Turing degrees and Muchnik degrees of recursively bounded DNR functions*

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Contents

1 Introduction

Let φ_i, $i \in \mathbb{N}$ be a standard enumeration of the 1-place partial recursive functions $\varphi :\subseteq \mathbb{N} \to \mathbb{N}$. A function $Y : \mathbb{N} \to \mathbb{N}$ is said to be *diagonally nonrecursive* (with respect to the given enumeration), abbreviated *DNR*, if $\forall i\, (Y(i) \neq \varphi_i(i))$. Such a Y is said to be *recursively bounded* if there exists a recursive function $p : \mathbb{N} \to \mathbb{N}$ such that $\forall i\, (Y(i) < p(i))$. In this situation it is known that the growth rate of p has a strong influence on the Turing degree of Y. For example,

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it follows from [1] (see also [15, §10]) that the Turing degrees of elementary-recursively bounded DNR functions form a proper subclass of the Turing degrees of primitive-recursively bounded DNR functions. Additional results in this vein may be found in [11, Chapter 3], and still other results may be obtained by translating theorems about partial randomness [8, 9] into the context of recursively bounded DNR functions [12, 13].

In this note we exposit two striking results along these lines due to Joseph S. Miller. Roughly speaking, the results are as follows. Let $p : \mathbb{N} \to \mathbb{N}$ be a nondecreasing recursive function such that $p(0) \geq 2$.

1. If $\sum_i p(i)^{-1} < \infty$, then every Martin-Löf random real computes a p-bounded DNR function.

2. If $\sum_i p(i)^{-1} = \infty$, then no Martin-Löf random real computes a p-bounded DNR function unless it is Turing complete.

Note that 2 may be viewed as a vast generalization of a theorem of Stephan [20]. Combining results 1 and 2, we see that $\sum_i p(i)^{-1} < \infty$ if and only if the Turing upward closure of the set of p-bounded DNR functions is of full measure.

In order to formulate results 1 and 2 precisely, we find it convenient to replace the class DNR by the closely related class LDNR of *linearly DNR* functions. As a by-product of this move, we use LDNR to identify some specific, natural Muchnik degrees in $\mathcal{E}_{\mathrm{w}}$ which are associated with 1 and 2.

In our exposition of Miller's results, we draw heavily on the ideas of Bienvenu and Porter [3]. Of course [3] contains many other interesting results concerning other topics such as shift-complexity. Our intention here is to break down the proofs of Miller's results into easily manageable components.

2 When $\sum_i p(i)^{-1} < \infty$

Let $\mathbb{N} = \{0, 1, 2, , \ldots\}$ = the natural numbers. Let MLR $= \{X \in \{0,1\}^{\mathbb{N}} \mid X$ is Martin-Löf random$\}$. The following theorem is a slight generalization of [3, Theorem 7.6(i)]. See also Kurtz's earlier result in [10, Proposition 3].

Definition 2.1. Given a function $p : \mathbb{N} \to \mathbb{N}$, we write

$$\prod p = \prod_{i=0}^{\infty} \{j \mid j < p(i)\} = \{Y \in \mathbb{N}^{\mathbb{N}} \mid \forall i\,(Y(i) < p(i))\}$$

denoting the set of p-bounded functions.

Theorem 2.2 (Miller)**.** Let $p : \mathbb{N} \to \mathbb{N}$ be a recursive function such that $\forall i\,(p(i) \geq 2)$ and $\sum_{i=0}^{\infty} p(i)^{-1} < \infty$. Let $\psi : \subseteq \mathbb{N} \to \mathbb{N}$ be a partial recursive function. Then $(\forall X \in \mathrm{MLR})\,(\exists Y \leq_{\mathrm{T}} X)\,(Y \in \prod p$ and $Y \cap \psi = \emptyset)$.

Proof. For each i let $q(i)$ be such that $2^{q(i)} \leq p(i) < 2^{q(i)+1}$. Note that $q : \mathbb{N} \to \mathbb{N}$ is recursive and $\forall i\,(q(i) \geq 1)$. For all $X \in \{0,1\}^{\mathbb{N}}$ define $\Psi^X : \mathbb{N} \to \mathbb{N}$ by $\Psi^X(i) =$

$\sum_{j<q(i)} X(j)2^j < 2^{q(i)}$. Let $U_i = \{X \mid \Psi^X(i) = \psi(i)\}$, and let λ be the fair coin probability measure on $\{0,1\}^{\mathbb{N}}$. Clearly U_i is uniformly Σ^0_1 and $\lambda(U_i) \le 2^{-q(i)}$, hence $\sum_i \lambda(U_i) \le \sum_i 2^{-q(i)} = 2\sum_i 2^{-q(i)-1} < 2\sum_i p(i)^{-1} < \infty$. Hence by Solovay's Lemma [16, Lemma 3.5] we have $(\forall X \in \mathrm{MLR})\, \exists n\, (\forall i \ge n)\, (X \notin U_i)$, i.e., $(\forall X \in \mathrm{MLR})\, \exists n\, (\forall i \ge n)\, (\Psi^X(i) \ne \psi(i))$. Given $X \in \mathrm{MLR}$, fix such an n and define $Y : \mathbb{N} \to \mathbb{N}$ by

$$Y(i) = \begin{cases} 1 & \text{if } i < n \text{ and } \psi(i) = 0, \\ 0 & \text{if } i < n \text{ and } \psi(i) \ne 0, \\ \Psi^X(i) & \text{if } i \ge n. \end{cases}$$

