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A DEGREE-THEORETIC DEFINITION OF THE RAMIFIED ANALYTICAL HIERARCHY *

Carl G. JOCKUSCH, Jr.
Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL 61801, U.S.A.

Stephen G. SIMPSON
Department of Mathematics, Pennsylvania State University, University Park, PA 16802, U.S.A.

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Let D be the set of all (Turing) degrees, $\leqslant$ the usual partial ordering of D, and j the (Turing) jump operator on D. The following relations are shown to be first-order definable in the structure $\mathcal{D} = \langle D, \leqslant, j\rangle$: $\boldsymbol{d}_1$ is hyperarithmetical in $\boldsymbol{d}_2$, $\boldsymbol{d}_1$ is the hyperjump of $\boldsymbol{d}_2$, $\boldsymbol{d}_1$ is ramified analytical in $\boldsymbol{d}_2$ (Corollaries 4.6, 4.13, 4.16). A first-order, degree theoretic definition of the ramified analytical hierarchy is obtained (Theorem 5.6). A first-order sentence is found which is true in $\mathcal{D}$ if the universe is (a generic extension of) L, and false in $\mathcal{D}$ if $0^{\#}$ exists (Corollary 4.7). The question of whether the notion of uniform upper bound is degree theoretically definable is investigated (Section 6). Exact pairs of upper bounds are used to replace analytical definability by arithmetical definability (Theorem 3.1).

1. Introduction

This paper is a contribution to degree theory in the spirit of Kleene and Post [17]. Thus we are concerned exclusively with the structure of the upper semilattice of degrees (Turing degrees, degrees of recursive unsolvability) augmented by the jump operator. However, the questions which we study here are somewhat removed from those which have mainly occupied degree theorists over the past few years. We therefore begin with a general discussion which is intended to put the rest of the paper in perspective.

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Ever since Spector's construction of a minimal degree [31] and the Friedberg-Muchnik solution of Post's problem [24], degrees have been one of the most intensively developed parts of mathematical logic. A number of ingenious proof techniques have been devised. Indeed, it seems fair to say that degree theorists have been preoccupied almost exclusively with technique. "We regard an unsolved problem as interesting only if it seems likely that its solution requires a new trick." (Sacks [25, p. 169]).

A related phenomenon of the last four or five years is that the degrees occurring at low levels of the arithmetical hierarchy have taken over as the main objects of study. Especially the lowest level, the degrees of the Δ^0_2 sets, is emphasized; see for example Cooper's bibliography [3]. The most probable explanation for this phenomenon is that the Δ^0_2 sets offer the best arena for the development and refinement of new proof techniques. In the constricted world of the Δ^0_2 sets, number quantifiers are irrelevant. Delicate priority arguments emerge and take their place as the one indispensable technique. Thus the Δ^0_2 sets are studied, not for their intrinsic importance, but for the interest and importance of the methods employed in their study.

It seems to us that these methodological concerns constitute an entirely legitimate and reasonable justification for the study of the Δ^0_2 sets. Priority methods enrich all of mathematical logic, and they have first to be developed somewhere. However, for the present paper we have no need of such a justification.

Our viewpoint here is different. We want to step back and consider what intrinsic significance the degrees of unsolvability may have. Thus we have trying to practice degree theory as if it were a branch of science rather than an art form; we want to ask degree theoretic questions whose answers may be expected to be more important than the methods by which the answers are to be discovered and proved.

It seems to us that such intrinsic importance is to be found in the study of *degree theoretic hierarchies*, i.e. transfinite, degree theoretic iterations of the jump operator. Let D be the set of all degrees, $\leqslant$ the usual partial ordering on D, and $j : D \to D$ the jump operator on D. By a degree theoretic hierarchy we mean, precisely, a sequence of degrees $\langle \boldsymbol{d}_\alpha \mid \alpha < \theta \rangle$ where θ is a limit ordinal, such that

(i) $\boldsymbol{d}_0 = \boldsymbol{0}$;

(ii) $\boldsymbol{d}_{\alpha+1} = j(\boldsymbol{d}_\alpha)$ for all $\alpha < \theta$;

(iii) $\boldsymbol{d}_\alpha \leqslant \boldsymbol{d}_\beta$ for $\alpha \leqslant \beta < \theta$;

(iv) for λ a limit ordinal less than θ, $\boldsymbol{d}_\lambda$ is first order definable in the structure $\mathcal{D}_\lambda = \langle D, \leqslant, j, I_\lambda \rangle$ where $I_\lambda = \{\boldsymbol{d} \in D \mid \boldsymbol{d} \leqslant \boldsymbol{d}_\alpha$ for some $\alpha < \lambda\}$. Earlier work by Kleene, Spector [32], Enderton, Putnam [5], and Sacks [26] yields a natural, degree theoretic definition of the hyperarithmetical hierarchy with $\theta = \omega_1$ where ω_1 is the "constructive" ω_1 of Church and Kleene. In Section 5 below, this is extended to a natural, degree theoretic definition of the ramified analytical hierarchy with $\theta = \beta_0$ where β_0 is the ordinal of ramified analysis. (How are these uses of the word "natural" to be made precise? We have given some thought to this question but have not found an entirely satisfactory answer. One possibility is to strengthen clause (iv) in the definition of degree theoretic hierarchy by requiring a sequence of first-order formulas $\langle \psi_n \rangle_{n<\omega}$ such that with each limit ordinal $\lambda < \theta$ is associated a number n_λ defined as the least n such that ψ_n defines a degree $\boldsymbol{d}_\lambda$ in $\mathcal{D}_\lambda$, and n_θ is undefined. Another possibility is to require that the ψ_n have an "algebraic" flavor. By Lemma 5.1 the mentioned degree theoretic hierarchies are in fact natural in both of these senses.) The special case $\lambda = \omega_1$ is treated separately in Section 4 where in addition it is shown that $\boldsymbol{d}_{\omega_1}$, the degree of Kleene's $\mathcal{O}$, is first-order definable in the structure $\mathcal{D} = \langle D, \leqslant, j \rangle$.

Also in Section 4 is obtained a first-order sentence which is true in $\mathcal{D}$ if $V = \mathrm{L}$ or a generic extension of L, and false in $\mathcal{D}$ if $0^\#$ exists. This is the first result we know of to the effect that the first-order theory of $\mathcal{D}$ is not absolute in the sense of Gödel (although by Lachlan [21] the first-order theory of $\langle D, \leqslant \rangle$ is undecidable). Moreover, it shows clearly that the "global" structure of the continuum is interestingly reflected in the first-order theory of $\mathcal{D}$. Something like this possibility had been suggested by Boolos and Putnam [1] and by a lemma of Martin concerning Gale-Stewart games. The connection between the structure of the continuum and the first-order theory of $\mathcal{D}$ will be explored further in another paper by Simpson [29].

Our methods of proof are hardly new. The proof of the main theorem in Section 3 below goes back to Sacks [25, §8, 11] who goes back to Kleene and Post [17] and Spector [31]. (If there is any methodological novelty here at all, it is the observation that Jensen's "fine structure" [12] theory[1] is directly applicable to the study of degrees.) We believe

[1] Kleene and Post [17] used the term "fine structure" with a quite different meaning according to which Jensen's theory would have to be called "coarse structure". This does not detract from the fundamental importance of Jensen's work in [12].

however that the lack of novelty in the proofs is compensated for by the interest of the theorems.

2. Notation and prerequisities

We write $\omega = \{0, 1, 2, ...\}$ = the set of nonnegative integers; $\omega^\omega = \{f \mid f\colon \omega \to \omega\}$ = the set of all total one-place number theoretic functions; $2^\omega = \{X \mid X \subseteq \omega\}$ = the set of all subsets of ω. Letters $e, i, j, k, m, n, ...$ denote elements of ω. Letters $f, g, h, ...$ denote elements of ω^ω. Upper case Latin letters $A, B, C, ...$ denote subsets of ω. The characteristic function of A is denoted c_A. In Section 5, lower case Greek letters $\alpha, \beta, \gamma, ...$ are sometimes used to denote ordinal numbers.

One-place partial functions $\{i\}^f$, $\{i\}^{f^{(n)}}$, etc. are defined in the usual way. We write $f \leqslant_T A$ to mean that f is recursive in A, i.e. $\exists m \forall n$ $f(n) = \{m\}^A(n)$. Similarly for $f \leqslant_T g$, $A \leqslant_T B$, etc. We have a recursive pairing function $\oplus$ defined by

$$A \oplus B = \{2i \mid i \in A\} \cup \{2i+1 \mid i \in B\}$$

and $f \oplus g = h$ where $h(2i) = f(i)$, $h(2i+1) = g(i)$. We assume basic familiarity with the (relative) arithmetical hierarchy and the jump operator. In particular we assume Post's Hierarchy Theorem: for each $n < \omega$, A is Σ^0_{n+1} in B if and only if A is recursively enumerable in the n^{th} jump of B. For this theorem see Rogers [24, Chapters 14 and 15] or Shoenfield [27, §§7.5, 7.6]. In general our notation is drawn from Rogers [24]. A *degree* is an equivalence class under the relation $A \equiv_T B$, i.e. $A \leqslant_T B$ & $B \leqslant_T A$. Lower case boldface letters $\mathbf{a}, \mathbf{b}, \mathbf{c}, ...$ denote degrees. If $\mathbf{a}, \mathbf{b}$ are the degrees of A, B respectively then we write: $\mathbf{a} \leqslant \mathbf{b}$ for $A \leqslant_T B$; $\mathbf{a} \cup \mathbf{b}$ for the degree of $A \oplus B$; $\mathbf{0}$ for the degree of recursive functions; $\mathbf{a}'$ for the degree of the jump of A; $\mathbf{a}^{(n)}$ for the degree of the n^{th} jump of A; and $\mathbf{a}^{(\omega)}$ for the degree of the ω-jump of A, i.e. the recursive union of the n^{th} jumps of A [24, pp. 256–258]. The jump operator is important in degree theory because it expresses numerical quantification.

Our discussion of forcing in Section 3 is self contained. In Section 4 we assume knowledge of (relative) hyperarithmeticity. We write ω_1 for the least nonrecursive ordinal and ω_1^B for the least ordinal not recursive in B. Previous knowledge of hyperarithmeticity is *not* needed in order to understand the results in Section 5. However, one of the proofs there makes essential use of Davis's specific definition of the hyperarithmetical hierarchy in terms of H-sets. For this definition see Spector [32] or Rogers [24, §16.8]. Another of the proofs in Section 5 uses results of Jensen [12]; see also the exposition by Devlin [4]. Section 6 uses a result of Jockusch [14] but is otherwise self contained.

3. Exact pairs of upper bounds

Throughout this section, M is a countable, nonempty subset of ω^ω which is closed under relative recursiveness, pairing, and the jump operator. In symbols,

(i) $f \leqslant_T g \in M \to f \in M$,

(ii) $f, g \in M \to f \oplus g \in M$,

(iii) $f \in M \to \mathrm{jump}(f) \in M$.

We want to regard M as the range of the function variables in an ω-model of a fragment of second-order arithmetic; cf. Rogers [24, p. 385] and Shoenfield [27, p. 227]. See also Friedman [7] where the usefulness and naturalness of the closure conditions (i)–(iii) are emphasized.

