Degrees of unsolvability: a tutorial*

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Abstract

Given a problem P, one associates to P a *degree of unsolvability*, i.e., a quantity which measures the amount of algorithmic unsolvability which is inherent in P. We focus on two degree structures: the semilattice of *Turing degrees*, $\mathcal{D}_\mathrm{T}$, and its completion, $\mathcal{D}_\mathrm{w} = \widehat{\mathcal{D}_\mathrm{T}}$, the lattice of *Muchnik degrees*. We emphasize specific, natural degrees and their relationship to reverse mathematics. We show how Muchnik degrees can be used to classify tiling problems and symbolic dynamical systems of finite type. We describe how the category of sheaves over $\mathcal{D}_\mathrm{w}$ forms a model of intuitionistic mathematics, known as the *Muchnik topos*. This model is a rigorous implementation of Kolmogorov's nonrigorous 1932 interpretation of intuitionism as a "calculus of problems."

Contents

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1 Turing degrees

The existence of unsolvable[1] mathematical problems was discovered by Turing [83]. Indeed, Turing exhibited a *specific, natural example*[2] of such a problem: the halting problem for Turing machines. Later, in the 1950s and 1960s, it was discovered that there are specific, natural, unsolvable problems in virtually every branch of mathematics: *number theory* (Hilbert's Tenth Problem [16]), *geometry* (the homeomorphism problem for finite simplicial complexes, the diffeomorphism problem for compact manifolds [46, Appendix]), *group theory* (the word problem [1] and the triviality problem [51] for finitely presented groups), *combinatorics* (the problem of tileability of the plane with a finite set of tiles [7, 53]), *mathematical logic* (the validity problem for predicate calculus [14, 83], the decision problem for first-order arithmetic [81]), and even *elementary calculus* (the problem of integrability in finite terms [52]).

A scheme for classifying unsolvable problems was developed by Post [50] and Kleene/Post [37]. Two reals[3] X and Y are said to be *Turing equivalent* if each is computable using the other as a Turing oracle. The *Turing degree* of a real is its equivalence class under this equivalence relation. Each of the specific, natural, unsolvable problems mentioned above is a *decision problem* and may therefore be straightforwardly described or "encoded"[4] as a real. It was then shown that each of these problems is of the same Turing degree as the halting problem. This Turing degree is denoted $\mathbf{0}'$. Thus the specific Turing degree $\mathbf{0}'$ is extremely useful and important.

Given a real X, the Turing degree of X is denoted $\deg_{\mathrm{T}}(X)$. If $\mathbf{a} = \deg_{\mathrm{T}}(X)$ and $\mathbf{b} = \deg_{\mathrm{T}}(Y)$ are the Turing degrees of reals X and Y respectively, we write $X \le_{\mathrm{T}} Y$ or $\mathbf{a} \le \mathbf{b}$ to mean that Y is "at least as unsolvable as" X in the following sense: X is computable using Y as a Turing oracle. We also write $X <_{\mathrm{T}} Y$ or $\mathbf{a} < \mathbf{b}$ to mean that $X \le_{\mathrm{T}} Y$ and $Y \not\le_{\mathrm{T}} X$. Let $\mathcal{D}_{\mathrm{T}}$ be the set of all Turing degrees. Clearly $\le$ is a partial ordering of $\mathcal{D}_{\mathrm{T}}$, and every pair of degrees in $\mathcal{D}_{\mathrm{T}}$ has a *supremum*, i.e., a least upper bound. In other words, $\mathcal{D}_{\mathrm{T}}$ is a *semilattice*. Kleene and Post proved that there are infinitely many degrees in $\mathcal{D}_{\mathrm{T}}$ which are less than $\mathbf{0}'$, and there are uncountably many other degrees in $\mathcal{D}_{\mathrm{T}}$ which are incomparable with $\mathbf{0}'$. Thus $\mathcal{D}_{\mathrm{T}}$ has a rich algebraic structure. However, despite recent remarkable progress [63, 77], no one has yet discovered a specific, natural example of an unsolvable problem of Turing degree $\not\ge \mathbf{0}'$.

Given a real X, let X' be a real which encodes the halting problem *relative to* X, i.e., with X used as a Turing oracle. If $\mathbf{a}$ is the Turing degree of X, let $\mathbf{a}'$ be the Turing degree of X'. It can be shown that $\mathbf{a}'$ is independent of the choice of X such that $\deg_{\mathrm{T}}(X) = \mathbf{a}$. The operator $\mathbf{a} \mapsto \mathbf{a}' : \mathcal{D}_{\mathrm{T}} \to \mathcal{D}_{\mathrm{T}}$ is called the *Turing jump* operator. Generalizing Turing's proof of the unsolvability of the halting problem, one shows that $\mathbf{a} < \mathbf{a}'$. In other words, X' is "more unsolvable than" X. Inductively we write $\mathbf{a}^{(0)} = \mathbf{a}$ and $\mathbf{a}^{(n+1)} = (\mathbf{a}^{(n)})'$ for all natural numbers n. Extending this induction into the transfinite, it is possible to define $\mathbf{a}^{(\alpha)}$ where α ranges over a large initial segment of the ordinal numbers including the constructibly countable ordinal numbers. We then have $\mathbf{a}^{(\alpha)} < \mathbf{a}^{(\beta)}$ whenever $\alpha < \beta$. See [57, Part A] and [30, 64].