Then Y differs at most finitely from Ψ^X, hence $Y \le_{\mathrm{T}} X$, and it is also clear that $\forall i\, (Y(i) < 2^{q(i)} \le p(i)$ and $Y(i) \ne \psi(i))$. □

Definition 2.3. Let $\mathrm{DNR} = \{Y \in \mathbb{N}^{\mathbb{N}} \mid \forall i\, (Y(i) \ne \varphi_i(i))\}$ where φ_e, $e \in \mathbb{N}$ is some fixed standard enumeration of the 1-place partial recursive functions. Given $p : \mathbb{N} \to \mathbb{N}$, let $\mathrm{DNR}_p = \mathrm{DNR} \cap \prod p$, and let $\mathrm{DNR}_{\mathrm{REC}} = \{Y \mid Y \in \mathrm{DNR}_p$ for some recursive function $p\}$.

Corollary 2.4. Let $p : \mathbb{N} \to \mathbb{N}$ be a recursive function such that $\forall i\, (p(i) \ge 2)$ and $\sum_{i=0}^{\infty} p(i)^{-1} < \infty$. Then $(\forall X \in \mathrm{MLR})\, (\exists Y \in \mathrm{DNR}_p)\, (Y \le_{\mathrm{T}} X)$.

Proof. This is the special case of Theorem 2.2 with $\psi(i) \simeq \varphi_i(i)$. □

3 When $\sum_i p(i)^{-1} = \infty$

The following definition and theorem are slight generalizations of [3, Definition 4.1(i), Theorem 5.3].

Definition 3.1.

1. We write $\mathbb{N}^* = \bigcup_{n=0}^{\infty} \mathbb{N}^n$ denoting the set of finite sequences of natural numbers. We use σ as a variable ranging over $\mathbb{N}^*$. Let $[0,1]$ denote the unit interval in the real line, and let $\mathbb{Q}$ denote the set of rational numbers.

2. A *continuous semimeasure on* $\mathbb{N}^*$ is a function $M : \mathbb{N}^* \to [0,1]$ such that $\forall \sigma\, (M(\sigma) \ge \sum_{i \in \mathbb{N}} M(\sigma^\frown\langle i \rangle))$.

3. A continuous semimeasure M on $\mathbb{N}^*$ is said to be *left recursively enumerable*, abbreviated *left r.e.,* if there exists a recursive function $(s,\sigma) \mapsto M_s(\sigma) : \mathbb{N} \times \mathbb{N}^* \to \mathbb{Q}$ such that $\forall \sigma\, (M(\sigma) = \lim_s M_s(\sigma)$ and $\forall s\, (0 \le M_s(\sigma) \le M_{s+1}(\sigma)))$. We may safely assume that $\forall s\, (M_s$ is a continuous semimeasure on $\mathbb{N}^*$ and $\{\sigma \mid M_s(\sigma) > 0\}$ is finite).

4. A left r.e. continuous semimeasure M on $\mathbb{N}^*$ is said to be *universal* if for all left r.e. continuous semimeasures $\overline{M}$ on $\mathbb{N}^*$ we have $\exists c\, \forall \sigma\, (\overline{M}(\sigma) < c \cdot M(\sigma))$. It is straightforward to prove the existence of such an M.

5. Throughout this note we let M denote a fixed universal left r.e. continuous semimeasure on $\mathbb{N}^*$, and we fix $M_s(\sigma)$ as above. Our definitions and results will not depend on the choice of M and $M_s(\sigma)$.

6. Given $Q \subseteq \mathbb{N}^\mathbb{N}$ and $n \in \mathbb{N}$, let $Q{\upharpoonright}n = \{Y{\upharpoonright}n \mid Y \in Q\}$. Note that $Q{\upharpoonright}n$ is a subset of $\mathbb{N}^n$, which is a prefix-free subset of $\mathbb{N}^*$. For any prefix-free set $S \subseteq \mathbb{N}^*$ let $M(S) = \sum_{\sigma \in S} M(\sigma)$.

7. A set $Q \subseteq \mathbb{N}^\mathbb{N}$ is said to be *deep* if there exists a recursive function $r : \mathbb{N} \to \mathbb{N}$ such that $\forall n\, (M(Q{\upharpoonright}r(n)) \leq 2^{-n})$.