Accordingly, we define what it means for a relation $R \subseteq M^i \times \omega^j$ to be Σ^1_k over M. R is *arithmetical over M* if there exist an arithmetical relation A and a parameter $h \in M$ such that for all $\boldsymbol{f} \in M^i$, $\boldsymbol{n} \in \omega^j$,

$$R(\boldsymbol{f}, \boldsymbol{n}) \leftrightarrow A(\boldsymbol{f}, \boldsymbol{n}, h) .$$

R is Σ^1_0 *over M* if it is arithmetical over M. R is Π^1_k *over M* if $\neg R$ is Σ^1_k over M. R is Σ^1_{k+1} *over M* if there exists a relation S which is Π^1_k over M and such that for all $\boldsymbol{f} \in M^i$, $\boldsymbol{n} \in \omega^j$,

$$R(\boldsymbol{f}, \boldsymbol{n}) \leftrightarrow (\exists g \in M)\, S(\boldsymbol{f}, g, \boldsymbol{n}) .$$

R is Δ^1_k *over M* if it is both Σ^1_k over M and Π^1_k over M. R is *analytical over M* or *analytically definable over M* if it is Σ^1_k over M for some $k < \omega$.

The goal of this Section is to prove the following theorem.

Theorem 3.1. *Let M be a countable, nonempty subset of ω^ω closed under $\leqslant_T$, $\oplus$, and the jump operator. Let n be a positive integer, and let H be a subset of ω. Then the following two assertions are equivalent.*

(a) *H is Σ^1_n over M.*

(b) *H is Σ^0_{n+2} in $A \oplus B$ whenever $A, B \subseteq \omega$ are such that*

$$M = \{f \mid f \leqslant_T A \;\&\; f \leqslant_T B\} . \tag{1}$$

A pair $A, B \subseteq \omega$ such that (1) holds is called an *exact pair* for M. The construction of an exact pair for M was first carried out (in a representative special case) by Kleene and Post [17]; see also Spector [31, Theorem 3] and Rogers [24, p. 274]. For us, the relevance of exact pairs is that analytical definability over M is equivalent to arithmetical definability relative to arbitrary exact pairs for M. This idea is expressed in the above Theorem and will be mercilessly exploited in the next two Sections.

Our first lemma, 3.2 below, resembles Kleene's Normal Form Theorem for the analytical hierarchy (over ω^ω). See Rogers [24, p. 376] and Shoenfield [27, p. 173]. It is perhaps worth noting that the last two prefix transformations of [24, p. 375] and [27, p. 173] are not valid for analytical definability over M. They are not valid because M is not assumed to be a model of the countable choice schema of second order arithmetic; i.e. it is not assumed that

$$\forall k(\exists f \in M)\, R(k, f) \to (\exists f \in M)\, \forall k R(k, (f)_k)$$

even if R is analytical over M. This is why Lemma 3.2 is not trivial.

Lemma 3.2. *Let $R \subseteq M^i \times \omega^j$ be a relation which is Σ^1_n over M, n odd. Then there are a Π^0_1 relation T and a parameter $h \in M$ such that, for all $f \in M^i$ and $k \in \omega^j$,*

$$R(f, k) \leftrightarrow (\exists g_1 \in M)(\forall g_2 \in M) \ldots (\exists g_n \in M)\, T(g_1, g_2, \ldots, g_n, f, k, h)\,.$$

Similarly if n is nonzero even.

Proof. The cases $n > 1$ follow immediately from the case $n = 1$. So let R be Σ^1_1 over M. For concreteness, suppose

$$R(f, k) \leftrightarrow (\exists g \in M)\ \forall w \exists x \forall y \exists z S(g, w, x, y, z, f, k, h)$$

where $h \in M$ and S is recursive and w, x, y, z are number variables. The idea of the proof is to replace the quantifiers $\exists x$, $\exists z$ by Skolem functions. Let $g \in M$, $f \in M^i$, and $k \in \omega^j$ be such that $\forall w \exists x \forall y \exists z S(g, w, x, y, z, f, k, h)$ holds. Define

$$g_1(w) = \mu x \forall y \exists z S(g, w, x, y, z, f, k, h)$$

and

$$g_2(\langle w, y\rangle) = \mu z S(g, w, g_1(w), y, z, f, k, h)$$

where μ is the least number operator. Then $g_1, g_2 \in M$ since they are arithmetical in $g, f, h \in M$ and M is closed under pairing and the jump operator. Thus, for all $f \in M^i$ and $k \in \omega$, one has

$$R(f, k) \leftrightarrow (\exists g \in M)(\exists g_1, g_2 \in M)\ \forall w \forall y S(g, w, g_1(w), y, g_2(\langle w, y\rangle), f, k, h)\,.$$

Then by pairing one gets

$$R(f, k) \leftrightarrow (\exists g \in M)\ \forall w S'(g, w\ f, k, h)$$

where S' is recursive, so R is in the desired form and Lemma 3.2 is proved. □

The proof that (a) ⇒ (b) in the theorem is now straightforward. Namely, suppose $n \geqslant 1$ and $H \subseteq \omega$ is Σ^1_n over M, and let $A, B \subseteq \omega$ be such that $M = \{f \mid f \leqslant_T A\ \&\ f \leqslant_T B\}$. We want to see that H is arithmetical in $A \oplus B$. This is accomplished by taking the Σ^1_n definition of H and replacing each function quantifier $(\exists g \in M)$ by number quantifiers

$$\exists e_1\, \exists e_2(\{e_1\}^A \text{ and } \{e_2\}^B \text{ are total and equal and } \ldots)\,.$$

Also, the parameter $h \in M$ is replaced by $\{i_1\}^A$ where i_1, i_2 are such that $\{i_1\}^A = \{i_2\}^B = h$. It is left to the reader to verify that these transformations convert a Σ^1_n definition over M into a Σ^0_{n+2} definition relative to $A \oplus B$, provided Lemma 3.2 is applied first in order to get rid of excess number quantifiers.

The proof that (b) ⇒ (a) is a perfect set forcing argument in the spirit of Sacks [26]. Let S be the set of finite sequences of 0's and 1's. Elements of S will be called *strings* and σ, τ, μ, ν will be variables for strings. If $P: S \to S$, P is called a *tree* if for all $\sigma, \tau \in S$, $\sigma \subseteq \tau \leftrightarrow P(\sigma) \subseteq P(\tau)$. (We write $\sigma \subseteq \tau$ to mean that τ is an extension of σ.) If $A \subseteq \omega$, let $\bar{A}(n)$ be the string of length n extended by the characteristic function of A. If P is a tree and $A \subseteq \omega$, let $P(A)$ be the unique subset of ω whose characteristic function extends $P(\bar{A}(n))$ for every n. Any set of the form $P(A)$ is called a *branch* of P, and $[P]$ denotes the set of all branches of P. Clearly $[P]$ is a perfect closed subset of 2^ω in its usual topology. A tree whose range is included in the range of P is called a *subtree* of P. If Q is a subtree of P, then $[Q] \subseteq [P]$. If P is a tree, we identify it with the corresponding number theoretic function under a Gödel numbering of S. Sacks [26] calls a tree *pointed* if it is recursive in all of its branches. We strengthen this notion by calling a tree P *uniformly pointed*, if there is a number e such that $\{e\}^B = P$ for every branch B of P.

Technical Note. The notion of uniform pointedness seems to be closely analogous to the notion of uniform introreducibility [15]. In fact a construction similar in strategy to but much easier than A.H. Lachlan's construction of a set which is introreducible but not uniformly so [15] yields a tree P which is pointed but not uniformly so. To make P pointed, one constructs P recursively in $0'$ and arranges that the even part of each branch of P differs only finitely from B, where B is a fixed set of degree $0'$. To insure that P is not uniformly pointed, one arranges that if A is any set with least element n, then $\{n\}^{P(A)} \neq B$. This can easily be done since B is not recursive. The requirements do not conflict significantly, and so the priority method is not used. In contrast to this result, every pointed tree P has a uniformly pointed subtree Q of the same degree as P. To see this, one attempts to construct a descending chain $\langle P_n \rangle_{n<\omega}$ of subtrees of P, all recursive in P, such that $\{n\}^A \neq P$ for every branch A of P_{n+1}. The process must "get stuck" since P is pointed and from this it is easy to construct a uniformly pointed subtree of P of the same degree as P.

The following Lemma is essentially the same as [26, Proposition 3.2].

Lemma 3.3. *If P is a uniformly pointed tree recursive in the set A, then P has a uniformly pointed subtree Q of the same degree as A.*

Proof. We want Q to be a tree such that $Q(B) = P(A \oplus B)$ for all $B \subseteq \omega$. Hence we define

$$Q(\langle b_0, ..., b_n \rangle) = P(\langle a_0, b_0, a_1, b_1, ..., a_n, b_n \rangle)$$

where a_i is 0 or 1 according as $i \in A$. Since P is uniformly pointed, for any B

$$Q(B) = P(A \oplus B) \equiv_T P \oplus A \oplus B \tag{2}$$

where the Turing equivalence is uniform in B. From this it follows that $P \oplus A$, and hence Q, is uniformly recursive in all branches of Q, so Q is uniformly pointed. Clearly $Q \equiv_T Q(\emptyset)$, and $Q(\emptyset) \equiv_T P \oplus A \oplus \emptyset \equiv_T A$ by (2) and the assumption $P \leqslant_T A$. Hence $Q \equiv_T A$. □

We call a set of trees *arithmetical* if the corresponding set of number theoretic functions is an arithmetical subset of ω^ω. Clearly the set of all trees is arithmetical, in fact Π^0_1.

Lemma 3.4. *The set of all uniformly pointed trees is arithmetical.*

Proof. A tree P is uniformly pointed if and only if

(3) $\exists n \forall A (A \in [P] \to P = \{n\}^A)$.

The matrix of (3) is a Π^0_2 relation of A, P, and n. But it is a well-known fact that the class of Π^0_2 relations is closed under universal set quantification. (A closely related fact appears in Shoenfield [27, p. 187].) Hence (3) is arithmetical (in fact Σ^0_3). □

Remark. We do not know whether the set of pointed trees is arithmetical (although it is obviously Π^1_1). In fact the notion of uniform pointedness was introduced so that our forcing conditions could be arithmetically characterized.

Assume now that M is a countable, nonempty subset of ω^ω closed under $\leqslant_T$ and $\oplus$. The proof that (b) $\Rightarrow$ (a) will be obtained by applying (b) to pairs (A, B) which will be generic with respect to a certain notion of forcing which we now describe. A *condition* is a pair $(P_0, P_1) \in M \times M$ such that $P_0 \equiv_T P_1$ and each P_i is a uniformly pointed tree. A condition (Q_0, Q_1) *extends* (P_0, P_1) if Q_0 is a subtree of P_0 and Q_1 is a subtree of P_1. We write $[P_0, P_1]$ for $[P_0] \times [P_1]$. Our *forcing language* is roughly first-order arithmetic with two free set variables $\underset{\sim}{A}$, $\underset{\sim}{B}$. More precisely, for each number m we have a numeral m and for each primitive recursive relation $R(A, B, v)$ ($R \subseteq 2^\omega \times 2^\omega \times \omega^k$, $k \geqslant 0$) we have a $(k+2)$-place relation symbol $\underset{\sim}{R}$. A typical *atomic formula* is $\underset{\sim}{R}(\underset{\sim}{A}, \underset{\sim}{B}, s)$, where $\underset{\sim}{A}$, $\underset{\sim}{B}$ are the two set variables and s is a sequence of number variables and numerals. Arbitrary *formulas* are built up from atomic formulas using negation ($\neg$) and the number quantifier $(\exists x)$. We now inductively classify certain formulas as Σ^0_n or Π^0_n. This classification will not be literally identical to the usual one, but it will be obvious that a formula which is Σ^0_n [Π^0_n] in the usual sense can be effectively translated into one which is Σ^0_n [Π^0_n] in our sense and vice versa, for $n \geqslant 1$. A formula is Σ^0_0 if it is atomic. A formula is Π^0_n if it has the form $\neg\psi$ where ψ is Σ^0_n. A formula is Σ^0_{n+1} if it has the form $(\exists x)\psi$ where ψ is Π^0_n. A formula is a *sentence* if it is Σ^0_n or Π^0_n for some n and has no free number variables. We now define the relation $(P, Q) \Vdash \phi$ (read (P, Q) *forces* ϕ) for conditions (P, Q) and sentences ϕ. The definition is orthodox except at the three lowest quantifier levels on the Π side where it is taken to coincide with truth in order to facilitate the proof of Lemma 3.8 on the definability of forcing. Explictly, if $\phi(\underset{\sim}{A}, \underset{\sim}{B})$ is a Σ^0_0 or Π^0_n sentence for $n \leqslant 2$, $(P, Q) \Vdash \phi(\underset{\sim}{A}, \underset{\sim}{B})$ means that $\phi(A, B)$ is true for all $(A, B) \in [P, Q]$. If ϕ is a Σ^0_{n+1} sentence (and so of the form $(\exists x)\ \psi(x))$, $(P, Q) \Vdash \phi$ means that $(P, Q) \Vdash \psi(\underline{m})$ for some numeral $\underline{m}$. If ϕ is a Π^0_n sentence for $n \geqslant 3$ (and so of the form $\neg\psi$), $(P, Q) \Vdash \phi$ means that there is no condition (P', Q') extending (P, Q) such that $(P', Q') \Vdash \psi$. As usual, a set of conditions is called *dense* if every condition is extended by some condition in the set.

