0}^{(2n)} \mid n \in \omega\}$$

is first-order definable in $\mathfrak{D}$. To define addition and multiplication on Ω it suffices to define them on 2Ω. We define $+$ on 2Ω as follows: $2m \leqq 2n$ and

$2m + 2n = 2k$ if and only if $2m \leqq 2n$ and there exists a degree $\boldsymbol{d}$ such that the following holds. Use $\boldsymbol{d}$ to define a binary relation $R = R_d$ on 2Ω by $R(\mathbf{0}^{(2i)}, \mathbf{0}^{(2j)})$ if and only if

$$\exists \boldsymbol{a} \leqq \boldsymbol{d}(\boldsymbol{a} \cup \mathbf{0}^{(2i-2)} < \boldsymbol{a} \cup \mathbf{0}^{(2i)} = \boldsymbol{a} \cup \mathbf{0}^{(2m)} = \mathbf{0}^{(2j)} \text{ and } \boldsymbol{a} > \mathbf{0}) .$$

Then R is an order-reversing one-one correspondence between $\{\mathbf{0}^{(2i)} \mid 1 \leqq i \leqq m\}$ and $\{\mathbf{0}^{(2j)} \mid n < j \leqq k\}$. The correctness of this definition follows from Lemma 3.3. Now armed with $+$ we define multiplication on 2Ω as follows: $2m \leqq 2n$ and $2m \cdot 2n = 4k$ if and only if $2m \leqq 2n$ and there exists a degree $\boldsymbol{d}$ such that R_d is the order-reversing one-one correspondence between $\{\mathbf{0}^{(2i)} \mid 1 \leqq i \leqq m\}$ and $\{\mathbf{0}^{(4ni)} \mid 1 \leqq i \leqq m\}$ and $R_d(\mathbf{0}^{(2)}, \mathbf{0}^{(4k)})$ holds. Again we are justified by Lemma 3.3. This completes the proof of Lemma 3.4.

We define a special mapping $\Gamma: D \longrightarrow 2^\omega$ by

$$\Gamma(\boldsymbol{a})(n) = \begin{cases} 1 & \text{if } \mathbf{0}^{(n+1)} \leqq \boldsymbol{a} \cup \mathbf{0}^{(n)}, \\ 0 & \text{otherwise} . \end{cases}$$

LEMMA 3.5. *Γ is onto*, i.e., *given $g \in 2^\omega$ we can find a degree $\boldsymbol{a}$ such that $\Gamma(\boldsymbol{a}) = g$.*

The proof of Lemma 3.5 is based on the following sublemma.

SUBLEMMA 3.6. *Suppose $P \in \mathcal{P}(c)$. Then we can find $P_0', P_1' \subseteqq P$ such that $P_0', P_1' \in \mathcal{P}(\boldsymbol{c}')$ and*

(i) *$\boldsymbol{c}' \nleqq \boldsymbol{a} \cup \boldsymbol{c}$ whenever $\boldsymbol{a} = \deg(f), f \in [P_0']$;*

(ii) *$\boldsymbol{c}' \leqq \boldsymbol{a} \cup \boldsymbol{c}$ whenever $\boldsymbol{a} = \deg(f), f \in [P_1']$.*

Proof. Let $\boldsymbol{c} = \deg(h)$. Let S_n be the set of $P(\mu)$ such that either $[n]^{P(\mu)\oplus h}(n)$ is defined, or $[n]^{P(\nu)\oplus h}(n)$ is undefined for all $\nu \supseteqq \mu$. Clearly the S_n are uniformly recursive in $\boldsymbol{c}'$ and dense open in P. Hence we can find $P_0' \subseteqq P$ recursive in $\boldsymbol{c}'$ such that $P_0'(\sigma) \in S_n$ whenever $lh(\sigma) = n$. Thus $(f \oplus h)^*$ is recursive in $f \oplus h^*$ for all $f \in [P_0']$, so (i) holds a fortiori. Define $P_1' \subseteqq P$ by $P_1'(\sigma) = P(\sigma \oplus h^*)$ for all σ. Then P_1' is recursive in $\boldsymbol{c}'$, and clearly h^* is recursive in $f \oplus h$ for all $f \in [P_1']$, so (ii) holds. This proves the sublemma.