Theorem 3.2 ([3, Theorem 5.3])**.** Let $p : \mathbb{N} \to \mathbb{N}$ be a recursive function, and let $Q \subseteq \prod p$ be deep and Π^0_1. Then $(\forall X \in \mathrm{MLR})\, (\forall Y \in Q)\, (Y \leq_\mathrm{T} X \Rightarrow 0' \leq_\mathrm{T} X)$.

Proof. A *difference test* is a pair of sequences U_n, V_n, $n \in \mathbb{N}$ of uniformly Σ^0_1 subsets of $\{0,1\}^\mathbb{N}$ such that $\forall n\, (\lambda(U_n \setminus V_n) \leq 2^{-n})$. A real $X \in \{0,1\}^\mathbb{N}$ is said to be *difference random* [6] if for all such difference tests we have $\exists n\, (X \notin U_n \setminus V_n)$. We shall use the following result of Franklin and Ng [6]: X is difference random if and only if X is Martin-Löf random and $\not\geq_\mathrm{T} 0'$.

Let p and Q be as in the hypothesis of Theorem 3.2. Let r be a recursive function such that $\forall n\, (M(Q{\upharpoonright}r(n)) \leq 2^{-n})$. Since p and r are recursive and Q is a Π^0_1 subset of $\prod p$, it follows by König's Lemma that $Q{\upharpoonright}r(n)$ is Π^0_1 uniformly in n. Given a partial recursive functional Φ, consider the left r.e. continuous semimeasure M_Φ on $\mathbb{N}^*$ given by $M_\Phi(\sigma) = \lambda(\{X \in \{0,1\}^\mathbb{N} \mid \Phi^X{\upharpoonright}|\sigma| = \sigma\})$. Since M is a universal left r.e. continuous semimeasure on $\mathbb{N}^*$, let c_Φ be a constant such that $\forall \sigma\, (M_\Phi(\sigma) \leq c_\Phi \cdot M(\sigma))$. Let

$$U_n = \{X \in \{0,1\}^\mathbb{N} \mid (\forall i < r(n))\, (\Phi^X(i)\downarrow)\}$$

and let $V_n = \{X \in U_n \mid \Phi^X{\upharpoonright}r(n) \notin Q{\upharpoonright}r(n)\}$. Then U_n and V_n are uniformly Σ^0_1 and $\lambda(U_n \setminus V_n) = M_\Phi(Q{\upharpoonright}r(n)) \leq c_\Phi \cdot M(Q{\upharpoonright}r(n)) \leq c_\Phi \cdot 2^{-n}$. We now see that if $\Phi^X \in Q$ then X is not difference random, so by [6] $X \in \mathrm{MLR}$ implies $0' \leq_\mathrm{T} X$. Since Φ is an arbitrary partial recursive functional, Theorem 3.2 is proved. $\square$

Theorem 3.3 ([3, Theorem 7.6(ii)])**.** Let p be a recursive function such that $\sum_{i=0}^\infty p(i)^{-1} = \infty$. Then, we can effectively find a partial recursive function $\psi :\subseteq \mathbb{N} \to \mathbb{N}$ such that the Π^0_1 set $Q = \{Y \in \prod p \mid Y \cap \psi = \emptyset\}$ is deep.

Proof. We may safely assume that $p(i) > 0$ for all i, because otherwise $Q = \emptyset$. Since p is recursive and $\sum_i p(i)^{-1} = \infty$, let $r : \mathbb{N} \to \mathbb{N}$ be recursive such that $\sum_{r(n) \leq i < r(n+1)} p(i)^{-1} > 2^n$ holds for all n. We shall have $\psi = \bigcup_s \psi_s$ where ψ_s is defined recursively by stages, as follows.

Stage 0. Let $\psi_0 = \emptyset$.

Stage $s+1$. Let $Q_s = \{Y \in \prod p \mid Y \cap \psi_s = \emptyset\}$ and let $n = (s+1)_0 =$ the largest n such that 2^n is a divisor of $s+1$. There are three cases.

Case 1. If $M_s(Q_s{\upharpoonright}r(n+1)) \leq 2^{-n}$ then do nothing, i.e., $\psi_{s+1} = \psi_s$.

Case 2. Otherwise, if $\{i \mid r(n) \leq i < r(n+1)\} \subseteq \mathrm{dom}(\psi_s)$ then again do nothing, i.e., $\psi_{s+1} = \psi_s$.

Case 3. Otherwise, pick an i such that $r(n) \le i < r(n+1)$ and $i \notin \mathrm{dom}(\psi_s)$. For each $j < p(i)$ let $Q_s^j = \{X \in Q_s \mid X(i) = j\}$. Thus $Q_s = \bigcup_{j<p(i)} Q_s^j$ and $Q_s{\upharpoonright}r(n+1) = \bigcup_{j<p(i)} Q_s^j{\upharpoonright}r(n+1)$ and these unions are disjoint unions. Since $M_s(Q_s{\upharpoonright}r(n+1)) > 2^{-n}$, there is at least one $j < p(i)$ such that $M_s(Q_s^j{\upharpoonright}r(n+1)) > 2^{-n}p(i)^{-1}$. Pick such a j and let $\psi_{s+1} = \psi_s \cup \{\langle i, j\rangle\}$.