Lemma 3.5. *If ϕ is a Σ^0_n sentence, the set of conditions which force ϕ or force $\neg\phi$ is dense.*

Proof. For $n \geqslant 3$ this is obvious from the definition of forcing for Π^0_n sentences. We prove it for $n = 2$ since the proofs for $n < 2$ are similar but easier. The proof for $n = 2$ is an adaptation of the main argument of [25, § 11]; see also the Σ^0_2 case of [26, Lemma 3.1]. Let ϕ be a Σ^0_2 sentence. Then ϕ has the form $(\exists x)\neg(\exists y)\psi$ where $\psi(\underline{A}, \underline{B}, x, y)$ is an atomic formula corresponding to a primitive recursive relation $R(A, B, x, y)$. Let $R(\sigma, \tau, x, y)$ mean that from the information $c_A \supseteq \sigma$, $c_B \supseteq \tau$ it may be computed that $R(A, B, x, y)$ holds. We may assume that $R(\sigma, \tau, x, y)$ is a recursive relation of σ, τ, x, y and that if $\sigma' \supseteq \sigma$, $\tau' \supseteq \tau$, and $R(\sigma, \tau, x, y)$ then $R(\sigma', \tau', x, y)$. Let (P_0, P_1) be a condition. We want to find a condition (Q_0, Q_1) which extends (P_0, P_1) and either forces ϕ or forces $\neg\phi$.

Case 1. $(\exists n)(\exists\sigma_0)(\exists\tau_0)\neg(\exists\sigma \supseteq \sigma_0)(\exists\tau \supseteq \tau_0)(\exists y)R(P_0(\sigma), P_1(\tau), n, y)$. Choose n, σ_0, τ_0 as in the Case hypothesis and define $Q_0(\sigma) = P_0(\sigma_0 * \sigma)$, $Q_1(\tau) = P_1(\tau_0 * \tau)$ for all σ, τ. (We write $\sigma * \tau$ for the concatenation σ followed by τ.) Then (Q_0, Q_1) is a condition and $(Q_0, Q_1) \Vdash \neg(\exists y)\ \psi(\underline{A}, \underline{B}, \underline{n}, y)$ so $(Q_0, Q_1) \Vdash \phi$.

Case 2. Not Case 1. In this case, one constructs a condition (Q_0, Q_1) so that whenever $\mathrm{lh}(\sigma) = \mathrm{lh}(\tau) = n + 1$, then

(4) $\qquad (\exists y)\,R(Q_0(\sigma), Q_1(\tau), n, y)$

holds. If this is done, it follows at once that $(Q_0, Q_1) \Vdash \neg\phi$. We define $Q_0(\sigma)$, $Q_1(\tau)$ simultaneously by induction on $\mathrm{lh}(\sigma) = \mathrm{lh}(\tau)$. Let $Q_i(\emptyset) = P_i(\emptyset)$. Assume inductively that $Q_i(\sigma)$ is defined whenever $\mathrm{lh}(\sigma) = n$, $i = 0$ or 1. If $\mathrm{lh}(\sigma) = n$, the *initial candidates* for $Q_i(\sigma * 0)$, $Q_i(\sigma * 1)$ are any two (effectively chosen) incompatible extensions of $Q_i(\sigma)$ which are in the range of P_i. The value of $Q_i(\sigma * j)$ is then the result of 2^{n+1} successive extensions of the initial candidate for $Q_i(\sigma * j)$, one for each string of length $n + 1$. These extensions are carried out by successively considering each pair (σ, τ) of strings of length $n + 1$ and extending the present candidates for $Q_0(\sigma)$, $Q_1(\tau)$ to strings μ, ν in the range of P_0, P_1, respectively so that $(\exists y) R(\mu, \nu, n, y)$ holds. This can be done since we are not in Case 1. Then (4) holds since $Q_0(\sigma)$, $Q_1(\tau)$ extend all candidates for those values. Q_i is a tree since the initial candidates are incompatible, and clearly Q_i is a subtree of P_i. Finally Q_i can be constructed recursively in $P_0 \oplus P_1$. From this it follows that (Q_0, Q_1) is a condition. Lemma 3.5 is proved. □

Definition 3.6. A sequence of conditions $\langle (P_n, Q_n) \rangle_{n<\omega}$ is called *M-generic* if (P_{n+1}, Q_{n+1}) extends (P_n, Q_n) for all n, and for every sentence ϕ of the forcing language which is Σ_k^0 for some k, there exists n such that $(P_n, Q_n) \Vdash \phi$ or $(P_n, Q_n) \Vdash \neg\phi$. A pair of sets $A, B \subseteq \omega$ is called *M-generic* if there is an M-generic sequence of conditions $\langle (P_n, Q_n) \rangle_n$ such that $(A, B) \in \bigcap_n [P_n, Q_n]$. Standard arguments show that if $\langle (P_n, Q_n) \rangle_n$ is an M-generic sequence with limit (A, B) and $\phi(\underline{A}, \underline{B})$ is a sentence of the forcing language, then $\phi(A, B)$ is true if and only if $(P_n, Q_n) \Vdash \phi(\underline{A}, \underline{B})$ for some n.

Lemma 3.7. *Let (A, B) be an M-generic pair. Then*

$$\{f \mid f \leqslant_T A \;\&\; f \leqslant_T B\} \subseteq M\,.$$

Proof. Assume (A, B) is M-generic and $f = \{e_0\}^A = \{e_1\}^B$. Then there is a condition (P, Q) with $A \in [P]$, $B \in [Q]$ such that $(P, Q) \Vdash \phi$ where ϕ is a Π_2^0 sentence expressing that $\{e_0\}^A = \{e_1\}^B$. Thus if C is any branch of P, $\{e_0\}^C = \{e_1\}^B$ holds, so $\{e_0\}^C = \{e_0\}^A = f$. Thus $f(x) = y \leftrightarrow (\exists\sigma)$ $\{e_0\}^{P(\sigma)}(x) = y$, so $f \leqslant_T P$. Hence $f \in M$ as required. □

If Φ is a set of sentences of the forcing language, *forcing for* Φ is $\{(P, Q, \phi) \mid \phi \in \Phi \;\&\; (P, Q) \Vdash \phi\}$, thought of as a subset of $\omega^\omega \times \omega^\omega \times \omega$. The following lemma classifies forcing for Σ_n^0 and Π_n^0 sentences in the analytical hierarchy over M.

Lemma 3.8. If Φ is either the set of Π_2^0 sentences or the set of Σ_3^0 sentences, forcing for Φ is a relation which is arithmetical (in fact Σ_3^0) over M. If $n \geqslant 1$, forcing for Π_{n+2}^0 sentences is Π_n^1 over M. If $n \geqslant 2$, forcing for Σ_{n+2}^0 sentences is Σ_n^1 over M.

Proof. (The definitions of "arithmetical over M" and "Σ_n^1 [Π_n^1] over M" were given at the beginning of this Section. Although these definitions allowed parameters from M, we shall not use parameters in the present proof.) First let Φ be the set of Π^0_2 sentences. If ϕ is a Π^0_2 sentence and $P, Q \in M$ then $(P, Q) \Vdash \phi$ if and only if

(i) $P \equiv_T Q$ and P, Q are uniformly pointed trees,

(ii) $(\forall A)(\forall B)(A \in [P] \ \& \ B \in [Q] \to \phi(A, B))$.

Clause (i) is arithmetical by Lemma 3.4. The matrix of clause (ii) is a Π^0_2 relation of P, Q, A, B (in the usual sense). But the class of Π^0_2 relations is closed under universal set quantification; see Shoenfield [27, p. 187]. Hence clause (ii) is arithmetical (in fact Π^0_2), uniformly in ϕ. So forcing for Π^0_2 sentences is arithmetical over M. The proof for general Φ is now a straightforward induction in the order $\Pi^0_2, \Sigma^0_3, \Pi^0_3, \ldots$. For $n \geq 2$ the Σ^0_{n+1} case follows from the Π^0_n case by raising the number quantifier in the definition of forcing to a function quantifier (over M). (As remarked in the proof of (a) $\Rightarrow$ (b), the number quantifier may not simply be disregarded because the countable axiom of choice may not hold in M.) For $n \geq 3$, the Π^0_n case follows from the Σ^0_n case, Lemma 3.4, and easy quantifier manipulations. $\square$

The following lemma will essentially complete the proof that (b) $\Rightarrow$ (a).

Lemma 3.9. *Let $\mathcal{H}$ be a countable family of subsets of ω. There is an M-generic pair (A, B) such that*

(i) $M = \{f \mid f \leq_T A \ \& \ f \leq_T B\}$,

(ii) for each $H \in \mathcal{H}$ and $n \geq 1$,

$$H \text{ is } \Sigma^0_{n+2} \text{ in } A \oplus B \Rightarrow H \text{ is } \Sigma^1_n \text{ over } M.$$

Proof. Let $\mathcal{H} = \{H_i \mid i \in \omega\}$, $M = \{f_i \mid i \in \omega\}$. Let $\{\psi_i \mid i \in \omega\}$ be the set of sentences of the forcing language which are Σ^0_n for some $n \geq 0$, and let $\{\phi_i \mid i \in \omega\}$ be the set of formulas of the forcing language which are Σ^0_{n+2} for some $n \geq 1$ and have exactly one free number variable. We define an M-generic sequence of conditions $\{(P_s, Q_s)\}$ by induction on s. Let (P_0, Q_0) be any condition, e.g. $P_0(\sigma) = Q_0(\sigma) = \sigma$ for all σ. Now assume inductively that (P_s, Q_s) has been defined. To define (P_{s+1}, Q_{s+1}), consider three cases.

Case 1. $s \equiv 0 \bmod 3$. Let $s = 3i$. Let (P_{s+1}, Q_{s+1}) be any condition (P, Q) extending (P_s, Q_s) such that $(P, Q) \Vdash \psi_i$ or $(P, Q) \Vdash \neg\psi_i$. Such a condition exists by Lemma 3.5.

Case 2. $s \equiv 1 \bmod 3$. Let $s = 3i + 1$. Let (P_{s+1}, Q_{s+1}) be any condition (P, Q) extending (P_s, Q_s) such that $f_i \leq_T P$. Such a condition exists by

Lemma 3.3 and the closure of M under pairing.