Now to prove Lemma 3.5, let $g \in 2^\omega$ be given. We shall define a nested sequence $\langle P_n \rangle_{n \in \omega}$ such that $P_n \in \mathcal{P}(\mathbf{0}^{(n)})$ for all n. Let P_0 be the identity tree. If $g(n) = 0$ choose $P_{n+1} \subseteqq P_n$ by 3.6 (i). If $g(n) = 1$ choose $P_{n+1} \subseteqq P_n$ by 3.6 (ii). Finally let $\boldsymbol{a} = \deg(f)$, $f \in \bigcap_n [P_n]$. By construction $\mathbf{0}^{(n+1)} \leqq \boldsymbol{a} \cup \mathbf{0}^{(n)}$ if and only if $g(n) = 1$ so we are done.

THEOREM 3.7. *Let φ be a formula in the language of analysis containing i free function variables and j free number variables. Then we can effectively find a formula φ^* in the language of degree theory with jump,* *such that for all* $\boldsymbol{d}_1, \cdots, \boldsymbol{d}_i \in D$ *and* $n_1, \cdots, n_j \in \omega$,

$$\mathfrak{A} \models \varphi[\Gamma(\boldsymbol{d}_1), \cdots, \Gamma(\boldsymbol{d}_i), n_1, \cdots, n_j]$$

if and only if

$$\mathfrak{D} \models \varphi^*[\boldsymbol{d}_1, \cdots, \boldsymbol{d}_i, \boldsymbol{0}^{(n_1)}, \cdots, \boldsymbol{0}^{(n_j)}] .$$

Proof. Straightforward using Lemmas 3.2, 3.4, and 3.5.

The next lemma is a useful variant of 3.5. Note that if $\boldsymbol{a}$ is any degree then $\deg(\Gamma(\boldsymbol{a})) \leqq \boldsymbol{a}^{(\omega)}$.

LEMMA 3.8. *For any degree* $\boldsymbol{b} \geqq \boldsymbol{0}^{(\omega)}$ *there exists a degree* $\boldsymbol{a}$ *such that*

$$\deg(\Gamma(\boldsymbol{a})) = \boldsymbol{a}^{(\omega)} = \boldsymbol{a} \cup \boldsymbol{0}^{(\omega)} = \boldsymbol{b} .$$

SUBLEMMA 3.9. *Suppose* $Q \in \mathcal{P}(\boldsymbol{c})$. *Then for any* $i \in \omega$ *we can find* $P \subseteq Q$, $P \in \mathcal{P}(\boldsymbol{c}^{(i+85)})$, *and* $m \in \{0, 1\}$ *such that* $[P] \subseteq \{f \mid f^{(\omega)}(i) = m\}$.

Proof. This is easily seen by a "local forcing" argument. See Lemma 3.1 of Sacks [21].

Now to prove Lemma 3.8, let $\boldsymbol{b} = \deg(g)$ be given. We shall construct a nested sequence of perfect trees $\langle P_k \rangle_{k \in \omega}$. There will also be a recursive sequence of integers $\langle n_k \rangle_{k \in \omega}$ such that $P_k \in \mathcal{P}(\boldsymbol{0}^{(n_k)})$. Put $n_0 = 0$ and let P_0 be the identity tree. If $k = 2i$, put $n_{k+1} = n_k + i + 85$ and use Sublemma 3.9 to find $P_{k+1} \subseteq P_k^{g(i)}$ and $m \in \{0, 1\}$ such that $[P_{k+1}] \subseteq \{f \mid f^{(\omega)}(i) = m\}$. If $k = 2i + 1$, put $n_{k+1} = n_k + 1$ and use Sublemma 3.6 as in the proof of Lemma 3.5, i.e., $P_{k+1} \subseteq P_k$ insures that $\boldsymbol{0}^{(n_k+1)} \leqq \boldsymbol{a} \cup \boldsymbol{0}^{(n_k)}$ if and only if $g(i) = 1$ where of course $\boldsymbol{a} = \deg(f)$, $f \in \bigcap_k [P_k]$. The entire construction is recursive in $\boldsymbol{0}^{(\omega)}$ with the use of $g(i)$ only at stages $2i + 1$ and $2i + 2$. Thus $\boldsymbol{a}^{(\omega)} \leqq \boldsymbol{b}$. On the other hand $g(i) = m$ if and only if $f \in [P_{2i}^m]$ so $\boldsymbol{b} \leqq \boldsymbol{a} \cup \boldsymbol{0}^{(\omega)}$. Finally $g(i) = \Gamma(\boldsymbol{a})(n_{2i+1})$ so $\boldsymbol{b} \leqq \deg(\Gamma(\boldsymbol{a}))$. This completes the proof.

