ADVANCES IN MATHEMATICS 53, 265–290 (1984)

A Dual Form of Ramsey's Theorem

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Let $k \in \omega$, where ω is the set of all natural numbers. Ramsey's Theorem deals with colorings of the k-element subsets of ω. Our dual form deals with colorings of the k-element partitions of ω. Let $(\omega)^k$ (respectively $(\omega)^\omega$) be the set of all partitions of ω having exactly k (respectively infinitely many) blocks. Given $X \in (\omega)^\omega$ let $(X)^k$ be the set of all $Y \in (\omega)^k$ such that Y is coarser than X. *Dual Ramsey Theorem.* If $(\omega)^k = C_0 \cup \cdots \cup C_{l-1}$ where each C_i is Borel then there exists $X \in (\omega)^\omega$ such that $(X)^k \subseteq C_i$ for some $i < l$. *Dual Galvin–Prikry Theorem.* Same as before with k replaced by ω. We also obtain dual forms of theorems of Ellentuck and Mathias. Our results also provide an infinitary generalization of the Graham–Rothschild "parameter set" theorem [*Trans. Amer. Math. Soc.* **159** (1971), 257–292] and a new proof of the Halpern–Läuchli Theorem [*Trans. Amer. Math. Soc.* **124** (1966), 360–367].

1. INTRODUCTION

The purpose of this paper is to establish a combinatorial theorem which is in a certain sense the dual of Ramsey's Theorem. The original theorem of Ramsey is concerned with colorings of the k-element subsets of a fixed infinite set. Our dual form is concerned with colorings of the k-element partitions of a fixed infinite set.

We begin by recalling Ramsey's Theorem [32]. Let ω be the set of natural numbers. Ramsey's Theorem says that if the k-element subsets of ω are colored with finitely many colors, then there exists an infinite subset of ω all of whose k-element subsets have the same color. In order to state Ramsey's

* Research supported by NSF Grant MCS 79-07774.

† Research supported by NSF Grant MCS 8107867 and an Alfred P. Sloan Research Fellowship.

0001-8708/84 $7.50

Theorem more precisely, we introduce some notation. Let $[\omega]^\omega$ be the set of all infinite subsets of ω. For $k \in \omega$ and $X \in [\omega]^\omega$ let $[X]^k$ be the set of all k-element subsets of X. With this notation we have:

1.1. RAMSEY'S THEOREM. *If* $[\omega]^k = C_0 \cup \cdots \cup C_{l-1}$ *then there exists* $X \in [\omega]^\omega$ *such that* $[X]^k \subseteq C_i$ *for some* i.

We now state our dual form of Ramsey's Theorem. By a *partition of* ω we mean a collection of pairwise disjoint, nonempty subsets of ω whose union is all of ω. The elements of a partition of ω are called its *blocks*. An *infinite partition of* ω is a partition of ω having infinitely many blocks. A *k-element partition of* ω is a partition of ω having exactly k blocks. If X and Y are partitions of ω, we say that Y *is coarser than* X if each block of X is a subset of some block of Y. The dual form of Ramsey's Theorem reads as follows: if the k-element partitions of ω are colored in a "nice" way with finitely many colors, then there exists an infinite partition of ω such that all coarser k-element partitions of ω have the same color.

In order to state our dual form of Ramsey's Theorem more precisely, we introduce some more notation. Let $(\omega)^\omega$ be the set of all infinite partitions of ω. For $k \in \omega$ let $(\omega)^k$ be the set of all k-element partitions of ω. For $X \in (\omega)^\omega$ let $(X)^k$ be the set of all $Y \in (\omega)^k$ such that Y is coarser than X. If Y is any partition of ω, we may identify Y with a binary relation $R_Y \subseteq \omega \times \omega$, where $(m, n) \in R_Y$ if and only if m and n belong to the same block of Y. The set of all binary relations, $\{\text{true}, \text{false}\}^{\omega \times \omega}$, is a topological space, where $\{\text{true}, \text{false}\}$ is endowed with the discrete topology. Thus $(\omega)^k$ and $(\omega)^\omega$ become topological spaces under the topology inherited from the space of binary relations. We call a subset of $(\omega)^k$ or $(\omega)^\omega$ "nice" if it is a Borel set, i.e., it belongs to the σ-algebra generated by the open sets of the appropriate topology. With this understanding we have:

1.2. DUAL RAMSEY THEOREM. *If* $(\omega)^k = C_0 \cup \cdots \cup C_{l-1}$, *where each* C_i *is Borel, then there exists* $X \in (\omega)^\omega$ *such that* $(X)^k \subseteq C_i$ *for some* i.