Case 3. $s \equiv 2 \bmod 3$. Let $s = 3\langle i, j\rangle + 2$. In this case we attempt to find a condition (P, Q) extending (P_s, Q_s) which insures that $H_i \neq \{k \mid \phi_j(A, B, k)\}$. More precisely, if there exist (P, Q) extending (P_s, Q_s) and a number k such that $k \notin H_i$ and $(P, Q) \Vdash \phi_j(\underline{A}, \underline{B}, \underline{k})$, let (P_{s+1}, Q_{s+1}) be any such (P, Q). Similarly if there exist (P, Q) extending (P_s, Q_s) and $k \in H_i$ such that $(P, Q) \Vdash \neg\phi_j(\underline{A}, \underline{B}, \underline{k})$, let (P_{s+1}, Q_{s+1}) be such a (P, Q). If no (P, Q) of either sort exists, let $(P_{s+1}, Q_{s+1}) = (P_s, Q_s)$, and say that stage $s + 1$ is *vacuous*.

By the usual arguments there are unique sets A, B such that $(A, B) \in \bigcap_n [P_n, Q_n]$. Case 1 insures that (A, B) is an M-generic pair. Case 2 insures that $M \subseteq \{f \mid f \leqslant_T A \;\&\; f \leqslant_T B\}$, and the reverse inclusion follows from Lemma 3.7 and genericity. To verify (ii), assume that $H_i = \{k \mid \phi_j(A, B, k)\}$, where ϕ_j is a Σ^0_{n+2} formula. Let $s = 3\langle i, j\rangle + 2$. Since forcing equals truth for generic sets, stage $s + 1$ must have been vacuous. But then

$$k \in H_i \Leftrightarrow (\exists P, Q)\,[(P, Q) \text{ extends } (P_s, Q_s) \;\&\; (P, Q) \Vdash \phi_j(\underline{A}, \underline{B}, \underline{k})]\,.$$

Easy quantifier manipulations and Lemma 3.8 now imply that H_i is Σ^1_n over M. □

To prove (b) ⇒ (a) in Theorem 3.1, assume H is Σ^0_{n+2} in $A \oplus B$ whenever $M = \{f \mid f \leqslant_T A \;\&\; f \leqslant_T B\}$. Apply Lemma 3.9 with $\mathcal{H} = \{H\}$ and let (A, B) be the resulting generic pair. Since H is Σ^0_{n+2} in $A \oplus B$, H is Σ^1_n over M as required. □

Technical Notes. (1) The assumption that M is closed under jump was not used in the proof of (b) ⇒ (a). (It is possible to eliminate this assumption altogether by modifying the definitions of condition and Σ^1_1 over M.)

(2) A slight modification of the argument shows that H is Σ^0_2 in $A \oplus B$ whenever (A, B) is exact over M only if H is Σ^0_2 in f, for some $f \in M$ (provided M is closed under $\leqslant_T$ and $\oplus$). From this it follows that no ascending sequence of degrees has a 1-l.u.b. (as defined in [26]) unless the jumps of the degrees in the sequence are eventually constant.

4. The degree of Kleene's $\mathcal{O}$

Let D, $\leqslant$, and j be as defined in the Introduction. In this Section we investigate first order definability in the structure

$$\mathcal{D} = \langle D, \leqslant, j\rangle\,.$$

Definition 4.1. A set $I \subseteq D$ is said to be a *countable ideal* in $\mathcal{D}$ if
(i) I is countable and nonempty;
(ii) $\boldsymbol{a} \leqslant \boldsymbol{b} \in I \to \boldsymbol{a} \in I$;
(iii) $\boldsymbol{a}, \boldsymbol{b} \in I \to \boldsymbol{a} \cup \boldsymbol{b} \in I$;
(iv) $\boldsymbol{a} \in I \to \boldsymbol{a}' \in I$.

Definition 4.2. Let I be a countable ideal in $\mathcal{D}$ and let $\boldsymbol{a}, \boldsymbol{b}$ be degrees. The pair $\boldsymbol{a}, \boldsymbol{b}$ is said to be *exact over* I if

$$I = \{\boldsymbol{d} \in D \mid \boldsymbol{d} \leqslant \boldsymbol{a} \;\&\; \boldsymbol{d} \leqslant \boldsymbol{b}\}\,.$$

By Section 3 or Spector [31], we know that for any countable ideal there exists an exact pair. From our present point of view, this is important because it implies that quantification over all countable ideals is expressible in the first-order theory of $\mathcal{D}$.

Let I be a countable ideal and let

$$M_I = \{f \in \omega^\omega \mid \text{degree}\,(f) \in I\}\,.$$

Let n be a positive integer and let $H \subseteq \omega$ be a set of degree $\boldsymbol{h}$.

Theorem 4.3. *H is Δ^1_n over M_I if and only if $\boldsymbol{h} \leqslant (\boldsymbol{a} \cup \boldsymbol{b})^{(n+1)}$ whenever $\boldsymbol{a}, \boldsymbol{b}$ is an exact pair for I. Also H is analytical over M_I if and only if $\boldsymbol{h}$ is arithmetical in $\boldsymbol{a} \cup \boldsymbol{b}$ whenever $\boldsymbol{a}, \boldsymbol{b}$ is an exact pair for I.*

Proof. Immediate from Theorem 3.1 and Post's Hierarchy Theorem once the following uniformity is noted: if H is arithmetical in $A \oplus B$ whenever A, B is an exact pair for I, then there is an n such that H is Σ^0_{n+2} in $A \oplus B$ whenever A, B is an exact pair for I. The uniformity is proved by applying Lemma 3.9 with $\mathcal{H} = \{H\}$ and then using (a) $\Rightarrow$ (b) in Theorem 3.1 on the resulting pair A, B. We are grateful to H. Putnam for pointing out to us the need to establish this uniformity. □

Definition 4.4. Let $M \subseteq \omega^\omega$ be nonempty and closed under $\leqslant_T$ and $\oplus$. We say that M is an ω-model of the Δ^1_k *comprehension axiom* if the characteristic function c_X is in M whenever $X \subseteq \omega$ is Δ^1_k over M. We say that M is an ω-model of the *full comprehension schema* if M is an ω-model of Δ^1_k comprehension for all $k \in \omega$.

If I is any countable ideal, then clearly M_I is an ω-model of Δ^1_0 comprehension. The smallest ω-model of Δ_0^1 comprehension is

$$\mathrm{AR} = \{f \in \omega^\omega \mid f \text{ is arithmetical}\} .$$

By Kleene [16] and Kreisel [18], the smallest ω-model of Δ_1^1 comprehension is

$$\mathrm{HYP} = \{f \in \omega^\omega \mid f \text{ is hyperarithmetical}\} .$$

It is well known that there is no smallest ω-model of Δ_k^1 comprehension, $k \geqslant 2$, or of full comprehension.

Corollary 4.5. *There are first order sentences δ_k $(k \geqslant 0)$ such that $\langle D, \leqslant, j, I\rangle \models \delta_k$ if and only if M_I is an ω-model of Δ_k^1 comprehension. There is a first order sentence δ_∞ such that $\langle D, \leqslant, j, I\rangle \models \delta_\infty$ if and only if M_I is an ω-model of full comprehension.*

Corollary 4.6. *The relations "$\boldsymbol{d}_1$ is arithmetical in $\boldsymbol{d}_2$" and "$\boldsymbol{d}_1$ is hyperarithmetical in $\boldsymbol{d}_2$" are first-order definable in $\mathcal{D}$.*

Corollary 4.7. *There is a first-order sentence ϕ such that $\mathcal{D} \models \phi$ if $V = L$ or if V is a generic extension of L, and $\mathcal{D} \models \neg\phi$ if $0^\#$ exists.*

Proof. Let ϕ say that there exist arbitrarily large hyperdegrees which are not minimal covers. The conclusion follows from [28, Theorems 5.2 and 5.4]. □

Corollary 4.8. *The first-order theory of $\mathcal{D}$ is probably not provably absolute with respect to models of set theory containing all the ordinals.*

We need the word "probably" in this corollary because the existence of $0^\#$ is not known to be consistent with set theory. However, it will be shown by Simpson [29] that there is a first-order sentence ψ such that $\mathcal{D} \models \psi$ if and only if every element of ω^ω is constructible in the sense of Gödel. Hence the word "probably" is in fact unnecessary.

We now examine the (Turing) degree of Kleene's $\mathcal{O}$. It is well known that any two complete Π_1^1 subsets of ω are recursively isomorphic. In particular, the degree of $\mathcal{O}$ is characterized by the fact that it is a complete Π_1^1 set. What we want to do here is give some *degree theoretic* characterizations of the degree of $\mathcal{O}$. By degree theoretic we mean, of course, first order in $\mathcal{D}$.

Lemma 4.9. *Let X be a subset of ω. Then $X \leqslant_T \mathcal{O}$ if and only if X is Δ^1_2 over* HYP.

Proof. For later use we prove a little more: for all $n \geqslant 1$, $X \subseteq \omega$ is Σ^0_n in $\mathcal{O}$ if and only if X is Σ^1_{n+1} over HYP. First we formulate a general fact which really belongs in Section 3.

Sublemma 4.10. *Let $M \subseteq \omega^\omega$ be nonempty and closed under $\leqslant_T$ and $\oplus$. Suppose $H, X \subseteq \omega$, H is Δ^1_k over M, X is Σ^0_n in H, $n \geqslant 1$. Then X is Σ^1_{k+n-1} over M.*

The proof of 4.10 is straightforward. Now by the Hyperarithmetical Quantifier Theorem of Gandy [9] and Spector [30], $\mathcal{O}$ is Σ^1_1 over HYP, hence Δ^1_2 over HYP. From this and 4.10 it follows for all $n \geqslant 1$ that if X is Σ^0_n in $\mathcal{O}$ then X is Σ^1_{n+1} over HYP. For the converse we need the following well-known fact, due essentially to Kleene [16]: there is an enumeration $\mathrm{HYP} = \{f_i \mid i \in \mathcal{O}\}$ such that $\{\langle i_1, ..., i_n\rangle \mid i_1, ..., i_n \in \mathcal{O}\ \&\ P(f_{i_1}, ..., f_{i_n})\}$ is Π^1_1 whenever P is Π^1_1. See also Rogers [24, Theorem XLI, p. 418]. Suppose $X \subseteq \omega$ is Σ^1_{n+1} over HYP, say for concreteness $n = 2$. Thus

$$m \in X \leftrightarrow (\exists g_1 \in \mathrm{HYP})(\forall g_2 \in \mathrm{HYP})(\exists g_3 \in \mathrm{HYP}) A(g_1, g_2, g_3, m)$$

where A is arithmetical over HYP. Define

$$R(i_1, i_2, m) \leftrightarrow i_1, i_2 \in \mathcal{O}\ \&\ (\exists g_3 \in \mathrm{HYP}) A(f_{i_1}, f_{i_2}, g_3, m)\,.$$

Then R is Π^1_1 hence recursive in $\mathcal{O}$. We have

$$m \in X \leftrightarrow (\exists i_1 \in \mathcal{O})(\forall i_2 \in \mathcal{O}) R(i_1, i_2, m)$$

so X is Σ^0_2 in $\mathcal{O}$. This completes the proof of Lemma 4.9. □

Theorem 4.11. *The degree of $\mathcal{O}$ is the largest degree which is $\leqslant (\boldsymbol{a} \cup \boldsymbol{b})^{(3)}$ whenever the pair $\boldsymbol{a}, \boldsymbol{b}$ is exact over $\{\boldsymbol{d} \mid \boldsymbol{d}$ is hyperarithmetical$\}$.*

Proof. By 4.9 the degree of $\mathcal{O}$ is maximum among degrees of sets which are Δ^1_2 over HYP. From this and 4.3 the theorem is immediate. □

Corollary 4.12. *The degree of $\mathcal{O}$ is first order definable in $\mathcal{D}$.*

Proof. Immediate from 4.11 and 4.6. By relativizing 4.12 uniformly we obtain:

Corollary 4.13. *The relation "$\boldsymbol{d}_1$ is the hyperjump of $\boldsymbol{d}_2$" is first order definable in $\mathcal{D}$.*

Definition 4.14. $M \subseteq \omega^\omega$ is said to be a *β-model* if M is nonempty and, whenever $S \subseteq (\omega^\omega)^3$ is Σ^1_1 and $f_1, f_2 \in M$ and $(\exists g \in \omega^\omega)\, S(f_1, f_2, g)$, then $(\exists g \in M)\, S(f_1, f_2, g)$.