LEMMA 3.10. *The binary relation* $\{\langle \boldsymbol{a}, \boldsymbol{b} \rangle \mid \boldsymbol{a}^{(\omega)} = \boldsymbol{b}\}$ *is first-order definable in* $\mathfrak{D}$.

Proof. Immediate from Lemma 3.1 and the following theorem of Sacks [21]: $\boldsymbol{a}^{(\omega)}$ is the smallest degree of the form $\boldsymbol{d}^{(2)}$ such that $\boldsymbol{d}$ is an upper bound of $\{\boldsymbol{a}^{(n)} \mid n \in \omega\}$.

LEMMA 3.11. *The binary relation* $\{\langle \boldsymbol{a}, \boldsymbol{b} \rangle \mid \deg(\Gamma(\boldsymbol{a})) = \boldsymbol{b} \geqq \boldsymbol{0}^{(\omega)}\}$ *is first-order definable in* $\mathfrak{D}$.

Proof. By Lemma 3.8 a degree $\boldsymbol{b} \geqq \boldsymbol{0}^{(\omega)}$ can be characterized as the largest degree of the form $\deg(\Gamma(\boldsymbol{a}))$ where $\boldsymbol{a}^{(\omega)} = \boldsymbol{b}$. The relation $\{\langle f_1, f_2 \rangle \mid f_1$ is recursive in $f_2\}$ is of course definable in analysis, so by Theorem 3.7 the relation $\{\langle \boldsymbol{a}_1, \boldsymbol{a}_2 \rangle \mid \Gamma(\boldsymbol{a}_1)$ is recursive in $\Gamma(\boldsymbol{a}_2)\}$ is first-order definable in $\mathfrak{D}$. These observations plus Lemma 3.10 are easily combined to yield Lemma 3.11.

We are now ready for the principal theorem of this section.

THEOREM 3.12. *Let n be a positive integer and let R be a set of n-tuples of degrees $\geqq \mathbf{0}^{(\omega)}$. Suppose that*

$$R^* = \{\langle f_1, \cdots, f_n\rangle \mid R(\deg(f_1), \cdots, \deg(f_n))\}$$

is definable in analysis. Then R is first-order definable in $\mathfrak{D}$.

Proof. Immediate from Theorem 3.7 and Lemma 3.11.

COROLLARY 3.13. *Each of the following relations is first-order definable in $\mathfrak{D}$:*

(i) $\{\langle \boldsymbol{a}, \boldsymbol{b}\rangle \mid \boldsymbol{a}$ *is hyperarithmetical in* $\boldsymbol{b}\}$;

(ii) $\{\langle \boldsymbol{a}, \boldsymbol{b}\rangle \mid \boldsymbol{a}$ *is ramified analytical in* $\boldsymbol{b}\}$;

(iii) $\{\langle \boldsymbol{a}, \boldsymbol{b}\rangle \mid \boldsymbol{a}$ *is constructible from* $\boldsymbol{b}\}$;

(iv) *for each* $n \geqq 2$, $\{\langle \boldsymbol{a}, \boldsymbol{b}\rangle \mid \boldsymbol{a}$ *is* Δ^1_n *in* $\boldsymbol{b}\}$.

COROLLARY 3.14. *Assume that $0^\#$ exists. Then the degree of $0^\#$ is first-order definable in $\mathfrak{D}$.*

Remark 3.15. Corollaries 3.13 and 3.14 are somewhat unsatisfying because the first-order definitions provided by their proofs look extremely artificial from the viewpoint of degree theory. It is desirable to replace these definitions by degree-theoretically natural ones. In [9] we exhibited degree-theoretically natural definitions of 3.13 (i) and (ii) and of the degree of Kleene's 0. It is also possible to exhibit a degree-theoretically natural definition of 3.13 (iii). It would be interesting and worthwhile to do the same for 3.13 (iv) and other relations of hierarchy theory, and for the degree of $0^\#$.