In Case 3 we have $Q_{s+1} = Q_s \setminus Q_s^j$, hence $Q_{s+1}{\upharpoonright}r(n+1) = Q_s{\upharpoonright}r(n+1) \setminus Q_s^j{\upharpoonright}r(n+1)$, hence

$$\begin{aligned} M(Q_{s+1}{\upharpoonright}r(n+1)) &= M(Q_s{\upharpoonright}r(n+1)) - M(Q_s^j{\upharpoonright}r(n+1)) \\ &\le M(Q_s{\upharpoonright}r(n+1)) - M_s(Q_s^j{\upharpoonright}r(n+1)) \\ &< M(Q_s{\upharpoonright}r(n+1)) - 2^{-n}p(i)^{-1}. \end{aligned} \tag{1}$$

But $M(Q_0{\upharpoonright}r(n+1)) \le 1 < \sum_{r(n)\le i<r(n+1)} 2^{-n}p(i)^{-1}$, so from (1) we see that for each n Case 3 holds at fewer than $r(n+1) - r(n)$ many stages $s+1$ with $(s+1)_0 = n$, and Case 2 never holds. Hence Case 1 holds at stage $s+1$ for all sufficiently large s such that $(s+1)_0 = n$, hence $M_s(Q_s{\upharpoonright}r(n+1)) \le 2^{-n}$ for all such s, so letting $Q = \bigcap_s Q_s$ we have $M(Q{\upharpoonright}r(n+1)) \le 2^{-n}$, Q.E.D. □

Theorem 3.4 (Miller)**.** Let p be a recursive function such that $\sum_{i=0}^{\infty} p(i)^{-1} = \infty$. Then, we can find a partial recursive function $\psi :\subseteq \mathbb{N} \to \mathbb{N}$ such that $(\forall X \in \mathrm{MLR})\,(\forall Y \in \prod p)\,(\text{if } Y \cap \psi = \emptyset \text{ and } Y \le_{\mathrm{T}} X \text{ then } 0' \le_{\mathrm{T}} X)$.

Proof. This is immediate from Theorems 3.2 and 3.3. □

Corollary 3.5 (Stephan [20])**.** If $X \in \mathrm{MLR}$ is of PA-degree, then $0' \le_{\mathrm{T}} X$.

Proof. Applying Theorem 3.4 with $p(i) = 2$ for all i, we obtain a disjoint pair of recursively enumerable sets $A_0 = \{i \mid \psi(i) = 0\}$ and $A_1 = \{i \mid \psi(i) = 1\}$ with the following property: $(\forall Y \in \{0,1\}^{\mathbb{N}})\,(\text{if } Y \text{ separates } A_0 \text{ from } A_1 \text{ then } (\forall X \in \mathrm{MLR})\,(Y \le_{\mathrm{T}} X \Rightarrow 0' \le_{\mathrm{T}} X))$. The corollary follows, because any X which is of PA-degree computes a separating function for any disjoint pair of recursively enumerable sets. □

4 Linear universality

Despite Theorems 3.2 and 3.4, it is not clear whether the following holds:

> If $p : \mathbb{N} \to \mathbb{N}$ is nondecreasing and recursive and $\sum_{i=0}^{\infty} p(i)^{-1} = \infty$, then $(\forall X \in \mathrm{MLR})\,(\forall Y \in \mathrm{DNR}_p)\,(Y \le_{\mathrm{T}} X \Rightarrow 0' \le_{\mathrm{T}} X)$.

The difficulty here is that, depending on our choice of a standard enumeration of the partial recursive functions, there may or may not exist a one-to-one recursive function $r : \mathbb{N} \to \mathbb{N}$ such that $\forall i\,(\psi(i) \simeq \varphi_{r(i)}(r(i)))$ and $\sum_{i=0}^{\infty} p(r(i))^{-1} = \infty$. See also the remarks of Bienvenu and Porter concerning their [3, Definition 7.5].

However, as we shall explain in this section and the next, the statement displayed above holds if we replace DNR functions by *linearly DNR* functions.

Definition 4.1. Let $\psi :\subseteq \mathbb{N} \to \mathbb{N}$ be a partial recursive function. We say that ψ is *universal* if for all partial recursive functions $\varphi :\subseteq \mathbb{N} \to \mathbb{N}$ there exists a recursive function $r : \mathbb{N} \to \mathbb{N}$ such that $\forall i\,(\varphi(i) \simeq \psi(r(i)))$. We say that ψ is *linearly universal* if it is "universal via linear functions," i.e., for all partial recursive functions $\varphi :\subseteq \mathbb{N} \to \mathbb{N}$ there exist constants $a, b \in \mathbb{N}$ such that $\forall i\,(\varphi(i) \simeq \psi(ai + b))$.