Let $\mathbf{0}$ be the bottom degree in $\mathcal{D}_{\mathrm{T}}$, i.e., the Turing degree of any solvable decision problem. We

[1] By *unsolvable* we mean algorithmically unsolvable, i.e., not solvable by a Turing program.

[2] We are not offering a rigorous definition of what is meant by "specific and natural." However, it is well known that considerations of specificity and naturalness play an important role in mathematics. Without such considerations, it would be difficult or impossible to pursue the ideal of "exquisite taste" in mathematical research, as famously enunciated by von Neumann.

[3] In this paper we take *reals* to be points in the Baire space $\mathbb{N}^{\mathbb{N}}$, i.e., functions $X : \mathbb{N} \to \mathbb{N}$ where $\mathbb{N} = \{0, 1, 2, \ldots\} =$ the natural numbers.

[4] More specifically, each of the mentioned problems amounts to the question of deciding whether or not a given finite string of symbols from a fixed finite alphabet belongs to a particular set of such strings. The problem is then identified with the characteristic function of the set of Gödel numbers of the finite strings which belong to the set.

then have a transfinite hierarchy of specific, natural Turing degrees

$$\mathbf{0} < \mathbf{0}' < \mathbf{0}'' < \cdots < \mathbf{0}^{(\alpha)} < \mathbf{0}^{(\alpha+1)} < \cdots$$

where α ranges over a large initial segment of the ordinal numbers [64]. Moreover, this hierarchy of specific, natural Turing degrees has been useful for the classification of unsolvable mathematical problems. See for instance [47] and [54, §14.8] and §4 below. However, no other specific, natural Turing degrees are known.

The semilattice $\mathcal{D}_{\mathrm{T}}$ is large and complicated, so it is reasonable to examine subsemilattices which are hopefully more manageable. One such subsemilattice has been studied in great depth. A Turing degree is said to be *recursively enumerable*[5] if it is the Turing degree of the characteristic function of a subset of $\mathbb{N}$ which is the range of a recursive function. Let $\mathcal{E}_{\mathrm{T}}$ be the subsemilattice of $\mathcal{D}_{\mathrm{T}}$ consisting of the recursively enumerable Turing degrees. The top and bottom degrees in $\mathcal{E}_{\mathrm{T}}$ are $\mathbf{0}'$ and $\mathbf{0}$. It is known that $\mathcal{E}_{\mathrm{T}}$ is structurally rich. Two key results due to Sacks [55, 56] are the *Splitting Theorem*[6] and the *Density Theorem*[7], and many other results have been obtained [40, 41, 62, 78]. For instance, the Turing degree of the first-order theory of $\mathcal{E}_{\mathrm{T}}$ is $\mathbf{0}^{(\omega)}$ [49]. However, except for $\mathbf{0}'$ and $\mathbf{0}$ no specific, natural, recursively enumerable Turing degrees are known.

2 Muchnik degrees

There are many specific, natural, unsolvable problems to which it is impossible to assign a Turing degree.

As an example, let T be an effectively essentially undecidable theory. For instance, we could take $T = \mathsf{PA} = \mathsf{Z}_1 =$ first-order arithmetic, or $T = \mathsf{Z}_2 =$ second-order arithmetic [66], or $T = \mathsf{ZFC} =$ Zermelo/Fraenkel set theory [28], or $T = \mathsf{Q} =$ Robinson's arithmetic [81], or $T =$ any consistent recursively axiomatizable extension of one of these. Consider the problem $\mathrm{C}(T)$ of "finding" a complete and consistent theory which extends T. A solution of the problem would be any such theory. Lindenbaum's Lemma implies that such theories exist, and by [81] no such theory is algorithmically decidable.[8] In this sense the problem $\mathrm{C}(T)$ is algorithmically unsolvable. On the other hand, the problem $\mathrm{C}(T)$ cannot correspond to a Turing degree, because for any solution X of $\mathrm{C}(T)$ there exists a solution Y of $\mathrm{C}(T)$ such that $Y <_{\mathrm{T}} X$.

In order to overcome this limitation of the Turing degrees, we now extend $\mathcal{D}_{\mathrm{T}}$ to its completion, $\mathcal{D}_{\mathrm{w}}$, the lattice of Muchnik degrees.