4. Inhomogeneity with jump

We begin this section by reviewing the background of the so-called homogeneity problem. Suppose that we have a theorem of the form $\mathfrak{D} \models \varphi$ where $\mathfrak{D} = \langle D, \cup, j\rangle$ and φ is a sentence in the language of degree theory with jump. If the proof of this theorem uses only "standard methods," then it is usually possible to generalize the theorem in a routine way by *relativization*. This means that for any fixed degree $\boldsymbol{c} = \deg(h)$ we can insert h at appropriate places in the proof of $\mathfrak{D} \models \varphi$ to obtain a proof of a new theorem $\mathfrak{D}_c \models \varphi$ where $\mathfrak{D}_c$ is the substructure of $\mathfrak{D}$ whose universe is $\{\boldsymbol{a} \mid \boldsymbol{a} \geqq \boldsymbol{c}\}$. For example, the existence of a minimal degree is expressed by $\mathfrak{D} \models \mu$ where μ is a certain first-order sentence. Relativization to $\boldsymbol{c}$ yields $\mathfrak{D}_c \models \mu$ which expresses the existence of a minimal cover of $\boldsymbol{c}$. (See also Rogers [18, §9.3].)

More examples are provided by the results of Sections 2 and 3, all of which relativize straightforwardly. Thus for any degree $\boldsymbol{c}$ there is a faithful translation of analysis into the first-order theory of $\mathfrak{D}_c$ in terms of the set

$$\Omega_c = \{\boldsymbol{c}^{(n)} \mid n \in \omega\}$$

and the mapping

$$\Gamma_c\colon \{\boldsymbol{a} \mid \boldsymbol{a} \geqq \boldsymbol{c}\} \longrightarrow 2^\omega$$

defined by

$$\Gamma_c(\boldsymbol{a})(n) = \begin{cases} 1 & \text{if } \boldsymbol{c}^{(n+1)} \leqq \boldsymbol{a} \cup \boldsymbol{c}^{(n)}, \\ 0 & \text{otherwise}. \end{cases}$$

It is important to note that the same translation works for all $\boldsymbol{c}$.

The literature of degree theory contains many more examples of relativization. In fact, we can make a blanket claim that all theorems in the literature of the form $\mathfrak{D} \models \varphi$ relativize to $\mathfrak{D}_c \models \varphi$ for all $\boldsymbol{c}$. This phenomenon led Rogers [18, p. 261] to formulate the *strong homogeneity conjecture* which says that for any $\boldsymbol{c}$ the structures $\mathfrak{D}$ and $\mathfrak{D}_c$ are isomorphic.

Unfortunately, the strong homogeneity conjecture is false. It was shown by Feiner [3] that $\mathfrak{D}$ is not isomorphic to $\mathfrak{D}_{\boldsymbol{0}^{(6)}}$. A slight extension of Feiner's argument shows that if $\mathfrak{D}_a$ and $\mathfrak{D}_b$ are isomorphic then $\boldsymbol{a} \leqq \boldsymbol{b}^{(6)}$ and $\boldsymbol{b} \leqq \boldsymbol{a}^{(6)}$ so in particular $\boldsymbol{a}^{(\omega)} = \boldsymbol{b}^{(\omega)}$. (See also Yates [31, §5] and Jockusch and Solovay [11].)

Since strong homogeneity fails, it is natural to propose the *homogeneity conjecture*: for any degree $\boldsymbol{c}$ the structures $\mathfrak{D}_c$ and $\mathfrak{D}$ are *elementarily equivalent*, i.e., have the same first-order theory. This conjecture is refuted by the next theorem.

THEOREM 4.1. *There exists a degree* $\boldsymbol{b}$ *such that* $\mathfrak{D}_b$ *is not elementarily equivalent to* $\mathfrak{D}$.