If M is a β-model then clearly M is closed under $\leqslant_T$, $\oplus$, and the jump operator. If $M \subseteq \omega^\omega$ is nonempty and closed under $\leqslant_T$, $\oplus$, and hyperjump, then M is a β-model by Kleene's Basis Theorem (Rogers [24, Corollary XLII(b), p. 420]). The converse is false since by Friedman [7] there exists a β-model consisting entirely of functions of lower hyperdegree than $\mathcal{O}$. On the other hand, it is easy to see that any β-model of Δ^1_2 (in fact Π^1_1) comprehension is closed under hyperjump.

Corollary 4.15. *There are first order sentences β_n, $n \geqslant 2$, such that $\langle D, \leqslant, j, I\rangle \models \beta_n$ if and only if M_I is a β-model of Δ^1_n comprehension. There is a first order sentence β_∞ such that $\langle D, \leqslant, j, I\rangle \models \beta_\infty$ if and only if M_I is a β-model of full comprehension.*

Proof. Let β_n (β_∞) say that M_I satisfies Δ^1_n (full) comprehension and is closed under hyperjump. □

Corollary 4.16. *The relation "$\boldsymbol{d}_1$ is ramified analytical in $\boldsymbol{d}_2$" is first order definable in $\mathcal{D}$.*

Proof. By a result of Gandy and Putnam, $\boldsymbol{d}_1$ is ramified analytical in $\boldsymbol{d}_2$ if and only if $\boldsymbol{d}_1$ belongs to the smallest β-model of full comprehension containing $\boldsymbol{d}_2$. (The ramified analytical hierarchy will be developed in detail in Section 5.) □

We end this Section with another degree theoretic characterization of the degree of $\mathcal{O}$. It will not be used in the rest of the paper, but we mention it for its independent interest. A degree $\boldsymbol{a}$ is said to be *minimal over* a countable ideal I if $I \subseteq \{\boldsymbol{d} \mid \boldsymbol{d} \leqslant \boldsymbol{a}\}$ and there is no $\boldsymbol{b} < \boldsymbol{a}$ with $I \subseteq \{\boldsymbol{d} \mid \boldsymbol{d} \leqslant \boldsymbol{b}\}$.

Theorem 4.17. *The degree of $\mathcal{O}$ is the largest degree which is $\leqslant \boldsymbol{a}^{(3)}$ whenever $\boldsymbol{a}$ is minimal over $\{\boldsymbol{d} \mid \boldsymbol{d}$ is hyperarithmetical$\}$.*

Proof. Part of the proof is a forcing argument similar to that of Section 3.

Define a condition to be a hyperarithmetical, uniformly pointed tree. Define genericity as in Section 3. The proof of Lemma 3.9 gives the following: if $X \subseteq \omega$ is not Δ^1_2 over HYP then there exists a generic set $A \subseteq \omega$ such that HYP $\subsetneq \{f \mid f \leqslant_T A\}$ and X is not Δ^0_4 in A. An argument of Sacks ([25, §8] and [26, sentence preceding Proposition 3.2]) shows that the degree of A is minimal over $\{\boldsymbol{d} \mid \boldsymbol{d}$ is hyperarithmetical$\}$. By Lemma 4.9 this proves half of what we want. It remains to show that $\mathcal{O}$ is Δ^0_4 in any set whose degree is minimal over $\{\boldsymbol{d} \mid \boldsymbol{d}$ is hyperarithmetical$\}$. Let B be any such set. By Gandy [9] or Spector [30] there is a Π^0_1 relation P such that

$$n \in \mathcal{O} \leftrightarrow (\exists f \in \mathrm{HYP})\, P(n, f)\,.$$

By Spector [30] or Harrison [11] this P can be taken to have the following additional property: if $P(n, f)$ holds and $f \notin$ HYP, then there is a pseudohierarchy recursive in f. In particular the degree of such an f will be an upper bound for $\{\boldsymbol{d} \mid \boldsymbol{d}$ is hyperarithmetical$\}$ but cannot be minimal over it. Hence

$$n \in \mathcal{O} \leftrightarrow (\exists f \leqslant_T B)\, P(n, f)\,.$$

so $\mathcal{O}$ is in fact Σ^0_3 in B. This completes the proof. □

Remark. There are two slightly stronger degree theoretic characterizations of the degree of $\mathcal{O}$, to wit:

(i) it is the smallest degree of the form $(\boldsymbol{a} \cup \boldsymbol{b})^{(3)}$ where the pair $\boldsymbol{a}, \boldsymbol{b}$ is exact over $\{\boldsymbol{d} \mid \boldsymbol{d}$ is hyperarithmetical$\}$;

(ii) it is the smallest degree of the form $\boldsymbol{a}^{(3)}$ where $\boldsymbol{a}$ is minimal over $\{\boldsymbol{d} \mid \boldsymbol{d}$ is hyperarithmetical$\}$.

For the proof, construct a pair (A, B) which is generic over HYP only with respect to Σ^0_3 and Π^0_3 sentences of the forcing language. The construction is carried out recursively in $\mathcal{O}$ so that the complete Σ^0_3 set relative to $A \oplus B$ is recursive in $\mathcal{O}$. By 3.1, 4.9, and 4.17 this gives the desired results.

5. Master codes and the ramified analytical hierarchy

As in Section 3, let M be a countable, nonempty subset of ω^ω which is closed under $\leqslant_T$, $\oplus$, and the jump operator. Let n be a positive integer. A set $H \subseteq \omega$ is a Δ^1_n *master code* for M if for all $X \subseteq \omega$, $X \leqslant_T H$ if and only if X is Δ_n^1 over M. The notion of Δ_n^1 master code may seem strange, but there are at least two familiar examples.

Example 1. Let 3.5^e be a limit notation in Kleene's $\mathcal{O}$. Then $H_{3.5^e}$ is a Δ_1^1 master code for

$$\{f \in \omega^\omega \mid f \leqslant_T H_n \text{ for some } n <_{\mathcal{O}} 3.5^e\}.$$

See Case III in the proof of Lemma 5.5 below.

Example 2. Kleene's $\mathcal{O}$ itself is a Δ_2^1 master code for

$$\mathrm{HYP} = \{f \in \omega^\omega \mid f \text{ is hyperarithmetical}\}.$$

This is the content of Lemma 4.9. Note that HYP cannot have a Δ_1^1 master code since it is a model of the Δ_1^1 comprehension axiom.

We shall give more examples later in this Section.

If H is a Δ_n^1 master code for M then the degree of H is maximum among the degrees of the sets which are Δ_n^1 over M. Thus, the degree of a Δ_n^1 master code for M is uniquely determined if it exists. Moreover, an M which satisfies Δ_n^1 comprehension cannot have a Δ_n^1 master code. The notion of Δ_n^1 master code is degree theoretically interesting in view of the following lemma.

Lemma 5.1. *Let I be a countable ideal in $\mathcal{D}$. There are first-order formulas $\psi_n(x)$ such that $\boldsymbol{d} \in D$ is the degree of a Δ_n^1 master code for M_I if and only if $\langle D, \leqslant, j, I\rangle \models \psi_n[\boldsymbol{d}]$.*

Proof. Explicitly, $\psi_n[\boldsymbol{d}]$ says that $\boldsymbol{d}$ is the largest degree which is $\leqslant (\boldsymbol{a} \cup \boldsymbol{b})^{(n+1)}$ whenever the pair $\boldsymbol{a}, \boldsymbol{b}$ is exact over I. That this works is immediate from 4.3. □

The purpose of this Section is to point out that the ramified analytical hierarchy is the best of all possible worlds as far as the existence of master codes is concerned. Precisely, let M be a nonzero level of the ramified analytical hierarchy and let n be a positive integer such that M is not a model of the Δ_n^1 comprehension axiom, then we shall show that M has a Δ_n^1 master code (Lemma 5.5). This will lead via 5.1 to a degree theoretic characterization of the ramified analytical hierarchy (Theorem 5.6).

The ramified analytical hierarchy has been studied extensively by

H. Putnam and his colleagues [2]. It will be defined and its principal properties reviewed after a few pages. But first we must discuss the *constructible hierarchy* of Gödel. We work in set theory without urelements. For our purposes it is best to define the constructible hierarchy as follows.

$$L_0 = HF = \{x \mid x \text{ is a hereditarily finite set}\} .$$

$$L_\lambda = \bigcup \{L_\alpha \mid \alpha < \lambda\}$$

for limit ordinals λ.

$$L_{\alpha+1} = \{X \subseteq L_\alpha \mid X \text{ is first-order definable over } \langle L_\alpha, \in\rangle\} .$$

Here the first order definitions over L_α are allowed to mention parameters, i.e. constants denoting arbitrary elements of L_α. The *constructible universe*, L, is the union of the L_α for all ordinals α. The constructible hierarchy increases through all the ordinals since $\omega + \alpha$ is a subset of L_α. Hence L is a proper class. Gödel proved that $\langle L, \in\rangle$ is a model of set theory plus the generalized continuum hypothesis. Gödel's proof was analyzed by Lévy [23].

We need the Lévy hierarchy of formulas in the language of set theory. A formula is Σ_0 if it is built up from atomic formulas $x = y$, $x \in y$ using propositional connectives &, $\neg$ and bounded quantifiers $\exists u(u \in x \ \& \ ...)$. Here $x, y, ...$ are set variables. A formula is Π_k if it is of the form $\neg\phi$ where ϕ is Σ_k. A formula is Σ_{k+1} if it is of the form $\exists x \phi$ where ϕ is Π_k. A set $X \subseteq L_\alpha$ is $\Sigma_k(L_\alpha)$ if there are a Σ_k formula $\phi(x, y)$ and a parameter $b \in L_\alpha$ such that

$$X = \{a \in L_\alpha \mid \langle L_\alpha, \in\rangle \models \phi[a, b]\} .$$

Similarly for $\Pi_k(L_\alpha)$. A set $X \subseteq L_\alpha$ is $\Delta_k(L_\alpha)$ if it is both $\Sigma_k(L_\alpha)$ and $\Pi_k(L_\alpha)$. Thus

$$L_{\alpha+1} = \bigcup_{k<\omega} \Sigma_k(L_\alpha) = \bigcup_{k<\omega} \Delta_k(L_\alpha) .$$

Let α be an ordinal and n a positive integer. In this paper only, a $\Delta_n(L_\alpha)$ *master code* is a set $H \subseteq \omega$ such that for all $X \subseteq \omega$, $X \leqslant_T H$ if and only if X is $\Delta_n(L_\alpha)$. The next lemma is essentially due to R.B. Jensen. It is a remarkable refinement of the Main Technical Lemma of Boolos and Putnam [1].

Lemma 5.2. *Let α be an ordinal and n a positive integer such that not every $\Delta_n(L_\alpha)$ subset of ω is an element of L_α. Then there exists a $\Delta_n(L_\alpha)$ master code.*

Proof. It is perhaps worthwhile here to remark that, for $\alpha = 0$, Lemma 5.2 says simply that for each $n \geqslant 1$ there exists $H \subseteq \omega$ such that for all $X \subseteq \omega$, $X \leqslant_T H$ if and only if X is Δ^0_n. This is a well-known corollary of Post's Hierarchy Theorem. Lemma 5.2 is true for arbitrary α but we shall give the proof only for the case when α is greater than ω and p.r. closed; see Jensen and Karp [13]. (This seems to be a reasonable compromise since 5.2 is applied below only in a situation where α is a limit of admissible ordinals, hence α is greater than ω and p.r. closed.) In this case $\mathrm{L}_\alpha = J_\alpha$ so we can apply the results of Jensen [12]; see also Devlin [4].