Dual Ramsey Theorem 1.2 will be proved in Section 2 except for a lemma whose proof will be postponed until Section 6.

In Section 4 we shall go on to obtain an "infinite exponent" version of Theorem 1.2. This is a dual form of the Galvin–Prikry Theorem [10]. For $X \in (\omega)^\omega$ let $(X)^\omega$ be the set of all $Y \in (\omega)^\omega$ such that Y is coarser than X. Then we have:

1.3. DUAL GALVIN–PRIKRY THEOREM. *If* $(\omega)^\omega = C_0 \cup \cdots \cup C_{l-1}$, *where each* C_i *is Borel, then there exists* $X \in (\omega)^\omega$ *such that* $(X)^\omega \subseteq C_i$ *for some* i.

Besides proving Theorems 1.2 and 1.3, we shall also explore the extent to which the hypothesis "each C_i Borel" in Theorems 1.2 and 1.3 can be weakened. That this hypothesis cannot be dropped entirely is shown by the following counterexample.

1.4. COUNTEREXAMPLE. *There exists a coloring* $(\omega)^2 = C_0 \cup C_1$ *such that for all* $X \in (\omega)^\omega$ *neither* $(X)^2 \subseteq C_0$ *nor* $(X)^2 \subseteq C_1$.

To see this, let $(\omega)^\omega = \{X_\alpha : \alpha < 2^{\aleph_0}\}$ be a well ordered list of all the infinite partitions of ω. We construct C_0 and C_1 by transfinite induction. At stage 0 put $C_0^0 = C_1^0 = \varnothing$. At stage $\alpha + 1$ note inductively that $|C_0^\alpha \cup C_1^\alpha| < 2^{\aleph_0}$ so we can choose Y_0^α, $Y_1^\alpha \in (X_\alpha)^2 \backslash (C_0^\alpha \cup C_1^\alpha)$ such that $Y_0^\alpha \neq Y_1^\alpha$. Put $C_i^{\alpha+1} = C_i^\alpha \cup \{Y_i^\alpha\}$, $i = 0, 1$. At limit stages $\beta < 2^{\aleph_0}$ put $C_i^\beta = \bigcup \{C_i^\alpha : \alpha < \beta\}$, $i = 0, 1$. Finally put $C_0 = \bigcup \{C_0^\alpha : \alpha < 2^{\aleph_0}\}$ and $C_1 = (\omega)^2 \backslash C_0$. Clearly $Y_i^\alpha \in (X_\alpha)^2 \backslash C_{1-i}$ for $i = 0, 1$, so we have our counterexample.

The above construction made essential use of the Axiom of Choice. We shall shown in Section 5 that any proof of the existence of a counterexample must use the Axiom of Choice. Namely, there is a model of Zermelo–Fraenkel set theory without the Axiom of Choice in which Theorems 1.2 and 1.3 remain true even when the hypothesis "each C_i Borel" is dropped entirely. We obtain this result by dualizing a well known forcing construction of Mathias [24].

It is interesting to note that many well known combinatorial theorems can be deduced as corollaries of the main results of this paper. For instance, Theorem 2.2 is a slight generalization of the Dual Ramsey Theorem 1.2 in which partitions are replaced by A-partitions where A is a finite alphabet. Thus Theorem 2.2 may be viewed as an infinitary analog of a fairly difficult theorem of finite combinatorics due to Graham and Rothschild [11]. In Section 3 we deduce the Graham–Rothschild Theorem as a corollary of our infinitary result. We also deduce Ramsey's Theorem [32], the Halpern–Läuchli Theorem [14], and an "infinite-dimensional" generalization of the Halpern–Läuchli Theorem due to Laver [21]. (Unfortunately, Hindman's Theorem [16] does not seem to be easily deducible from the results in this paper. However, Hindman's Theorem as well as its topological generalization due to Milliken [26] are easily deduced from a theorem of Carlson [5] which is closely related to the results of Section 6. See Theorem 6.9 and Remark 6.10 below.)