A *mass problem* is defined to be a set of reals.[9] The idea here is that a mass problem P "represents" (i.e., is the solution set of) the problem of "finding" or "computing" some real X which belongs to P. Accordingly, a mass problem P is said to be *unsolvable* if it contains no Turing computable real, i.e., if $P \cap \mathrm{REC} = \emptyset$ where $\mathrm{REC} = \{X \mid X \text{ is computable}\}$. Following the same idea, we generalize the notion of Turing reducibility as follows. For mass problems P and Q, we say that P is *Muchnik reducible* to Q, abbreviated $P \leq_{\mathrm{w}} Q$, if every solution of Q can be used as a Turing oracle to compute some solution of P. In other words, $P \leq_{\mathrm{w}} Q$ if and only if $\forall Y\,(Y \in Q \Rightarrow \exists X\,(X \in P \text{ and } X \leq_{\mathrm{T}} Y))$.[10] We say that P is *Muchnik equivalent* to Q, abbreviated $P \equiv_{\mathrm{w}} Q$, if $P \leq_{\mathrm{w}} Q$ and $Q \leq_{\mathrm{w}} P$. The *Muchnik degree* of P, written $\deg_{\mathrm{w}}(P)$, is the equivalence class of P under $\equiv_{\mathrm{w}}$. Let $\mathcal{D}_{\mathrm{w}}$ be the set of all Muchnik degrees, partially ordered by letting $\deg_{\mathrm{w}}(P) \leq \deg_{\mathrm{w}}(Q)$ if and only if $P \leq_{\mathrm{w}} Q$. It is easy

[5] A.k.a., computably enumerable [79].

[6] The Sacks Splitting Theorem says that $\mathcal{E}_{\mathrm{T}}$ satisfies $\forall x\,(x > 0 \Rightarrow \exists u\,\exists v\,(u < x \text{ and } v < x \text{ and } \sup(u,v) = x))$.

[7] The Sacks Density Theorem says that $\mathcal{E}_{\mathrm{T}}$ satisfies $\forall x\,\forall y\,(x < y \Rightarrow \exists z\,(x < z < y))$.

[8] When speaking of decidable and undecidable theories, we identify a theory with the characteristic function $X \in \{0,1\}^{\mathbb{N}}$ of the set of Gödel numbers of theorems of the theory.

[9] This concept is from Medvedev [42]. As in footnote 3 a *real* is a function $X \in \mathbb{N}^{\mathbb{N}}$.

[10] This is Muchnik's notion of *weak reducibility* [44, Definition 2].

to see that $\mathcal{D}_\mathrm{w}$ is a lattice. Given a real X, we identify X with the mass problem $\{X\}$ = the singleton set whose only member is X. Thus $\deg_\mathrm{T}(X) = \deg_\mathrm{w}(\{X\})$ and $\mathcal{D}_\mathrm{T}$ is now a subset of $\mathcal{D}_\mathrm{w}$.

The relationship between $\mathcal{D}_\mathrm{T}$ and $\mathcal{D}_\mathrm{w}$ may be viewed as an instance of a general construction. Namely, for any partially ordered set $\mathcal{K}$, let $\widehat{\mathcal{K}}$ be the set of upwardly closed subsets $U \subseteq \mathcal{K}$, partially ordered by reverse inclusion, i.e., $U \le V$ if and only if $U \supseteq V$. Identifying $a \in \mathcal{K}$ with the upwardly closed set $U_a = \{x \in \mathcal{K} \mid x \ge a\} \in \widehat{\mathcal{K}}$, we see that $\mathcal{K}$ is a subordering of $\widehat{\mathcal{K}}$, i.e., $a \le b$ if and only if $U_a \le U_b$. Thus $\widehat{\mathcal{K}}$ is a complete and completely distributive lattice, the *completion* of $\mathcal{K}$. There is a unique isomorphism of $\mathcal{D}_\mathrm{w}$ onto $\widehat{\mathcal{D}_\mathrm{T}}$ which extends the identity map on $\mathcal{D}_\mathrm{T}$, and in this sense $\mathcal{D}_\mathrm{w}$ is the completion of $\mathcal{D}_\mathrm{T}$. The upshot here is that Muchnik degrees can be identified with upwardly closed sets of Turing degrees.[11] This remark will be important in §5 below.

In the above example, let us identify $\mathrm{C}(T)$ with the mass problem $\{X \mid X$ is a complete and consistent extension of $T\}$. Under this identification, $\mathrm{C}(T)$ is Muchnik reducible to the halting problem.[12] However, the halting problem is not Muchnik reducible to $\mathrm{C}(T)$, because the halting problem has a Turing degree while $\mathrm{C}(T)$ does not. Thus, letting $\mathbf{1}$ = the Muchnik degree of $\mathrm{C}(T)$, we have $\mathbf{0} < \mathbf{1} < \mathbf{0}'$. Furthermore, the particular Muchnik degree $\mathbf{1} = \deg_\mathrm{w}(\mathrm{C}(T))$ can be characterized abstractly in a way which does not depend on T. We now see that $\mathbf{1}$ is a very specific, very natural, very important Muchnik degree which is not a Turing degree.