Example 4.2. Let φ_e, $e \in \mathbb{N}$ be a standard enumeration of the 1-place partial recursive functions. The partial recursive function ψ defined by $\psi(i) \simeq \varphi_i(i)$ is universal. The partial recursive function ψ defined by $\psi(2^e(2i+1)) \simeq \varphi_e(i)$ is linearly universal.

Lemma 4.3. If a partial recursive function $\psi :\subseteq \mathbb{N} \to \mathbb{N}$ is linearly universal, then it is "uniformly linearly universal." More precisely, there exist primitive recursive functions $a, b : \mathbb{N} \to \mathbb{N} \setminus \{0\}$ such that $\forall e\,\forall i\,(\varphi_e(i) \simeq \psi(a(e)i + b(e)))$.

Proof. Fix an index $\widehat{e}$ such that $\forall e\,\forall i\,(\varphi_{\widehat{e}}(2^e(2i+1)) \simeq \varphi_e(i))$. Since ψ is linearly universal, fix constants $\widehat{a}, \widehat{b} \in \mathbb{N}$ such that $\forall i\,(\varphi_{\widehat{e}}(i) \simeq \psi(\widehat{a}i + \widehat{b}))$. For all e and all i we have $\varphi_e(i) \simeq \varphi_{\widehat{e}}(2^e(2i+1)) \simeq \psi(\widehat{a}2^e(2i+1) + \widehat{b})$, so we may take $a(e) = 2^{e+1}\widehat{a}$ and $b(e) = 2^e\widehat{a} + \widehat{b}$. Since $\varphi_{\widehat{e}}$ is not a constant function, we have $\widehat{a} > 0$, hence $a(e) > 0$ and $b(e) > 0$ for all e. □

The next two theorems improve the conclusions of Theorems 3.3 and 3.4 by saying that they hold for any ψ which is linearly universal.

Theorem 4.4 ([3, Theorem 7.6(ii)])**.** Let p be a nondecreasing recursive function such that $\sum_{i=0}^{\infty} p(i)^{-1} = \infty$. Let ψ be a partial recursive function which is linearly universal. Then the Π^0_1 set $Q = \{Y \in \prod p \mid Y \cap \psi = \emptyset\}$ is deep.

Proof. Let $\overline{e} \in \mathbb{N}$ be given. Since ψ is linearly universal, let $\overline{a} = a(\overline{e})$ and $\overline{b} = b(\overline{e})$ where $a, b : \mathbb{N} \to \mathbb{N}$ are fixed primitive recursive functions as given by Lemma 4.3. Thus we have $\varphi_{\overline{e}}(i) \simeq \psi(\overline{a}i + \overline{b})$ for all i. Define $\overline{p} : \mathbb{N} \to \mathbb{N}$ by $\overline{p}(i) = p(\overline{a}i + \overline{b})$. Since p is recursive and nondecreasing with $\sum_i p(i)^{-1} = \infty$, we claim that $\overline{p}$ is likewise recursive and nondecreasing with $\sum_i \overline{p}(i)^{-1} = \infty$. To see this, note that for all i and j we have $\overline{a}i + \overline{b} \le \overline{a}i + \overline{b} + j$, hence $\overline{p}(i) = p(\overline{a}i + \overline{b}) \le p(\overline{a}i + \overline{b} + j)$, hence $\overline{p}(i)^{-1} \ge p(\overline{a}i + \overline{b} + j)^{-1}$, hence $\overline{a}\,\overline{p}(i)^{-1} \ge \sum_{j<\overline{a}} p(\overline{a}i + \overline{b} + j)^{-1}$, hence $\overline{a} \sum_i \overline{p}(i)^{-1} \ge \sum_i \sum_{j<\overline{a}} p(\overline{a}i + \overline{b} + j)^{-1} = \sum_{j \ge \overline{b}} p(j)^{-1} = \infty$, hence $\sum_i \overline{p}(i)^{-1} = \infty$ as claimed. But then, applying Theorem 3.3 to $\overline{p}$, we can effectively find a partial recursive function $\overline{\psi} : \mathbb{N} \to \mathbb{N}$ such that the Π^0_1 set $\overline{Q} = \{\overline{Y} \in \prod \overline{p} \mid \overline{Y} \cap \overline{\psi} = \emptyset\}$ is deep.