In Jensen's terms, the hypothesis of 5.2 is that $\eta^n_\alpha = 1$. Hence, by Jensen's results, there is a $\Delta_n(\mathrm{L}_\alpha)$ mapping $f : \omega \xrightarrow{\text{onto}} \mathrm{L}_\alpha$. Define

$$H = \{\langle i, j, k\rangle \mid i, j, k \in \omega \;\&\; \langle \mathrm{L}_\alpha, \in\rangle \models \phi_i[f(j), f(k)]\}$$

where $\langle \phi_i(x, y) \mid i \in \omega\rangle$ is a primitive recursive enumeration of the Π_{n-1} formulas with two free variables. Then $H \subseteq \omega$ and H is $\Delta_n(\mathrm{L}_\alpha)$. Furthermore, if $X \subseteq \omega$ is $\Sigma_n(\mathrm{L}_\alpha)$ then X is recursively enumerable in H. So H is a $\Delta_n(\mathrm{L}_\alpha)$ master code. For later use, note that for all $i \in \omega$, a subset of ω is $\Sigma_{n+i}(\mathrm{L}_\alpha)$ if and only if it is Σ^0_{i+1} in H. Hence the i^{th} jump of H is a $\Delta_{n+i}(\mathrm{L}_\alpha)$ master code. □

Technical Note. The subject of Δ_n master codes has not previously been discussed in the published literature. We therefore record some further information here. We use the nomenclature of Jensen. Let $\langle J_\alpha, \in, A\rangle$ be amenable, $A \subseteq J_\alpha$, and put $\rho = \rho^1_{\alpha,A}$ and $\eta = \eta^1_{\alpha,A}$. Then the following two assertions are equivalent:

(i) there is a $\Sigma_1(J_\alpha, A)$ mapping from a subset of J_ρ onto J_α;

(ii) there is a $\Delta_1(J_\alpha, A)$ mapping from J_η onto J_α.

Suppose that these assertions hold. Then $\rho < \eta$ and there is no κ with $\omega \cdot \rho < \kappa < \omega \cdot \eta$ and $\langle J_\alpha, \in\rangle \models$ "κ is a cardinal". Let λ be the least ordinal such that there is a $\Delta_1(J_\alpha, A)$ mapping from λ onto an unbounded subset of $\omega \cdot \alpha$. Then $\eta = \max\{\rho, \lambda\}$. Furthermore there exists $H \subseteq J_\eta$ which is $\Delta_1(J_\alpha, A)$ and such that the following holds. If $\lambda < \eta$, then

$$\Sigma_k(J_\eta, H) = P(J_\eta) \cap \Sigma_{k+1}(J_\alpha, A)$$

for all $k \geqslant 1$. If $\lambda = \eta$, then

$$\Sigma_k(J_\eta, H) = P(J_\eta) \cap \Sigma_k(J_\alpha, A)$$

for all $k \geqslant 1$; moreover, the structure $\langle J_\eta, \in, H\rangle$ is admissible. Some of these statements are essentially due to Jensen. The proofs will appear in a book on admissibility which Simpson is preparing.

We now pass from the language of set theory to the language of second-order arithmetic. The *ramified analytical hierarchy* is a hierarchy of subsets of ω^ω. Define

$$M_0 = \emptyset ,$$
$$M_\lambda = \bigcup \{M_\alpha \mid \alpha < \lambda\} \text{ for limit ordinals } \lambda ,$$
$$M_{\alpha+1} = \{ f \in \omega^\omega \mid \text{graph(f) is analytically definable over } M_\alpha \} .$$

(These analytical definitions are allowed to mention parameters, i.e. constants denoting arbitrary elements of M_α. An unpublished theorem of Putnam says that every element of M_α is analytically definable in M_α without parameters. Hence it would be possible to dispense with the parameters, although we shall not do so here.) There is a smallest ordinal β such that $M_{\beta+1} = M_\beta$. This countable ordinal is called β_0. Evidently M_{β_0} is a model of full comprehension. Gandy and Putnam have shown by an inner model construction that M_{β_0} is the smallest β-model of full comprehension.

Lemma 5.3. *For all* $\alpha \leqslant \beta_0 + 1$,

$$M_\alpha = \mathrm{L}_\alpha \cap \omega^\omega .$$

The proof is in Boolos-Putnam [1]. An ordinal α is said to be *locally countable* if $\langle \mathrm{L}_\alpha, \in \rangle \models \forall x \exists f$ (f is a mapping of natural numbers onto x). It is shown in [1] that every $\alpha \leqslant \beta_0$ is locally countable. For α a locally countable ordinal we define $M_\alpha = \mathrm{L}_\alpha \cap \omega^\omega$. This is harmless since by 5.3 it disagrees with our previous definition of M_α only when $\alpha > \beta_0 + 1$. The following lemma is well-known, but we sketch a proof anyway.

Lemma 5.4. *Let α be a locally countable ordinal which is a limit of smaller admissible ordinals. Let n be a positive integer. A relation* $R \subseteq M_\alpha^i \times \omega^j$ is $\Sigma_n(\mathrm{L}_\alpha)$ *if and only if it is* Σ^1_{n+1} *over* M_α.

Proof (sketch). Hereditarily countable sets x can be coded by elements of ω^ω; for instance, a code for x can be taken to be a function $f \in \omega^\omega$ such that, for some mapping $i : \omega \xrightarrow{\text{onto}} \mathrm{TC}(\{x\})$,

$$f(\langle m, n \rangle) = \begin{cases} 0 & \text{if } i(m) \in i(n), \\ 1 & \text{otherwise} , \end{cases}$$

where $\mathrm{TC}(\{x\})$ is the transitive closure of $\{x\}$. The set of all codes is Π^1_1 (over ω^ω). By local countability, every $x \in \mathrm{L}_\alpha$ has a code in M_α. Since α is a limit of smaller admissible ordinals, M_α is closed under the hyperjump operation. Hence the set of all codes in M_α is Π^1_1 over M_α. Let S be the mapping which takes a code f to the set encoded by f. Thus $\mathrm{L}_\alpha = \{S(f) \mid f$ is a code & $f \in M_\alpha\}$. We claim that for each $R \subseteq \mathrm{L}^l_\alpha$ which is $\Sigma_0(\mathrm{L}_\alpha)$, there is $P \subseteq M^l_\alpha$ which is Δ^1_1 over M_α and such that, for all codes $f_1, ..., f_l \in M_\alpha$, $P(f_1, ..., f_l)$ if and only if $R(S(f_1), ..., S(f_l))$. This is proved by induction on the number of symbols in the Σ_0 formula defining R. Now for each $R \subseteq \mathrm{L}_\alpha$ let

$$R^* = \{f \in M_\alpha \mid f \text{ is a code } \& \ S(f) \in R\}.$$

We claim that R is $\Sigma_n(\mathrm{L}_\alpha)$ if and only if R^* is Σ^1_{n+1} over M_α. This is easily proved by induction on n and establishes Lemma 5.4. □

Technical Note. Lemma 5.4 becomes false if one weakens the hypothesis that α is a limit of smaller admissible ordinals. However, 5.4 can be generalized in a slightly different direction. Let $M \subseteq \omega^\omega$ be a β-model and let $A = \{S(f) \mid f$ is a code & $f \in M\}$. Then A is a transitive set, and a relation on M is $\Sigma_n(A)$, $n \geqslant 1$, if and only if it is Σ^1_{n+1} over M. (If M is nonempty and closed under $\leqslant_T$, $\oplus$, and hyperjump, then M is a β-model. But the converse is false by Friedman [7].)

We come now to the lemma which is the main point of this Section.

Lemma 5.5. *Let α be a nonzero, locally countable ordinal. Let n be a positive integer. Suppose that M_α does not satisfy the Δ^1_n comprehension axiom. Then M_α has a Δ^1_n master code.*

Proof. There are three cases.

Case I. α is a limit of smaller admissible ordinals. The conclusion is immediate by Lemmas 5.2 and 5.4. Note that in this Case we must have $n \geqslant 2$ since M_α satisfies Δ^1_1 (in fact Π^1_1) comprehension.

Case II. α is admissible but not a limit of smaller admissibles. Let $\beta < \alpha$ be such that there is no admissible ordinal between α and β. Let $B \in M_\alpha$ be a code for β. Then clearly $\omega^B_1 = \alpha$. Furthermore

$$M_\alpha = \{f \in \omega^\omega \mid f \text{ is hyperarithmetic in } B\}.$$

This is the relativization to B of the fact that $M_{\omega_1} = \mathrm{HYP}$ (cf. [1,16]).

By similarly relativizing Lemma 4.9 we see that O^B is a Δ^1_2 master code for M_α and more generally, for all $i < \omega$, the ith jump of O^B is a Δ^1_{i+2} master code for M_α. Here again M_α satisfies Δ^1_1 comprehension so there is no Δ^1_1 master code.

Case III. α is not admissible and not a limit of smaller admissibles. As in Case II, let $B \in M_\alpha$ be such that $\alpha < \omega_1^B$. Hence $\omega \cdot \alpha$ is a limit ordinal less than ω_1^B. Let $\omega \cdot \alpha = |3 \cdot 5^e|^B$ where $3 \cdot 5^e$ is a limit notation in O^B (cf. [32]). Thus $M_\alpha = \{f | f \leqslant_T H_n^B$ for some $n <_{O^B} 3 \cdot 5^e\}$ ([1,16]). We claim that $H^B_{3 \cdot 5^e}$ is a Δ^1_1 master code for M_α and more generally, for all $i < \omega$, a subset of ω is Σ^0_{i+1} in $H^B_{3 \cdot 5^e}$ if and only if it is Σ^1_{i+1} over M_α. That $H^B_{3 \cdot 5^e}$ is Δ^1_1 over M_α is clear from [5]. The rest is proved by the same technique as in the proof of Lemma 4.9, using Sublemma 4.10 and the following fact: there is an enumeration $M_\alpha = \{f_i | i \in \omega\}$ such that $\{\langle i_1, ..., i_n \rangle \mid R(f_{i_1}, ..., f_{i_n})\}$ is recursive in $H^B_{3 \cdot 5^e}$ whenever R is arithmetical. This completes the proof of Lemma 5.5. □

Corollary. *Suppose that α is a nonzero, locally countable ordinal, $1 \leqslant n < \omega$, and $H \subseteq \omega$ is a Δ^1_n master code for M_α. Then for all $i < \omega$, the i^{th} jump of H is a Δ^1_{n+i} master code for M_α.*

We are now ready to give our level by level, degree theoretic characterization of the ramified analytical hierarchy. A sequence of degrees $\langle \boldsymbol{d}_\alpha | \alpha < \theta \rangle$ is defined degree theoretically as follows. Put $\boldsymbol{d}_0 = \boldsymbol{0}$ and $\boldsymbol{d}_{\alpha+1} = \boldsymbol{d}'_\alpha$ = jump of $\boldsymbol{d}_\alpha$. Let λ be a limit ordinal such that $\boldsymbol{d}_\alpha$ have been defined for all $\alpha < \lambda$. Put $I_\lambda = \{\boldsymbol{d} \mid (\exists \alpha < \lambda)\, \boldsymbol{d} \leqslant \boldsymbol{d}_\alpha\}$. Let m_λ be the least integer m such that there is a largest degree $\boldsymbol{d}$ which is $\leqslant (\boldsymbol{a} \cup \boldsymbol{b})^{(m)}$ for all pairs $\boldsymbol{a}, \boldsymbol{b}$ which are exact over I_λ. Let $\boldsymbol{d}_\lambda$ be this largest degree. Let θ be the least ordinal such that $\boldsymbol{d}_\theta$ is undefined.