Proof. Let $\boldsymbol{b}$ be any degree such that $\boldsymbol{b}^{(\omega)} > \boldsymbol{0}^{(\omega)}$ and $\boldsymbol{b}$ is definable in analysis (e.g., we may take $\boldsymbol{b} = \boldsymbol{0}^{(\omega)}$). By the relativized version of Theorem 3.7, there is a formula $\psi(x)$ such that for all $\boldsymbol{a}$ and $\boldsymbol{c}$, $\mathfrak{D}_c \models \psi[\boldsymbol{a}]$ if and only if $\deg(\Gamma_c(\boldsymbol{a})) = \boldsymbol{b}^{(\omega)}$. By the relativized version of Lemma 3.10, there is a formula $\theta(x)$ such that for all $\boldsymbol{a}$ and $\boldsymbol{c}$, $\mathfrak{D}_c \models \theta[\boldsymbol{a}]$ if and only if $\boldsymbol{a} \geqq \boldsymbol{c}$ and $\boldsymbol{a}^{(\omega)} = \boldsymbol{c}^{(\omega)}$. Let φ be the sentence $\exists x(\psi(x) \wedge \theta(x))$. The relativized version of Lemma 3.8 yields a degree $\boldsymbol{a} \geqq \boldsymbol{b}$ such that $\deg(\Gamma_b(\boldsymbol{a})) = \boldsymbol{a}^{(\omega)} = \boldsymbol{b}^{(\omega)}$. Hence $\mathfrak{D}_b \models \varphi$. On the other hand $\mathfrak{D} \models \neg\varphi$ since $\deg(\Gamma(\boldsymbol{a})) \leqq \boldsymbol{a}^{(\omega)}$ for all $\boldsymbol{a}$. This completes the proof.

A slight modification of this proof yields the following result:

THEOREM 4.2. *Assume* $V = L$. *If* $\mathfrak{D}_a$ *and* $\mathfrak{D}_b$ *are elementarily equi-*

valent, then $\mathbf{a}^{(\omega)} = \mathbf{b}^{(\omega)}$.

In contrast to this result, we have the following easy consequence of a lemma of Martin [16]:

THEOREM 4.3. *Assume* PD. *Then there exists a degree* $\mathbf{a}$ *such that* $\mathfrak{D}_{\mathbf{a}}$ is *elementarily equivalent to* $\mathfrak{D}_{\mathbf{b}}$ *for all* $\mathbf{b} \geqq \mathbf{a}$.

5. First-order theory without jump

In this section we translate analysis into the first-order theory of $\langle D, \cup \rangle$. The rough idea behind the translation is as follows. By Theorem 3.7 we have the desired result for $\langle D, \cup, j \rangle$. In the proof of 3.7, j was used only to establish the existence of certain configurations of degrees which encode analysis. But once we know that such configurations exist, we can list their essential properties and speak about them in the language of $\langle D, \cup \rangle$ via Lemma 3.1.

We say that $\mathbf{b}$ is *n-minimal over* $\mathbf{a}$ if $\{\mathbf{d} \mid \mathbf{a} \leqq \mathbf{d} \leqq \mathbf{b}\}$ is a linear ordering of size $n + 1$. An $\mathbf{a}$*-tower* is a sequence of degrees $\langle \mathbf{a}_n \rangle_{n \in \omega}$ such that

$$\mathbf{a} = \mathbf{a}_0 < \mathbf{a}_1 < \cdots < \mathbf{a}_n < \cdots$$

and $\mathbf{a}_n$ is n-minimal over $\mathbf{a}$ for each n. A k-ary relation $R \subseteqq D^k$ is said to be *weakly definable* if it is first-order definable in $\langle D, \cup \rangle$ allowing parameters from D. An $\mathbf{a}$-tower $\langle \mathbf{a}_n \rangle_{n \in \omega}$ is *good* if

(i) the relations $\{\langle \mathbf{a}_m, \mathbf{a}_n, \mathbf{a}_k \rangle \mid m + n\}$ and $\{\langle \mathbf{a}_m, \mathbf{a}_n, \mathbf{a}_k \rangle \mid m \cdot n = k\}$ are weakly definable;

(ii) there exists a degree $\mathbf{c}$ such that the relation $\{\langle \mathbf{a}_n, \mathbf{d} \rangle \mid \mathbf{d}$ is n-minimal over $\mathbf{c}\}$ is weakly definable.

LEMMA 5.1. *There exists a good* $\mathbf{0}$*-tower*.