In addition to the Muchnik degree $\mathbf{1}$ and the Turing degrees $\mathbf{0}^{(\alpha)}$ for ordinal numbers $\alpha = 0, 1, 2, \ldots, \omega, \omega{+}1, \ldots$, there are many other specific, natural Muchnik degrees. Here are some examples and references.

1. Let λ be the fair coin probability measure on $\{0,1\}^\mathbb{N}$. A set $S \subseteq \{0,1\}^\mathbb{N}$ is said to be *effectively null* if $S \subseteq \bigcap_n U_n$ for some uniformly effectively open sequence of sets U_n such that $\lambda(U_n) \le 2^{-n}$ for all n. A real $Z \in \{0,1\}^\mathbb{N}$ is said to be *Martin-Löf random* [19, 48] if it does not belong to any effectively null set. Let $\mathbf{r}_1 = \deg_\mathrm{w}(\{Z \in \{0,1\}^\mathbb{N} \mid Z$ is Martin-Löf random$\})$. It is not difficult to show that $\mathbf{0} < \mathbf{r}_1 < \mathbf{1}$.

2. More generally, for any constructibly countable ordinal number α, let $\mathbf{r}_\alpha = \deg_\mathrm{w}(\{Z \mid (\forall \xi < \alpha)\,(Z$ is Martin-Löf random relative to $0^{(\xi)})\})$. It is not difficult to show that $\mathbf{0} = \mathbf{r}_0 < \mathbf{r}_1 < \mathbf{r}_2 < \cdots < \mathbf{r}_\alpha < \mathbf{r}_{\alpha+1} < \cdots$. Moreover, each $\mathbf{r}_\alpha$ for $\alpha \ge 2$ is incomparable with $\mathbf{1}$.

3. A partial recursive function $\psi :\subseteq \mathbb{N} \to \mathbb{N}$ is said to be *universal* if for each partial recursive function $\varphi :\subseteq \mathbb{N} \to \mathbb{N}$ there exists a recursive function $p : \mathbb{N} \to \mathbb{N}$ such that $\varphi(n) \simeq \psi(p(n))$ for all n.[13] Fix such a function ψ and let $\mathbf{d} = \deg_\mathrm{w}(\{Z \in \mathbb{N}^\mathbb{N} \mid Z \cap \psi = \emptyset\})$ and $\mathbf{d}_\mathrm{REC} = \deg_\mathrm{w}(\{Z \in \mathbb{N}^\mathbb{N} \mid Z \cap \psi = \emptyset$ and Z is recursively bounded$\})$. Clearly $\mathbf{d}$ and $\mathbf{d}_\mathrm{REC}$ are independent[14] of our choice of ψ. By [3, 29] we have $\mathbf{0} < \mathbf{d} < \mathbf{d}_\mathrm{REC} < \mathbf{r}_1$.

4. Given a recursive function $f : \mathbb{N} \to \mathbb{N}$, define $Z \in \{0,1\}^\mathbb{N}$ to be *f-complex* if
$$\exists c\, \forall n\, (\mathrm{K}(Z \restriction \{1, \ldots, n\}) > f(n) - c)$$
where K denotes Kolmogorov complexity. In this way each specific, natural,[15] recursive function f gives rise to a specific, natural Muchnik degree $\mathbf{k}_f = \deg_\mathrm{w}(\{Z \in \{0,1\}^\mathbb{N} \mid Z$ is f-complex$\})$, and

[11] For a more precise statement, see [5, Theorem 5.8].

[12] This follows from a theorem of Kleene [36, page 398]. See also [32, 61].

[13] Here $E_1 \simeq E_2$ means that E_1 and E_2 are both undefined or both defined and equal.

[14] Let φ_n, $n \in \mathbb{N}$ be a fixed, standard, partial recursive enumeration of the partial recursive functions. A function $Z \in \mathbb{N}^\mathbb{N}$ is said to be *diagonally nonrecursive* [3, 26, 29, 35, 69] if $Z \cap \psi = \emptyset$ where ψ is the well known *diagonal function*, defined by $\psi(n) \simeq \varphi_n(n)$. Letting $\mathrm{DNR} = \{Z \in \mathbb{N}^\mathbb{N} \mid Z$ is diagonally nonrecursive$\}$ and $\mathrm{DNR}_\mathrm{REC} = \{Z \in \mathrm{DNR} \mid Z$ is recursively bounded$\}$, we have $\mathbf{d} = \deg_\mathrm{w}(\mathrm{DNR})$ and $\mathbf{d}_\mathrm{REC} = \deg_\mathrm{w}(\mathrm{DNR}_\mathrm{REC})$.