Our construction of $\overline{\psi}$ given $\overline{e}$ is uniform in the following sense: there is a primitive recursive function which maps an arbitrary $\overline{e}$ to an index of the corresponding partial recursive function $\overline{\psi}$. Therefore, by the Recursion Theorem (a.k.a., the Recursion-Theoretic Fixed Point Theorem, see [14, §11.2]) we can find an $\overline{e}$ which is an index of the corresponding $\overline{\psi}$. For this $\overline{e}$ and for all i we have $\overline{\psi}(i) \simeq \varphi_{\overline{e}}(i) \simeq \psi(\overline{a}i + \overline{b})$. Thus the recursive functional $Y \mapsto \overline{Y}$ given by $\overline{Y}(i) = Y(\overline{a}i + \overline{b})$ maps Q into $\overline{Q}$. Since $\overline{Q}$ is deep, it follows by [3, Theorem 6.4] that Q is deep, Q.E.D. □

Theorem 4.5 (essentially due to Miller)**.** Let $p : \mathbb{N} \to \mathbb{N}$ be a nondecreasing recursive function such that $\sum_{i=0}^{\infty} p(i)^{-1} = \infty$. Let $\psi : \subseteq \mathbb{N} \to \mathbb{N}$ be a partial recursive function which is linearly universal. Then

$$(\forall X \in \mathrm{MLR})\,(\forall Y \in \textstyle\prod p)\,(\text{if } Y \cap \psi = \emptyset \text{ and } Y \leq_{\mathrm{T}} X \text{ then } 0' \leq_{\mathrm{T}} X).$$

Proof. This is immediate from Theorems 3.2 and 4.4. □

5 Some Muchnik degrees in $\mathcal{E}_{\mathrm{w}}$

Recall from [17, 18, 19] that $\mathcal{E}_{\mathrm{w}}$ is the lattice of Muchnik degrees of nonempty Π^0_1 subsets of $\{0,1\}^{\mathbb{N}}$. Recall also from [15, 17, 18, 19] that $\mathbf{r}_1 = \deg_{\mathrm{w}}(\mathrm{MLR}) \in \mathcal{E}_{\mathrm{w}}$ and $\mathbf{0} < \mathbf{r}_1 < \mathbf{1}$, where $\mathbf{0}$ and $\mathbf{1}$ are the bottom and top Muchnik degrees in $\mathcal{E}_{\mathrm{w}}$. The purpose of this section is to define and discuss some specific, natural Muchnik degrees in $\mathcal{E}_{\mathrm{w}}$ which are associated with Theorems 2.2 and 4.5.

Definition 5.1. A function $Y : \mathbb{N} \to \mathbb{N}$ is said to be *linearly DNR* if $Y \cap \psi = \emptyset$ for some linearly universal partial recursive function $\psi : \subseteq \mathbb{N} \to \mathbb{N}$. We write $\mathrm{LDNR} = \{Y \in \mathbb{N}^{\mathbb{N}} \mid Y \text{ is linearly DNR}\}$ and $\mathrm{LDNR}_{\mathrm{REC}} = \{Y \in \mathrm{LDNR} \mid Y \in \prod p \text{ for some recursive function } p\}$. Given $p : \mathbb{N} \to \mathbb{N}$ let $\mathbf{d}_p = \deg_{\mathrm{w}}(\mathrm{LDNR}_p)$ be the Muchnik degree of $\mathrm{LDNR}_p = \mathrm{LDNR} \cap \prod p$.

Remark 5.2.

1. It is easy to see that $\deg_{\mathrm{w}}(\mathrm{LDNR}) = \deg_{\mathrm{w}}(\mathrm{DNR}) = \mathbf{d}$ and $\deg_{\mathrm{w}}(\mathrm{LDNR}_{\mathrm{REC}}) = \deg_{\mathrm{w}}(\mathrm{DNR}_{\mathrm{REC}}) = \mathbf{d}_{\mathrm{REC}}$, and by [1, 15, 17] these Muchnik degrees belong to $\mathcal{E}_{\mathrm{w}}$ and we have $\mathbf{0} < \mathbf{d} < \mathbf{d}_{\mathrm{REC}} < \mathbf{r}_1$. Moreover $\mathbf{d} = \inf_p \mathbf{d}_p$ where p ranges over all functions, and $\mathbf{d}_{\mathrm{REC}} = \inf_p \mathbf{d}_p$ where p ranges over all recursive functions.

2. Note that LDNR and $\mathrm{LDNR}_{\mathrm{REC}}$ are independent of the choice of a standard enumeration of the partial recursive functions. Moreover, LDNR_p and $\mathbf{d}_p$ are also independent of this choice, provided p is nondecreasing. In particular, the Muchnik degree $\mathbf{d}_p$ is specific and natural[1] provided p is specific, natural, and nondecreasing. This would not be the case if we had based our definition of $\mathbf{d}_p$ on DNR instead of LDNR. By using LDNR instead of DNR, we can now sharpen the observations in [15, §10].

3. Let p be nondecreasing and unbounded such that $p(0) \geq 2$. Let ψ be a linearly universal partial recursive function. Is the Muchnik degree of $Q = \{Y \in \prod p \mid Y \cap \psi = \emptyset\}$ independent of the choice of ψ? If so, then we could define $\mathbf{d}_p$ more simply as $\mathbf{d}_p = \deg_{\mathrm{w}}(Q)$. Our actual definition of $\mathbf{d}_p$ circumvents this question, at the cost of extra complication.