We end this introduction with some historical remarks. In August 1981, subsequent to some conversations with Klaus Leeb [22], Simpson developed a series of conjectures which are stated as Theorems 2.2, 4.1, 5.7 and 5.8 below. Simpson's chief inspiration came from the theorems of Galvin–Prikry [10], Graham–Rothschild [11], and Paris–Harrington [29]. When Simpson tried to prove his conjectures, he succeeded only in establishing the special case $k = 3$ of Theorem 1.2. His proof of this special case used Hindman's Theorem [16]. Simpson also managed to reduce all of his conjectures to a certain infinitary Hales–Jewett [13] type conjecture which is stated below as Lemma 2.4. These reductions due to Simpson are presented in Sections 2, 3, 4 and 5 below. But Simpson's attempts to prove the key Lemma 2.4 met with no success. At that point Simpson communicated his conjectures to several people including Ron Graham and Leo Harrington.

Later, in July 1982, Simpson and Carlson met at the AMS Recursion Theory Institute which was held at Cornell University. In several conversations Simpson told Carlson of his conjectures and of his attempts to prove them. In particular Simpson described the key role of Lemma 2.4 and mentioned the relevance of Hindman's Theorem [16]. Carlson and Simpson discussed these matters further at an AMS meeting in Toronto in August 1982.

Shortly after the Toronto meeting, Carlson obtained a proof of Lemma 2.4 and indeed of the stronger Theorem 6.3. It is essentially that proof of Lemma 2.4 which we present below in Section 6. Subsequently, in October 1982, Carlson [4] devised a more difficult proof which yields a still stronger result, namely a common generalization of Lemma 2.4 and Hindman's Theorem [16] as well as the Hindman–Milliken Theorem [26]. (See Theorem 6.9 and Remark 6.10 below.) This more difficult proof of Carlson's was circulated in manuscript form by Prikry [31]. Carlson plans to publish it in a separate paper [5] which will also contain further results obtained by the same method.

2. Proof of the Dual Ramsey Theorem

The purpose of this section is to prove the Dual Ramsey Theorem 1.2. We find it convenient to prove a more general theorem in which partitions are replaced by A-partitions.

2.1. Definition. Let A be a fixed finite set of symbols which is disjoint from ω. We refer to A as a *finite alphabet.* An *A-partition* of ω is a collection of pairwise disjoint, nonempty subsets of $A \cup \omega$ called *blocks*, whose union is all of $A \cup \omega$, and such that each block contains at most one element of A. A *free block* is a block which is disjoint from A.

Let $(\omega)_A^\omega$ be the set of all A-partitions of ω having infinitely many free blocks. For $k \in \omega$ let $(\omega)_A^k$ be the set of all A-partitions of ω having exactly k free blocks. (Equivalently, $(\omega)_A^k$ is the set of all A-partitions of ω having exactly $|A| + k$ blocks. Here $|A|$ is the cardinality of A.) If X and Y are A-

partitions of ω, we say that Y is *coarser than* X if each block of X is a subset of some block of Y. For $X \in (\omega)_A^\omega$ we write

$$(X)_A^\omega = \{Y \in (\omega)_A^\omega \colon Y \text{ is coarser than } X\}$$

and

$$(X)_A^k = \{Y \in (\omega)_A^k \colon Y \text{ is coarser than } X\}.$$

The main result of this section is the following:

2.2. THEOREM. *Let A be a finite alphabet. If $(\omega)_A^k = C_0 \cup \cdots \cup C_{l-1}$, where each C_i is Borel, then there exists $X \in (\omega)_A^\omega$ such that $(X)_A^k \subseteq C_i$ for some i.*

The Dual Ramsey Theorem 1.2 is a special case of Theorem 2.2 obtained by taking $A = \varnothing =$ the empty set.

Before proving Theorem 2.2 we must develop some notation. We conform to the usual practice of identifying $n \in \omega$ with the set of all smaller natural numbers, i.e., $n = \{0, 1,..., n-1\}$. For $X \in (\omega)_A^\omega$ we write $s \prec X$ to mean that s is a *segment of* X, i.e., $s = X[n]$ for some $n \in \omega$, where

$$X[n] = \{x \cap (A \cup n) \colon x \in X\} \setminus \{\varnothing\}.$$

In this case we write $lh(s) = n$ and $\#(s) = |\{x \in s \colon x \subseteq n\}|$.