[15] For example, $f(n)$ could be $n/2$ or $n/3$ or $\sqrt{n}$ or $\sqrt[3]{n}$ or $\log_2 n$ or $\log_3 n$ or $\log_2 \log_2 n$, etc., or f could be the inverse Ackermann function.

there is also $\mathbf{k}_{\rm REC} = \deg_{\rm w}(\{Z \in \{0,1\}^{\mathbb{N}} \mid Z$ is f-complex for some unbounded recursive function $f\})$. By [35] we have $\mathbf{k}_{\rm REC} = \mathbf{d}_{\rm REC}$, and by [19, Theorem 6.2.3] we have $\mathbf{k}_1 = \mathbf{r}_1$ where $1 : \mathbb{N} \to \mathbb{N}$ is the identity function. Building on the methods of Miller [43], Hudelson [27] has shown that $\mathbf{d}_{\rm REC} < \mathbf{k}_f < \mathbf{k}_g \leq \mathbf{r}_1$ holds for many pairs of unbounded recursive functions f, g. In particular, this holds whenever $\forall n\,(f(n) \leq f(n+1) \leq f(n)+1$ and $f(n) + 2\log_2 f(n) \leq g(n) \leq n)$.

5. Let $\mathrm{MLR}^X = \{Z \in \{0,1\}^{\mathbb{N}} \mid Z$ is Martin-Löf random relative to $X\}$. We say that X *is LR-reducible to* Y, abbreviated $X \leq_{\rm LR} Y$, if $\mathrm{MLR}^X \supseteq \mathrm{MLR}^Y$ [19, 48]. Letting $\mathbf{b}_\alpha = \deg_{\rm w}(\{Y \mid 0^{(\alpha)} \leq_{\rm LR} Y\})$, it is not difficult to show that $\mathbf{0} = \mathbf{b}_0 < \mathbf{b}_1 < \mathbf{b}_2 < \cdots < \mathbf{b}_\alpha < \mathbf{b}_{\alpha+1} < \cdots$. On the other hand, by [72] we know that the Muchnik degrees $\mathbf{b}_\alpha$ for $\alpha \geq 1$ are incomparable with the Muchnik degrees $\mathbf{d}$, $\mathbf{1}$, and $\mathbf{r}_\alpha$ for all $\alpha \geq 1$.

6. A partial recursive function $\psi :\subseteq \mathbb{N} \to \mathbb{N}$ is said to be *linearly universal* if it is "universal via linear functions," i.e., for each partial recursive function $\varphi :\subseteq \mathbb{N} \to \mathbb{N}$ there exist $a, b \in \mathbb{N}$ such that $\varphi(n) \simeq \psi(an+b)$ for all n. Let $D = \{Z \in \mathbb{N}^{\mathbb{N}} \mid Z \cap \psi = \emptyset$ for some linearly universal partial recursive function $\psi\}$, and let $D_{\rm REC} = \{Z \in D \mid Z$ is recursively bounded$\})$. Clearly $\deg_{\rm w}(D) = \mathbf{d}$ and $\deg_{\rm w}(D_{\rm REC}) = \mathbf{d}_{\rm REC}$ where $\mathbf{d}$ and $\mathbf{d}_{\rm REC}$ are as above. However, letting $D_h = \{Z \in D \mid Z$ is h-bounded$\}$ where h is a specific recursive function, we get a family of Muchnik degrees $\mathbf{d}_h = \deg_{\rm w}(D_h)$ which are of considerable interest [68, §10] [34]. In particular, for any unbounded recursive function h such that $\forall n\,(1 \leq h(n) \leq h(n+1))$ we know by [3, 26, 76] and [8, §7.3] that $\mathbf{d}_{\rm REC} < \mathbf{d}_h < \mathbf{1}$, and if $\sum_n h(n)^{-1} < \infty$ then $\mathbf{d}_h < \mathbf{r}_1$, and if $\sum_n h(n)^{-1} = \infty$ then $\mathbf{d}_h$ is incomparable with $\mathbf{r}_\alpha$ for all $\alpha \geq 1$. Also of interest is the Muchnik degree $\mathbf{d}_{\rm slow} = \deg_{\rm w}(\{Z \mid Z \in D_h$ for some recursive function h such that $\forall n\,(h(n) \leq h(n+1))$ and $\sum_n h(n)^{-1} = \infty\})$.

7. There are many other examples of specific, natural Muchnik degrees. See for instance the Computability Menagerie[16] [33]. Our choice of examples in this paper is oriented toward §3 below.