4. Clearly $\forall i\,(p(i) \leq q(i))$ implies $\mathbf{d}_q \leq \mathbf{d}_p$. There are many open questions here concerning specific, natural Muchnik degrees in $\mathcal{E}_{\mathrm{w}}$. For instance, letting $p(i) = \max(i^2, 1)$ and $q(i) = \max(i^3, 1)$, do we have $\mathbf{d}_q < \mathbf{d}_p$?

[1]For an explanation of what we mean by *specific and natural*, see [19, footnote 2].

Lemma 5.3. The predicates "φ_e is linearly universal" and "Y is linearly DNR" are Σ^0_3.

Proof. Fix an index $\widehat{e}$ such that $\varphi_{\widehat{e}}$ is linearly universal. Then for all e, φ_e is linearly universal if and only if $\exists a\,\exists b\,\forall i\,(\varphi_{\widehat{e}}(i) \simeq \varphi_e(ai+b))$. A Tarski/Kuratowski computation [14, §14.3] shows that this predicate is Σ^0_3. Moreover, $Y \in \mathrm{LDNR}$ if and only if $\exists e\,(\varphi_e$ is linearly universal and $Y \cap \varphi_e = \emptyset)$, which is again Σ^0_3. □

Theorem 5.4. Let $p : \mathbb{N} \to \mathbb{N}$ be a nondecreasing recursive function such that $p(0) \geq 2$. Then $\mathbf{d}_p \in \mathcal{E}_{\mathrm{w}}$. Moreover, $\mathbf{d}_p \leq \mathbf{r}_1$ if and only if $\sum_{i=0}^{\infty} p(i)^{-1} < \infty$, and $\mathbf{d}_p \geq \mathbf{r}_1$ if and only if p is bounded, in which case $\mathbf{d}_p = \mathbf{1}$.

Proof. Lemma 5.3 implies that LDNR_p is Σ^0_3, and our assumption $\forall i\,(p(i) \geq 2)$ implies that LDNR_p includes a nonempty Π^0_1 subset of $\{0,1\}^{\mathbb{N}}$. It follows by the Σ^0_3 Embedding Lemma (see [17, Lemma 3.3] or [18, §3.3]) that $\mathrm{LDNR}_p \equiv_{\mathrm{w}} D_p$ for some nonempty Π^0_1 set $D_p \subseteq \{0,1\}^{\mathbb{N}}$. Thus $\mathbf{d}_p = \deg_{\mathrm{w}}(D_p) \in \mathcal{E}_{\mathrm{w}}$. Theorem 2.2 tells us that $\sum_i p(i)^{-1} < \infty$ implies $\mathbf{d}_p \leq \mathbf{r}_1$. Theorem 4.5 tells us that $\sum_i p(i)^{-1} = \infty$ implies $\mathbf{d}_p \not\leq \mathbf{r}_1$. A theorem of Jockusch [10, Theorem 5] says that if p is bounded then $\mathbf{d}_p = \mathbf{1}$. A theorem of Greenberg and Miller [7] says that if p is unbounded then $\mathbf{d}_p \not\geq \mathbf{r}_1$. □

Definition 5.5. An *order function* is an unbounded nondecreasing recursive function $p : \mathbb{N} \to \mathbb{N}$ such that $p(0) \geq 2$. Let us say that p is *slow-growing* if $\sum_i p(i)^{-1} = \infty$, otherwise *fast-growing*. Define $\mathrm{LDNR}_{\mathrm{slow}} = \bigcup_p \mathrm{LDNR}_p$ and $\mathbf{d}_{\mathrm{slow}} = \deg_{\mathrm{w}}(\mathrm{LDNR}_{\mathrm{slow}}) = \inf_p \mathbf{d}_p$ where p ranges over all slow-growing order functions. We could define $\mathrm{LDNR}_{\mathrm{fast}}$ and $\mathbf{d}_{\mathrm{fast}}$ similarly, but this would give us nothing new, because we would have $\mathrm{LDNR}_{\mathrm{fast}} = \mathrm{LDNR}_{\mathrm{REC}}$ and $\mathbf{d}_{\mathrm{fast}} = \mathbf{d}_{\mathrm{REC}}$.

Theorem 5.6. For each slow-growing order function p, we have $\mathbf{d}_p \in \mathcal{E}_{\mathrm{w}}$ and $\mathbf{d}_{\mathrm{REC}} < \mathbf{d}_p < \mathbf{1}$ and $\mathbf{d}_p$ is incomparable with $\mathbf{r}_1$. And similarly, we have $\mathbf{d}_{\mathrm{slow}} \in \mathcal{E}_{\mathrm{w}}$ and $\mathbf{d}_{\mathrm{REC}} < \mathbf{d}_{\mathrm{slow}} < \mathbf{1}$ and $\mathbf{d}_{\mathrm{slow}}$ is incomparable with $\mathbf{r}_1$.