Theorem 5.6. *We have $\theta = \beta_0$ and*

$$I_\theta = \{\text{degree}\,(f) \mid f \textit{ is ramified analytical}\}\,.$$

The level by level correspondence is given by

$$(5) \qquad I_{\omega \cdot \alpha} = \{\text{degree}\,(f) \mid f \in M_\alpha\}$$

for $1 \leqslant \alpha \leqslant \beta_0$. Furthermore $m_{\omega \cdot \alpha} = k_\alpha + 2$ where k_α is the largest integer k such that M_α satisfies Δ^1_k comprehension. Also $\boldsymbol{d}_{\omega \cdot \alpha + i}$ is the degree of Δ^1_{k+i+1} master codes for M_α, $k = k_\alpha$.

Proof. The equation (5) will be proved by induction on α, $1 \leq \alpha \leq \beta_0$. For $\alpha = 1$ (5) says merely that I_ω is the set of all degrees of arithmetical functions. For limit ordinals α the induction is trivial since then $I_{\omega \cdot \alpha} = \bigcup_{\beta < \alpha} I_{\omega \cdot \beta}$ and $M_\alpha = \bigcup_{\beta < \alpha} M_\beta$. For the successor step, suppose that (5) holds and $\alpha < \beta_0$. Then Lemmas 5.1 and 5.5 imply that $m_{\omega \cdot \alpha} = k_\alpha + 2$ and for each $i < \omega$, $\boldsymbol{d}_{\omega \cdot \alpha + i} = \boldsymbol{d}^{(i)}_{\omega \cdot \alpha}$ is the degree of Δ^1_{k+i+1} master codes for M_α, $k = k_\alpha$. Hence

$$\begin{aligned} I_{\omega \cdot (\alpha+1)} &= \{\boldsymbol{d} \mid \boldsymbol{d} \text{ is arithmetical in } \boldsymbol{d}_{\omega \cdot \alpha}\} \\ &= \{\text{degree } (X) \mid X \text{ is analytically definable over } M_\alpha\} \\ &= \{\text{degree } (f) \mid f \in M_{\alpha+1}\}\,. \end{aligned}$$

This completes the proof.

Technical Note. For limit ordinals $\lambda < \beta_0$ the definition of $\boldsymbol{d}_\lambda$ can be sharpened somewhat. Namely one can construct an exact pair $\boldsymbol{a}, \boldsymbol{b}$ for I_λ such that $\boldsymbol{d}_\lambda = (\boldsymbol{a} \cup \boldsymbol{b})^{(m_\lambda)}$. Compare this with the Remark at the end of Section 4. We do not know whether in the definition of $\boldsymbol{d}_\lambda$ for arbitrary limit ordinals $\lambda < \beta_0$ the use of exact pairs of upper bounds can be replaced by the use of minimal upper bounds.

Examples (continued). Let us attempt to clarify Theorem 5.6 by focusing on some cases.

(1) We have $\boldsymbol{d}_\omega = \boldsymbol{0}^{(\omega)}$, the degree of the truth set of first order arithmetic. Theorem 5.6 characterizes this degree theoretically as the largest degree which is $\leq (\boldsymbol{a} \cup \boldsymbol{b})^{(2)}$ whenever $\boldsymbol{a}, \boldsymbol{b}$ are an exact pair of upper bounds for the arithmetical degrees.

(2) Let λ be a recursive limit ordinal. Then $\boldsymbol{d}_\lambda$ is the degree of $H_{3 \cdot 5^e}$ where $3 \cdot 5^e$ is a notation for λ. Spector [32] showed that $\boldsymbol{d}_\lambda$ is well-defined but he did not characterize $\boldsymbol{d}_\lambda$ degree theoretically, although there was probably little doubt that this could be done. Theorem 5.6 characterizes $\boldsymbol{d}_\lambda$ as the largest degree which is $\leq (\boldsymbol{a} \cup \boldsymbol{b})^{(2)}$ whenever the pair $\boldsymbol{a}, \boldsymbol{b}$ is exact over I_λ. Sacks [26] has already characterized $\boldsymbol{d}_\lambda$ in a stronger way as the smallest degree of the form $\boldsymbol{a}^{(2)}$ where $\boldsymbol{a}$ is an upper bound for I_λ. The use of exact pairs of upper bounds was unnecessary in this case because of the "predicative" nature of the hyperarithmetical hierarchy; see Kreisel [18,19]. The point here, due to Kleene [16], is that for $\alpha < \omega_1$ the master code for M_α can be given an analytical (in fact Δ^1_1) definition which is *invariant* in the sense that it defines the same set when

interpreted in an arbitrary M, $M_\alpha \subseteq M \subseteq \omega^\omega$. This is in sharp contrast to the behavior of M_{ω_1} = HYP where by [10] no sets are invariantly analytically definable except those which already belong to HYP. See also [5]. This is why for $\boldsymbol{d}_{\omega_1}$ we need exact pairs of upper bounds rather than just upper bounds.

(3) Let $\lambda = \omega_1$. Then $\boldsymbol{d}_\lambda$ is the degree of Kleene's $\mathcal{O}$; see also Section 4. There is no $\Delta_1(\mathrm{L}_\lambda)$ master code. Kleene's $\mathcal{O}$ is a complete $\Sigma_1(\mathrm{L}_\lambda)$ subset of ω and a $\Delta_2(\mathrm{L}_\lambda)$ master code. Also $M_\lambda = \mathrm{L}_\lambda \cap \omega^\omega$ = HYP and $\mathcal{O}$ is a Δ^1_2 master code for M_λ.

(4) For each $n < \omega$ let ω_n be the n^{th} admissible ordinal. Put $\lambda = \bigcup_{n<\omega} \omega_n$ It is well-known that λ is not admissible and that M_λ is the smallest β-model of Π^1_1 comprehension. Put $\mathcal{O}_\omega = \{\langle m, n\rangle \mid m \in \mathcal{O}_n\}$ where $\mathcal{O}_0 = \emptyset$ and $\mathcal{O}_{n+1}$ is the hyperjump of $\mathcal{O}_n$. Then $\mathcal{O}_\omega$ is a $\Delta_1(\mathrm{L}_\lambda)$ master code and a Δ^1_2 master code for M_λ. Theorem 5.6 characterizes $\boldsymbol{d}_\lambda$, the degree of $\mathcal{O}_\omega$, as the largest degree which is $\leqslant (\boldsymbol{a} \cup \boldsymbol{b})^{(3)}$ whenever $\boldsymbol{a}$, $\boldsymbol{b}$ are an exact pair of upper bounds for the degrees $\boldsymbol{d}_{\omega_n}$ = degree of $\mathcal{O}_n$, $1 \leqslant n < \omega$.

(5) Let $\lambda = \omega_1^{E_1}$ = the smallest admissible ordinal which is a limit of admissible ordinals. It is well known that λ is projectible into ω and that M_λ is the smallest β-model of Δ^1_2 comprehension (see Kripke [20]). Let $\mathcal{O}_\lambda$ be the complete $\Sigma_1(\mathrm{L}_\lambda)$ subset of ω. Equivalently,

$$\mathcal{O}_\lambda = \{\langle m, n\rangle \mid \{m\}^{E_1}(n) = 0\}$$

where E_1 is Tugue's functional; cf. [8,33]. Then $\boldsymbol{d}_\lambda$ is the degree of $\mathcal{O}_\lambda$. Theorem 5.6 characterizes $\boldsymbol{d}_\lambda$ as the largest degree which is $\leqslant (\boldsymbol{a} \cup \boldsymbol{b})^{(4)}$ whenever the pair $\boldsymbol{a}$, $\boldsymbol{b}$ is exact over the degrees of functions in $M_\lambda = \{f \in \omega^\omega \mid f \text{ is recursive in } E_1\}$.

(6) Let λ be the smallest nonprojectible admissible ordinal greater than ω. Then L_λ satisfies Σ_1 comprehension so there is no complete $\Sigma_1(\mathrm{L}_\lambda)$ subset of ω. Nevertheless there exists a $\Delta_2(\mathrm{L}_\lambda)$ master code which is also a Δ^1_3 master code for M_λ. This master code is denoted $\mathcal{O}_\lambda$. Then $\boldsymbol{d}_\lambda$ is the degree of $\mathcal{O}_\lambda$ and Theorem 5.6 characterizes $\boldsymbol{d}_\lambda$ as the largest degree which is $\leqslant (\boldsymbol{a} \cup \boldsymbol{b})^{(4)}$ whenever $\boldsymbol{a}$, $\boldsymbol{b}$ are exact over the degrees of functions in M_λ.

(7) Let λ be the smallest Σ_2 admissible ordinal. Then L_λ satisfies Δ_2 comprehension so there is no $\Delta_2(\mathrm{L}_\lambda)$ master code. Let $\mathcal{O}_\lambda$ be the complete $\Sigma_2(\mathrm{L}_\lambda)$ subset of ω. Then $\mathcal{O}_\lambda$ is a $\Delta_3(\mathrm{L}_\lambda)$ master code and a Δ^1_4 master code for M_λ, and $\boldsymbol{d}_\lambda$ is the degree of $\mathcal{O}_\lambda$. Theorem 5.6 characterizes $\boldsymbol{d}_\lambda$ degree theoretically as the largest degree which is $\leqslant (\boldsymbol{a} \cup \boldsymbol{b})^{(5)}$ whenever $\boldsymbol{a}$, $\boldsymbol{b}$ are an exact pair over the degrees of the functions in M_λ.

Remark. The degree theoretic hierarchy $\langle \boldsymbol{d}_\alpha \mid \alpha < \beta_0 \rangle$ can be extended naturally to a hierarchy of degrees $\langle \boldsymbol{d}_\alpha \mid \alpha < \aleph_1^L \rangle$ which are cofinal in the constructible degrees. For example, $\boldsymbol{d}_{\beta_0}$ is the degree of the truth set for $\langle L_{\beta_0}, \in \rangle$ and can be defined degree theoretically as the largest degree which is $\leqslant (\boldsymbol{a} \cup \boldsymbol{b})^{(\omega)}$ whenever the pair $\boldsymbol{a}, \boldsymbol{b}$ is exact over I_{β_0}. In general the n^{th} jumps, $n < \omega$, which were used in Theorem 5.6 must be replaced by ν^{th} jumps where ν can be any ordinal less than $\aleph_1^L$. These ν^{th} jumps are defined by combining the ideas of Leeds and Putnam [22] with the ideas of the present paper. The details will appear elsewhere.

Discussion. The work reported in this Section was inspired by the earlier work of H. Putnam and his colleagues [1,2,5,22]. Obviously we are very much in Putnam's debt. However, our work differs from that of Putnam in two respects. First, we have taken account of the Jensen Theory, which was not available to Putnam. Second, Putnam et al. employ the notion of uniform upper bound, which seems to be somewhat pathological. We do not know whether the notion of uniform upper bound is degree theoretically definable. This problem is investigated but not solved in Section 6. We have gotten around the problem here by eliminating uniform upper bounds in favor of exact pairs of upper bounds. This yields two significant improvements over [22].

(1) Our degrees $\boldsymbol{d}_\alpha$, $\alpha < \aleph_1^L$, are (Turing) degrees rather than arithmetical degrees.

(2) Our degrees are seen to be degree theoretic, while this is in doubt for the degrees used by Putnam.

Precisely, each of our degrees $\boldsymbol{d}_\alpha$, $\alpha < \aleph_1^L$, is seen to be definable in the structure $\mathcal{D}$ by a formula of $\mathcal{L}_{\aleph_1 \aleph_0}$, the infinitary logic with countable conjunctions and finite strings of quantifiers. (In particular, $\boldsymbol{d}_\alpha$ is seen to be necessarily fixed by all automorphisms of $\mathcal{D}$. But it is an open problem due to Rogers [24, p. 261] whether $\mathcal{D}$ has any nontrivial automorphisms.)