3 The lattices $\mathcal{E}_{\rm w}$ and $\mathcal{S}_{\rm w}$

The lattice $\mathcal{D}_{\rm w}$ is large and complicated, so it is desirable to consider more manageable sublattices. The smallest such sublattice which comes immediately to mind is the countable lattice $\mathcal{E}_{\rm w}$ consisting of the Muchnik degrees of nonempty, effectively closed subsets of $\{0,1\}^{\mathbb{N}}$. The explicit study of $\mathcal{E}_{\rm w}$ was undertaken only relatively recently [9, 11, 65, 67, 68] but was implicit in some much older literature [25, 31, 32, 58, 59]. By [59] the top and bottom degrees in $\mathcal{E}_{\rm w}$ are $\mathbf{1}$ and $\mathbf{0}$, and by [11] every countable distributive lattice is lattice-embeddable into $\mathcal{E}_{\rm w}$. The only Turing degree in $\mathcal{E}_{\rm w}$ is $\mathbf{0}$, but there is an obvious analogy

$$\frac{\mathcal{E}_{\rm w}}{\mathcal{D}_{\rm w}} = \frac{\mathcal{E}_{\rm T}}{\mathcal{D}_{\rm T}}$$

and indeed the Splitting Theorem and the Density Theorem hold for $\mathcal{E}_{\rm w}$ [9, 10]. The Turing degree of the first-order theory of $\mathcal{E}_{\rm w}$ is known to be $\geq \mathbf{0}'$ [60] and conjectured to be $= \mathbf{0}^{(\omega_1^{\rm CK}+\omega)}$ [15, page 127] [73, Remark 3.2.3].

An advantage of $\mathcal{E}_{\rm w}$ over $\mathcal{E}_{\rm T}$ is that $\mathcal{E}_{\rm w}$ contains a great variety of specific, natural Muchnik degrees in addition to its top and bottom degrees $\mathbf{1}$ and $\mathbf{0}$. In particular, it is not difficult [69, §3] to show that the Muchnik degrees $\mathbf{d}, \mathbf{d}_{\rm REC}, \mathbf{k}_f, \mathbf{r}_1, \mathbf{d}_h$, and $\mathbf{d}_{\rm slow}$ which were discussed in §2 belong to $\mathcal{E}_{\rm w}$.

[16] The inhabitants of the Computability Menagerie are downwardly closed sets of Turing degrees, and the complements of such sets are essentially the same thing as Muchnik degrees.

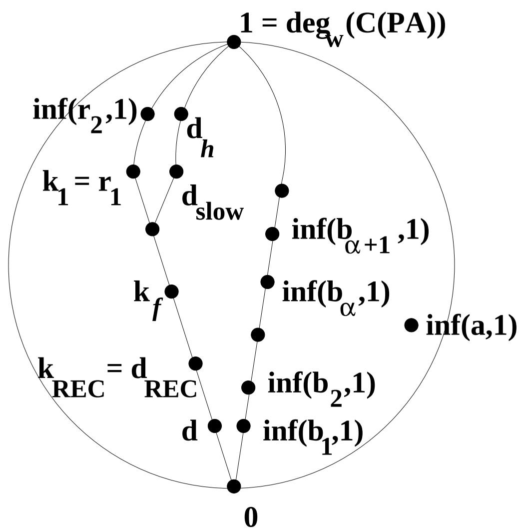

Figure 1: A picture of $\mathcal{E}_{\rm w}$.

Also of interest is the countable lattice $\mathcal{S}_{\rm w}$ consisting of the Muchnik degrees of nonempty, effectively closed subsets of $\mathbb{N}^{\mathbb{N}}$. An easy argument [73, Lemma 3.3.5] shows that $\mathcal{S}_{\rm w}$ has an alternative characterization as the lattice of Muchnik degrees of nonempty, lightface Σ^0_3 subsets of $\mathbb{N}^{\mathbb{N}}$. This is important, because it implies that $\mathcal{S}_{\rm w}$ contains many specific, natural Muchnik degrees beyond those which are already in $\mathcal{E}_{\rm w}$. In particular, the Muchnik degree $\mathbf{r}_2$ which was discussed in §2 belongs to $\mathcal{S}_{\rm w}$, as do the Turing degrees $\mathbf{0}^{(\alpha)}$ and the Muchnik degrees $\mathbf{b}_\alpha$ for all recursive ordinal numbers $\alpha < \omega_1^{\rm CK}$ [72].

Trivially $\mathcal{E}_{\rm w}$ is a sublattice of $\mathcal{S}_{\rm w}$, and by [73, Theorem 3.3.1] we know that $\mathcal{E}_{\rm w}$ is an initial segment of $\mathcal{S}_{\rm w}$. This is important, because it means that we have a specific, natural, lattice homomorphism $\mathbf{s} \mapsto \inf(\mathbf{s}, \mathbf{1}) : \mathcal{S}_{\rm w} \to \mathcal{E}_{\rm w}$. With this homomorphism, each of the specific, natural Muchnik degrees in $\mathcal{S}_{\rm w}$ has a specific, natural image in $\mathcal{E}_{\rm w}$. In particular, the Muchnik degrees $\inf(\mathbf{r}_2, \mathbf{1})$ [69, §3] and $\inf(\mathbf{b}_\alpha, \mathbf{1})$ for all ordinal numbers $\alpha < \omega_1^{\rm CK}$ [70, 72] belong to $\mathcal{E}_{\rm w}$.