Proof. By the Main Lemma (Theorem 2.1), let $A = \langle \mathbf{a}_n \rangle_{n \in \omega}$ be a $\mathbf{0}$-tower such that $\mathbf{a}_n \cup \mathbf{0}^{(2)} = \mathbf{0}^{(2n+2)}$ for all n. We claim that A is good. The set $\{\mathbf{a}_n \mid n \in \omega\}$ is a countable ideal and so by Lemma 3.1 is weakly definable. Hence the one-one correspondence $\{\langle \mathbf{a}_n, \mathbf{0}^{(2n)} \rangle \mid n \in \omega\}$ is weakly definable. Now part (i) of goodness is immediate from Lemma 3.4. For part (ii) put $\mathbf{c} = \mathbf{0}^{(\omega)}$ and use Lemma 3.3 to justify the following definition: $\mathbf{d}$ is n-minimal over $\mathbf{c}$ if and only if $\mathbf{d} \geqq \mathbf{c}$ and there exists a degree $\mathbf{e}$ such that the following holds. Use $\mathbf{e}$ to define a binary relation R by $R(\mathbf{0}^{(2i)}, \mathbf{b})$ if and only if

$$\exists \mathbf{a} \leqq \mathbf{e}(\mathbf{a} \cup \mathbf{0}^{(2i-2)} < \mathbf{a} \cup \mathbf{0}^{(2i)} = \mathbf{a} \cup \mathbf{0}^{(2n)} = \mathbf{b} \text{ and } \mathbf{a} > \mathbf{0}) .$$

Then R is an order-reversing one-one correspondence between $\{\mathbf{0}^{(2i)} \mid 1 \leqq i \leqq n\}$ and $\{\mathbf{b} \mid \mathbf{c} < \mathbf{b} \leqq \mathbf{d}\}$. This completes the proof of Lemma 5.1.

If $A = \langle \mathbf{a}_n \rangle_{n \in \omega}$ is an $\mathbf{a}$-tower, we define $\Gamma_A \colon D \longrightarrow 2^\omega$ by

$$\Gamma_A(\boldsymbol{b})(n) = \begin{cases} 1 & \text{if } \boldsymbol{a}_{n+1} \leqq \boldsymbol{b} \cup \boldsymbol{a}_n , \\ 0 & \text{otherwise} . \end{cases}$$

LEMMA 5.2. *For all* $f \in 2^\omega$ *there exists a* **0**-*tower* A *and a degree* $\boldsymbol{b}$ *such that* $\Gamma_A(\boldsymbol{b}) = f$.

Proof. Let $A = \langle \boldsymbol{a}_n \rangle_{n \in \omega}$ be as in the proof of Lemma 5.1. By Lemma 3.5 we can find $\boldsymbol{c}$ such that for all n, $\mathbf{0}^{(2n+4)} \leqq \boldsymbol{c} \cup \mathbf{0}^{(2n+2)}$ if and only if $f(n) = 1$. Put $\boldsymbol{b} = \boldsymbol{c} \cup \mathbf{0}^{(2)}$. It follows that $\boldsymbol{a}_{n+1} \leqq \boldsymbol{b} \cup \boldsymbol{a}_n$ if and and only if $f(n) = 1$, i.e., $\Gamma_A(\boldsymbol{b}) = f$.

THEOREM 5.3. *Let* φ *be a sentence in the language of analysis. Then we can effectively find a sentence* ψ *in the first-order language of semilattices, such that* $\mathfrak{A} \models \varphi$ *if and only if* $\langle D, \cup \rangle \models \psi$.

Proof. We abbreviate $\langle D, \cup \rangle \models \cdots$ as $\models_D \cdots$. By Lemma 5.1 let α, β, δ be first-order formulas such that there exist a **0**-tower $\langle \boldsymbol{a}_n \rangle_{n \in \omega}$ and degrees $\boldsymbol{c}, \boldsymbol{d}, \cdots$ such that

(i) $\{\langle \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3 \rangle \mid \models_D \alpha[\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{d}, \cdots]\} = \{\langle \boldsymbol{a}_m, \boldsymbol{a}_n, \boldsymbol{a}_k \rangle \mid m + n = k\}$;

(ii) $\{\langle \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3 \rangle \mid \models_D \beta[\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{d}, \cdots]\} = \{\langle \boldsymbol{a}_m, \boldsymbol{a}_n, \boldsymbol{a}_k \rangle \mid m \cdot n = k\}$;

(iii) $\{\langle \boldsymbol{e}_1, \boldsymbol{e}_2 \rangle \mid \models_D \delta[\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{c}, \boldsymbol{d}, \cdots]\} = \{\langle \boldsymbol{a}_n, \boldsymbol{b} \rangle \mid \boldsymbol{b} \text{ is } n\text{-minimal over } \boldsymbol{c}\}$.