Remark. It would be interesting to look at master codes and degree theoretic hierarchies for notions of degree other than that of Kleene–Post [17]. Some of the notions we have in mind are: many-one degrees [24], hyperdegrees, α-degrees and α-calculability degrees, Δ_2^1 degrees, L-degrees [28], Q-degrees, and Wadge degrees for subsets of ω^ω. (If $A, B \subseteq \omega^\omega$ we say A is *Wadge reducible* to B if there exists a continuous function $F\colon \omega^\omega \to \omega^\omega$ such that $A = F^{-1}(B)$.) There are also a number of notions of degree that arise from Kleene's theory of recursion in higher types. For each notion of degree, it is to be expected that some sort of master codes and degree theoretic hierarchies exist, but they would probably take different forms. These differences in the form of the hierarchies would yield significant insight into the nature of the various notions of degree.

6. Uniform upper bounds

If M is a countable set of functions and $\boldsymbol{a}$ is a degree, $\boldsymbol{a}$ is called a *uniform upper bound (u.u.b.)* of M if there is a binary function f of degree at most $\boldsymbol{a}$ such that $M = \{(f)_i \mid i \in \omega\}$, where $(f)_i(j) = f(i, j)$. (This is easily seen to be equivalent to the definition of u.u.b. in [2].) In [2], the ramified analytical hierarchy is analyzed in terms of u.u.b.'s. The purpose of the present section is to attempt to relate the notion of u.u.b. to the degree theoretic notions used in our analysis of that hierarchy. In particular we give a necessary condition for a degree $\boldsymbol{a}$ to be a u.u.b. to the class of all arithmetical functions, and also a sufficient condition. These two conditions are first-order conditions on $\boldsymbol{a}$, but unfortunately they do not appear to be equivalent. Throughout this Section the set of arithmetical functions will be denoted AR. Any results we state for AR are also valid (with essentially the same proof) for the class of functions which occur in any fixed proper initial segment of the hyperarithmetical hierarchy closed under the jump operation, i.e. for $\{f \mid (\exists n <_O 3 \cdot 5^e) f \leqslant_T H_n\}$ for $3 \cdot 5^e \in O$, replacing $0^{(\omega)}$ by the degree of $H_{3 \cdot 5^e}$.

The next proposition is analogous in statement and proof to the result of Enderton and Putnam [5] that $0^{(\omega)}$ is recursive in $\boldsymbol{a}''$ whenever all arithmetical functions are recursive in $\boldsymbol{a}$. A degree $\boldsymbol{a}$ is called a *sub-uniform upper bound (s.u.u.b.)* of a class M of functions if there is a binary function f of degree $\leqslant \boldsymbol{a}$ such that $M \subseteq \{(f)_i \mid i \in \omega\}$.

Proposition 6.1. *If $\boldsymbol{a}$ is a s.u.u.b. of* AR, *then $\boldsymbol{a}' \geqslant 0^{(\omega)}$.*

Proof. There is a Π^0_1 relation $P(n, f)$ on $\omega \times \omega^\omega$ such that for each n there is a unique f (denoted f_n) such that $P(n, f)$ holds. Furthermore $f_n \equiv_T 0^{(n)}$, uniformly in n (cf. [24, Exercise 16–98]). Let g be a function recursive in $\boldsymbol{a}$ such that every arithmetical function is $(g)_i$ for some i. Let $h(n) = (\mu i)\, P(n, (g)_i)$. Then h is a total function recursive in $\boldsymbol{a}'$, and $(g)_{h(n)} = f_n$ for all n. Since $0^{(n)} \equiv_T f_n$ uniformly in n, it follows that $\boldsymbol{a}' \geqslant 0^{(\omega)}$. □

Corollary 6.2. $\mathbf{0}^{(\omega)}$ *is the least element of* $\{\boldsymbol{a}' \mid \boldsymbol{a}$ *is a u.u.b. of* AR$\}$.

Proof. In view of Proposition 6.1, it suffices to show that there is degree $\boldsymbol{a}$ such that $\boldsymbol{a}$ is a u.u.b. of AR and $\boldsymbol{a}' \leqslant \mathbf{0}^{(\omega)}$. Such a degree $\boldsymbol{a}$ is easily obtained by combining the method of Kleene-Post [17] to construct u.u.b.'s of AR with that of Friedberg [6] to control the jump of a set being constructed. Alternatively, one may let $\boldsymbol{a} = \boldsymbol{b}'$ where $\boldsymbol{b}$ is any degree such that $\mathbf{0}^{(n)} \leqslant \boldsymbol{b}$ for all $n < \omega$ and $\boldsymbol{b}'' \leqslant \mathbf{0}^{(\omega)}$ [26]. Then $\boldsymbol{a}$ is a u.u.b. of AR by Corollary 6.5, which follows shortly. □

If $\boldsymbol{a}, \boldsymbol{b}$ are degrees, $\boldsymbol{a}$ is called a *high cover* of $\boldsymbol{b}$ if $\boldsymbol{a} \geqslant \boldsymbol{b}$ and $\boldsymbol{a}' \geqslant \boldsymbol{b}''$. [14, Theorem 1] (relativized) shows that if $\boldsymbol{a}$ is a high cover of $\boldsymbol{b}$, then $\boldsymbol{a}$ is a u.u.b. of the functions recursive in $\boldsymbol{b}$. We have $\boldsymbol{b}$ is an *upper bound* of AR if every arithmetical function is recursive in $\boldsymbol{b}$.

Theorem 6.3. *If* $\boldsymbol{a}$ *is a high cover of some upper bound* $\boldsymbol{b}$ *of* AR, *then* $\boldsymbol{a}$ *is a u.u.b. of* AR.

Proof. By the remark just before the theorem, the hypotheses of the theorem imply that $\boldsymbol{a}$ is a s.u.u.b. of AR. Thus the theorem is a consequence of the following lemma.

Lemma 6.4. *If* $\boldsymbol{a}$ *is a s.u.u.b. of* AR, *then* $\boldsymbol{a}$ *is a u.u.b. of* AR.

Proof. If $f, g \in \omega^\omega$ we say f *weakly majorizes* g if $f(n) \geqslant g(n)$ for all but finitely many n. Since $\boldsymbol{a}$ is a s.u.u.b. of AR, it is easy to see that there is a function f_M recursive in $\boldsymbol{a}$ which weakly majorizes all arithmetical functions. By the proof of Proposition 6.1 there are functions $g, f_n (n < \omega)$, and h such that g is binary and recursive in $\boldsymbol{a}$, h is recursive in $\boldsymbol{a}'$, $f_n \equiv_T \mathbf{0}^{(n)}$, and $(g)_{h(n)} = f_n$. By the Limit Lemma there is a binary function r recursive in $\boldsymbol{a}$ such that $\lim_{s\to\infty} r(n, s) = h(n)$ for all n. Now define

$$\hat{g}(\langle e, n, k\rangle, z) = \Phi_e((g)_{r(n,k+z)}; z)$$

if $\Phi_e((g)_{h(n,k+x)}; x)$ is defined in at most $k + f_M(x)$ steps for each $x \leqslant z$, otherwise let $\hat{g}(\langle e, n, k\rangle, z) = 0$. (Here $\Phi_e(f; x)$ is alternate notation for $\{e\}^f(x)$.) If $\Phi_e((g)_{r(n,k+x)}; x)$ is defined in at most $k + f_M(x)$ steps for all x, then $(\hat{g})_{\langle e,n,k\rangle}$ differs only finitely from $\Phi_e((g)_{h(n)})$, and otherwise $(\hat{g})_{\langle e,n,k\rangle}$ is 0 for all but finitely many arguments. Hence $(\hat{g})_{\langle e,n,k\rangle}$ is arithmetical for any fixed $\langle e, n, k\rangle$. Also if $\Phi_e((g)_{h(n)})$ is total, then the majorizing property of f_M insures that $(\hat{g})_{\langle e,n,k\rangle} = \Phi_e((g)_{h(n)})$ for all sufficiently large k. Since $\hat{g}$ is clearly recursive in $\boldsymbol{a}$, it witnesses that $\boldsymbol{a}$ is a u.u.b. of AR. □

It may be shown by extending the proof of Lemma 6.4 that $\boldsymbol{a}$ is a u.u.b. of AR iff there is a function recursive in $\boldsymbol{a}$ which weakly majorizes all arithmetical functions.

Let U be the set of u.u.b.'s of AR. We do not know whether U is first-order definable in $\mathcal{D} = \langle D, \leqslant, j\rangle$ nor even whether it is invariant under all automorphisms of this structure. Let U_1 be the set of upper bounds $\boldsymbol{a}$ of AR such that $\boldsymbol{a}' \geqslant \boldsymbol{0}^{(\omega)}$. Let U_2 be the set of degrees which are high covers of upper bounds of AR. U_1 and U_2 are each first-order definable in $\mathcal{D}$ and $U_2 \subseteq U \subseteq U_1$ by 6.1 and 6.3. We do not know whether either or both inclusions can be reversed. Also we do not know whether U, U_1 or U_2 contains a minimal upper bound to $\{\boldsymbol{0}^{(n)} \mid n \in \omega\}$. Finally we do not know whether U coincides with the set of u.u.b.'s of AR $\cap\ 2^\omega$ (the class of arithmetical sets) nor whether 6.1, 6.2, or 6.4 remain valid with AR replaced by AR $\cap\ 2^\omega$.

The following is an immediate corollary to Theorem 6.3.

Corollary 6.5. *If $\boldsymbol{b}$ is an upper bound of* AR, *$\boldsymbol{b}'$ is a u.u.b. of* AR.

Observe that Corollary 6.5 (and thus also Theorem 6.3 and Lemma 6.4) fail if AR is replaced by HYP, the set of hyperarithmetic functions. Indeed by [5, §3] and [1] there is a degree $\boldsymbol{b}$ such that $\boldsymbol{b}$ is an upper bound of HYP, but no degree containing a set Δ^1_1 in $\boldsymbol{b}$ is a u.u.b. of HYP. However using the notion of exact pair one may easily get an analogue to Theorem 6.3 which holds for more general classes of functions. Let M be a countable nonempty subset of ω^ω, closed under $\leqslant_T$ and $\oplus$.

Theorem 6.6. *If $\{\boldsymbol{a}, \boldsymbol{b}\}$ is exact over M and $\boldsymbol{c}$ is a high cover of $\boldsymbol{a}$ and of $\boldsymbol{b}$, then $\boldsymbol{c}$ is a u.u.b. of M.*

Proof. Let $M_{\boldsymbol{a}}$, $M_{\boldsymbol{b}}$ be the set of functions recursive in $\boldsymbol{a}$, $\boldsymbol{b}$, respectively, so $M = M_{\boldsymbol{a}} \cap M_{\boldsymbol{b}}$. By the remark before Theorem 6.3, $\boldsymbol{c}$ is a u.u.b. of $M_{\boldsymbol{a}}$ and of $M_{\boldsymbol{b}}$. Let f, g be functions recursive in $\boldsymbol{c}$ such that $M_{\boldsymbol{a}} = \{(f)_i \mid i \in \omega\}$ and $M_{\boldsymbol{b}} = \{(g)_j \mid j \in \omega\}$. Define $h(\langle i, j\rangle, z)$ to be $(f)_i(z)$ if $(f)_i(x) = (g)_j(x)$ for all $x \leqslant z$, and let $h(\langle i, j\rangle, z) = 0$ otherwise. Then $(h)_{\langle i,j\rangle} = (f)_i$ if

$(f)_i = (g)_j$ and otherwise $(h)_{\langle i,j \rangle}$ is almost everywhere 0. Hence

$$M = M_a \cap M_b = \{(h)_k : k \in \omega\}. \square$$

Remark. It can also be shown that if $\{a, b\}$ is exact over M, $c \geqslant a$, and $c' \geqslant (a \cup b)''$, then c is a u.u.b. of M.

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