Clearly $\mathcal{E}_{\rm T}$ is a subsemilattice of $\mathcal{S}_{\rm w}$, and by the Arslanov Completeness Criterion [29, Theorem 1] (see also [69, §5]) our homomorphism of $\mathcal{S}_{\rm w}$ onto $\mathcal{E}_{\rm w}$ is one-to-one when restricted to $\mathcal{E}_{\rm T}$. Thus we have a semilattice embedding $\mathbf{a} \mapsto \inf(\mathbf{a}, \mathbf{1}) : \mathcal{E}_{\rm T} \hookrightarrow \mathcal{E}_{\rm w}$ which carries the top and bottom degrees $\mathbf{0}', \mathbf{0} \in \mathcal{E}_{\rm T}$ to the top and bottom degrees $\mathbf{1}, \mathbf{0} \in \mathcal{E}_{\rm w}$. Unfortunately, the range of this embedding does not appear to contain any specific, natural Muchnik degrees other than $\mathbf{1}$ and $\mathbf{0}$. Thus the problem of finding a specific, natural, recursively enumerable Turing degree in the range $\mathbf{0} < \mathbf{a} < \mathbf{0}'$ remains open.

Figure 1 is a picture of $\mathcal{E}_{\rm w}$. In this picture, $\mathbf{a}$ is any recursively enumerable Turing degree in the range $\mathbf{0} < \mathbf{a} < \mathbf{0}'$. The black dots other than $\inf(\mathbf{a}, \mathbf{1})$ denote some of the specific, natural Muchnik degrees which we have discussed.

4 Applications

We briefly mention an application of $\mathcal{E}_{\rm w}$ to tiling problems. A *Wang tile* is a unit square with colored edges. Given a finite set A of Wang tiles, let P_A be the problem of tiling the plane with copies of tiles from A. More formally, P_A is the set of mappings $X : \mathbb{Z} \times \mathbb{Z} \to A$ such that for all $(i, j) \in \mathbb{Z} \times \mathbb{Z}$ the right edge of $X(i, j)$ matches the left edge of $X(i+1, j)$ and the top edge of $X(i, j)$ matches the bottom edge of $X(i, j+1)$. Clearly $\deg_{\rm w}(P_A) \in \mathcal{E}_{\rm w}$ provided $P_A \neq \emptyset$. It turns out [20, 74] that conversely, every Muchnik degree in $\mathcal{E}_{\rm w}$ is $\deg_{\rm w}(P_A)$ for some finite set A of Wang tiles. This result plus the existence of an infinite independent set of degrees in $\mathcal{E}_{\rm w}$ has a recursion-theory-free consequence for symbolic dynamics. Namely, there exists an infinite collection of 2-dimensional symbolic dynamical systems of finite type which are strongly independent of each other with respect to symbolic products, symbolic disjoint unions, and symbolic morphisms. For details see [74, §3].

We briefly mention the connection between degrees of unsolvability and reverse mathematics. From my book [66] it is clear that basic recursion-theoretic concepts such as Turing reducibility [66, Remark I.7.5], the Turing jump operator [66, Remark I.3.4], basis theorems [66, §§VII.1,VIII.2], the hyperarithmetical hierarchy [66, §VIII.3], the hyperjump [66, Remark I.5.4], and algorithmic randomness [66, §X.1] are highly relevant to reverse mathematics. More recently [72] it emerged that some advanced recursion-theoretic concepts such as LR-reducibility are also highly relevant to reverse mathematics. Beyond this, there is an obvious correspondence between the so called "Big Five" subsystems of Z_2 [66, Chapters II–VI] and certain degrees of unsolvability. Namely, the systems RCA_0, WKL_0, ACA_0, ATR_0, and Π^1_1-CA_0 correspond to the Muchnik degrees $\mathbf{0}$, $\mathbf{1}$, $\mathbf{0}'$, $\mathbf{0}^{(\alpha)}$ for $\alpha < \omega_1^{\mathrm{CK}}$, and $\mathbf{0}^{(\omega_1^{\mathrm{CK}})}$ respectively, where ω_1^{CK} is the least nonrecursive ordinal. In addition, the system WWKL_0 [66, §X.1] corresponds to the Muchnik degree $\mathbf{r}_1$.

5 The Muchnik topos

From Medvedev's 1955 paper introducing mass problems [42] and Muchnik's 1963 paper introducing Muchnik reducibility [44][17], it is evident that both authors were motivated by Kolmogorov's nonrigorous 1932 interpretation of intuitionistic propositional calculus as a "calculus of problems" [38, 39]. Kolmogorov's idea was to view intuitionistic propositions as "problems," and intuitionistic proofs of propositions as "solutions" of the corresponding "problems." Intuitionistic propositional connectives are then viewed as methods of combining "problems" to form new "problems." Two "problems" are viewed as being "equivalent" if from any solution of either of them a "solution" of the other can be "easily" or "immediately" extracted. We cannot expect the Law of the Excluded Middle to hold, because it would mean that for any proposition there should be an "easy" proof of either the proposition or its negation.