Proof. The statements concerning $\mathbf{d}_p$ follow directly from Theorem 5.4. To obtain the same conclusions for $\mathbf{d}_{\mathrm{slow}}$, first imitate the proof of Lemma 5.3 to show that $\mathrm{LDNR}_{\mathrm{slow}}$ is Σ^0_3, then imitate the proof of Theorem 5.4. □

Remark 5.7. Given an order function p, Khan [11, Theorems 3.13 and 3.15] has shown how to construct order functions p^+ and p^- such that $\mathbf{d}_{p^+} < \mathbf{d}_p < \mathbf{d}_{p^-}$. If p is a slow-growing order function, it should be possible to construct a slow-growing order function p^+ such that $\mathbf{d}_{p^+} < \mathbf{d}_p$. This would imply that $\mathbf{d}_{\mathrm{slow}} < \mathbf{d}_p$ for all slow-growing order functions p.

References

[1] Klaus Ambos-Spies, Bjørn Kjos-Hanssen, Steffen Lempp, and Theodore A. Slaman. Comparing DNR and WWKL. *Journal of Symbolic Logic*, 69(4):1089–1104, 2004. 2, 7

[2] A. Beckmann, V. Mitrana, and M. Soskova, editors. *Evolving Computability, Proceedings of CiE 2015* (Bucharest). Number 9136 in Lecture Notes in Computer Science. Springer, 2015. XV + 363 pages. 10

[3] Laurent Bienvenu and Christopher P. Porter. Deep Π^0_1 classes. *Bulletin of Symbolic Logic*, 22(2):249–286, 2016. 2, 3, 4, 5, 6

[4] Z. Chatzidakis, P. Koepke, and W. Pohlers, editors. *Logic Colloquium '02: Proceedings of the Annual European Summer Meeting of the Association for Symbolic Logic and the Colloquium Logicum, held in Münster, Germany, August 3–11, 2002*. Number 27 in Lecture Notes in Logic. Association for Symbolic Logic, 2006. VIII + 359 pages. 10

[5] J.-E. Fenstad, I. T. Frolov, and R. Hilpinen, editors. *Logic, Methodology and Philosophy of Science VIII*. Number 126 in Studies in Logic and the Foundations of Mathematics. North-Holland, 1989. XVII + 702 pages. 9

[6] Johanna N. Y. Franklin and Keng Meng Ng. Difference randomness. *Proceedings of the American Mathematical Society*, 139(1):345–360, 2011. 4

[7] Noam Greenberg and Joseph S. Miller. Diagonally non-recursive functions and effective Hausdorff dimension. *Bulletin of the London Mathematical Society*, 43(4):636–654, 2011. 8

[8] Kojiro Higuchi, W. M. Phillip Hudelson, Stephen G. Simpson, and Keita Yokoyama. Propagation of partial randomness. *Annals of Pure and Applied Logic*, 165(2):742–758, 2014. http://dx.doi.org/10.1016/j.apal.2013.10.006. 2

[9] W. M. Phillip Hudelson. Mass problems and initial segment complexity. *Journal of Symbolic Logic*, 79(1):20–44, 2014. 2

[10] Carl G. Jockusch, Jr. Degrees of functions with no fixed points. In [5], pages 191–201, 1989. 2, 8

[11] Mushfeq Khan. *Some results on algorithmic randomness and computability-theoretic strength*. PhD thesis, University of Wisconsin, 2014. VII + 93 pages. 2, 8

[12] Bjørn Kjos-Hanssen, Wolfgang Merkle, and Frank Stephan. Kolmogorov complexity and the recursion theorem. *Transactions of the American Mathematical Society*, 363(10):5465–5480, 2011. 2

[13] Bjørn Kjos-Hanssen and Stephen G. Simpson. Mass problems and Kolmogorov complexity. 4 October 2006. Preprint, 1 page. 2

[14] Hartley Rogers, Jr. *Theory of Recursive Functions and Effective Computability*. McGraw-Hill, 1967. XIX + 482 pages. 6, 8

[15] Stephen G. Simpson. Mass problems and randomness. *Bulletin of Symbolic Logic*, 11(1):1–27, 2005. 2, 7

[16] Stephen G. Simpson. Almost everywhere domination and superhighness. *Mathematical Logic Quarterly*, 53(4–5):462–482, 2007. 3

[17] Stephen G. Simpson. An extension of the recursively enumerable Turing degrees. *Journal of the London Mathematical Society*, 75(2):287–297, 2007. 7, 8

[18] Stephen G. Simpson. Mass problems associated with effectively closed sets. *Tohoku Mathematical Journal*, 63(4):489–517, 2011. 7, 8

[19] Stephen G. Simpson. Degrees of unsolvability: a tutorial. In [2], pages 83–94, 2015. 7

[20] Frank Stephan. Martin-Löf random and PA-complete sets. In [4], pages 341–347, 2006. 2, 5