Then by Lemma 3.1 we can write down a formula θ such that for *all* degrees $\boldsymbol{c}, \boldsymbol{d}, \cdots$, $\models_D \theta[\boldsymbol{c}, \boldsymbol{d}, \cdots]$ if and only if there exists a **0**-tower $A = \langle \boldsymbol{a}_n \rangle_{n \in \omega}$ such that (i), (ii), and (iii) hold. This A is necessarily good, and we refer to A as the good **0**-tower *encoded* by $\boldsymbol{c}, \boldsymbol{d}, \cdots$.

Now given φ we can write down a formula φ^* such that $\mathfrak{A} \models \varphi$ if and only if $\models_D \varphi^*[\boldsymbol{c}, \boldsymbol{d}, \cdots]$ whenever $\models_D \theta[\boldsymbol{c}, \boldsymbol{d}, \cdots]$. The idea here is that φ^* expresses φ in terms of the good **0**-tower encoded by $\boldsymbol{c}, \boldsymbol{d}, \cdots$. Thus first-order arithmetic is handled by α and β, while function quantifiers are handled by δ and the relativization to $\boldsymbol{c}$ of Lemmas 5.2 and 3.1. Finally let ψ be the sentence

$$\exists x, y, \cdots (\theta(x, y, \cdots) \wedge \varphi^*(x, , y, \cdots)) .$$

This completes the proof of Theorem 5.3.

COROLLARY 5.4. *There is a sentence* ψ *such that, provably in* ZFC, $\langle D, \cup \rangle \models \psi$ *if and only if every element of* 2^ω *is constructible.*

COROLLARY 5.5. *The first-order theory of* $\langle D, \cup \rangle$ *is not absolute with respect to models of set theory containing all the ordinals.*

COROLLARY 5.6. *The first-order theory of* $\langle D, \cup \rangle$ *is recursively isomorphic to the truth set of analysis* (i.e., *the set* E^ω *of* [18, p. 380]).

Proof. Theorem 5.3 says that E^ω is many-one reducible to the first-order theory of $\langle D, \cup \rangle$. The converse reducibility is obvious since $\{\langle f, g\rangle \mid f$ is recursive in $g\}$ is definable in analysis. Corollary 5.6 then follows by Chapter 7 of [18].

The questions which evoked Corollaries 5.5 and 5.6 were first raised in a paper on hyperdegrees [27]. It is perhaps worth remarking that all of the theorems of the present paper hold for hyperdegrees in place of degrees. In fact, Theorem 3.12 was first discovered in the context of hyperdegrees where the proof is somewhat simpler (cf. §§1 and 3 of [27]).

We end the paper with a non-exhaustive list of open questions. Some of these questions were suggested by Rogers [19].

QUESTION 5.7. Does there exist a degree $\boldsymbol{b}$ such that the semilattice of degrees $\geqq \boldsymbol{b}$ is not elementarily equivalent to $\langle D, \cup \rangle$?

QUESTION 5.8. Does there exist a degree $\boldsymbol{b}$ such that the semilattice of degrees $\geqq \boldsymbol{b}$ is not isomorphic to $\langle D, \cup \rangle$?

QUESTION 5.9. Do there exist degrees $\boldsymbol{a}, \boldsymbol{b}$ such that $\boldsymbol{a} \neq \boldsymbol{b}$ and the semilattices of degrees $\geqq \boldsymbol{a}$ and $\geqq \boldsymbol{b}$ are isomorphic?

QUESTION 5.10. Does $\langle D, \cup \rangle$ have automorphisms other than the identity?

QUESTION 5.11. Does there exist a degree other than **0** which is fixed by all automorphisms of $\langle D, \cup \rangle$?

QUESTION 5.12. Is the jump operator fixed by all automorphisms of $\langle D, \cup \rangle$?

QUESTION 5.13. Is the jump operator first-order definable in $\langle D, \cup \rangle$?

QUESTION 5.14. Is the set of constructible degrees first-order definable in $\langle D, \cup \rangle$?

THE PENNSYLVANIA STATE UNIVERSITY, UNIVERSITY PARK

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