Muchnik's rigorous implementation of Kolmogorov's idea [44, Theorem 4] is based on mass problems, Muchnik reducibility, and lattice operations in $\mathcal{D}_{\mathrm{w}}$. Given two Muchnik degrees $\mathbf{p}$ and $\mathbf{q}$, we interpret $\mathbf{p} \wedge \mathbf{q}$ as $\sup(\mathbf{p}, \mathbf{q})$, $\mathbf{p} \vee \mathbf{q}$ as $\inf(\mathbf{p}, \mathbf{q})$, $\mathbf{p} \Rightarrow \mathbf{q}$ as $\inf(\{\mathbf{x} \mid \sup(\mathbf{p}, \mathbf{x}) \geq \mathbf{q}\})$, "true" as $\mathbf{0}$, "false" as $\deg_{\mathrm{w}}(\emptyset)$, and $\mathbf{p} \vdash \mathbf{q}$ as $\mathbf{p} \geq \mathbf{q}$. For more details and references, see [73, §4] and [71, 80].

Recently Muchnik's interpretation of intuitionistic propositional calculus [44] has been extended to an interpretation of intuitionistic mathematics as a whole [5]. The extension is based on a category which we call the *Muchnik topos*. The idea here is to consider $\mathcal{D}_{\mathrm{T}}$ as a topological space in which the open sets are the upwardly closed sets of Turing degrees.[18] In general, for any topological space $\mathcal{T}$, a *sheaf* over $\mathcal{T}$ consists of a topological space $\mathcal{X}$ together with a local homeomorphism $p : \mathcal{X} \to \mathcal{T}$. A *sheaf morphism* from a sheaf $p : \mathcal{X} \to \mathcal{T}$ to a sheaf $q : \mathcal{Y} \to \mathcal{T}$ is a continuous function $f : \mathcal{X} \to \mathcal{Y}$ such that $p(x) = q(f(x))$ for all $x \in \mathcal{X}$. The sheaves and sheaf morphisms over $\mathcal{T}$ form a category called $\mathrm{Sh}(\mathcal{T})$. As noted by Fourman and Scott [24], $\mathrm{Sh}(\mathcal{T})$ is a topos and provides a model of intuitionistic higher-order logic in which the truth values are the open subsets of $\mathcal{T}$. The Muchnik topos is then the special case $\mathrm{Sh}(\mathcal{D}_{\mathrm{T}})$ with truth values in $\mathcal{D}_{\mathrm{w}}$. All of this background material concerning sheaves and intuitionistic higher-order logic is explained at length in our paper [5].

Within the Muchnik topos $\mathrm{Sh}(\mathcal{D}_{\mathrm{T}})$, there are two versions of the real number system $\mathbb{R}$: the sheaf $\mathbb{R}_C = \mathbb{R} \times \mathcal{D}_{\mathrm{T}}$ of *Cauchy reals*, and the sheaf $\mathbb{R}_M = \{(r, \mathbf{a}) \in \mathbb{R}_C \mid \deg_{\mathrm{T}}(r) \leq_{\mathrm{T}} \mathbf{a}\}$ of *Muchnik reals*. Roughly speaking, the difference between $\mathbb{R}_C$ and $\mathbb{R}_M$ is that a Cauchy real can exist anywhere within the topological space $\mathcal{D}_{\mathrm{T}}$, but a Muchnik real can exist only where we have enough Turing oracle

[17] See also the English translation in [45].

[18] This topological space was considered by Muchnik [44, page 1332] [45, page 52].

power to compute it. For precise definitions, see [5]. It turns out [5, Theorem 5.19] that the Muchnik topos satisfies a *Choice and Bounding Principle*:

$$(\forall x\, \exists y\, \Phi(x, y)) \Rightarrow \exists w\, \exists z\, \forall x\, (wx \leq_{\mathrm{T}} (x, z) \wedge \Phi(x, wx))$$

where x, y, z are variables ranging over Muchnik reals, w is a variable ranging over functions from Muchnik reals to Muchnik reals, and $\Phi(x, y)$ is any formula of intuitionistic higher-order logic in which w and z do not occur. Our Choice and Bounding Principle reflects a well known intuitonistic idea: if for all real numbers x there exists a real number y which bears a certain relationship to x, then there should be a function $x \mapsto y$ which computes such a y using x as a Turing oracle.

We feel that, among various interpretations of intuitionistic mathematics, our interpretation in terms of the Muchnik topos stands out because of its relationship to the ideas of Kolmogorov, Medvedev, and Muchnik.

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