By an *A-segment* we mean a segment of any $X \in (\omega)_A^\omega$. Thus an A-segment s is nothing more than an A-partition of $lh(s) \in \omega$, and $\#(s)$ is the number of free blocks of s. If s and t are A-segments, $s \prec t$ means that s is a segment of t, i.e., $s = t[n]$ for some $n < lh(t)$. Also $s \preccurlyeq t$ means that $s \prec t$ or $s = t$. Also $s \leqslant s'$ means that $lh(s) = lh(s')$ and s is coarser than s', i.e., each block of s' is a subset of some block of s. Finally $s \leqslant X$ means that $s \leqslant X[lh(s)]$, or equivalently $s \prec Y$ for some $Y \in (X)_A^\omega$. If $s \leqslant X$ we write

$$(s, X)_A^\omega = \{Y \in (X)_A^\omega \colon s \prec Y\}$$

and

$$(s, X)_A^k = \{Y \in (X)_A^k \colon s \prec Y\}.$$

We shall now prove Theorem 2.2 for $k = 0$.

2.3. LEMMA. *Let A be a finite alphabet. If $X \in (\omega)_A^\omega$ and $(X)_A^0 = C_0 \cup \cdots \cup C_{l-1}$, where each C_i is Borel, then there exists $Y \in (X)_A^\omega$ such that $(Y)_A^0 \subseteq C_i$ for some i.*

Proof. Note that $(X)_A^0$ is a compact Hausdorff space with basic open sets $(s, X)_A^0$, where $s \leqslant X$, $\#(s) = 0$. Since C_i is a Borel set, it has the property of

Baire (see Kuratowski [19]), i.e., there exist an open set O_i and dense open sets D_{in}, $n \in \omega$, such that

$$(C_i \backslash O_i) \cup (O_i \backslash C_i) \subseteq (X)^0_A \backslash \bigcap_{n \in \omega} D_{in},$$

We now perform a Baire category construction. Since $(X)^0_A = C_0 \cup \cdots \cup C_{l-1}$, the Baire Category Theorem implies that $O_0 \cup \cdots \cup O_{l-1}$ is a dense open set in $(X)^0_A$. Let t_0 and $i < l$ be such that $t_0 \leqslant X$ and $\#(t_0) = 0$ and $(t_0, X)^0_A \subseteq O_i$. Let t_{n+1} be such that $t_n < t_{n+1}$ and $\#(t_{n+1}) = n+1$ and $(s, X)^0_A \subseteq D_{in}$ for all $s \leqslant t_{n+1}$ with $\#(s) = 0$. Finally put $Y = \lim_n t_n =$ the unique $Y \in (X)^\omega_A$ such that $t_n < Y$ for all $n \in \omega$. By construction $(Y)^0_A \subseteq D_{in}$ for all $n \in \omega$. Since $t_0 < Y$ we have $(Y)^0_A \subseteq (t_0, X)^0_A \subseteq O_i$. Hence $(Y)^0_A \subseteq C_i$. This completes the proof of Lemma 2.3.

If s is any A-segment, we write $s^* = s \cup \{\{lh(s)\}\}$, i.e., s^* is the unique A-segment t such that $s < t$ and $lh(t) = lh(s) + 1$ and $\#(t) = \#(s) + 1$. For $X \in (\omega)^\omega_A$ let $(X)^*_A$ be the set of all A-segments s such that $\#(s) = 0$ and $s^* \leqslant X$. At a key point in the proof of Theorem 2.2 we shall need the following lemma.

2.4. Lemma. *Let A be a finite alphabet. If $Y \in (\omega)^\omega_A$ and $(Y)^*_A = C^*_0 \cup \cdots \cup C^*_{l-1}$ then there exists $Z \in (Y)^\omega_A$ such that $(Z)^*_A \subseteq C^*_i$ for some i.*

We postpone the proof of Lemma 2.4 until Section 6.

We now turn to the proof of Theorem 2.2. The proof will proceed by induction on k. The base step $k = 0$ has already been given as Lemma 2.3. The inductive step is given by the following lemma.

2.5. Lemma. *Assume that Theorem 2.2 holds for $(\omega)^k_{A+1}$, where $A+1$ denotes a finite alphabet of cardinality $|A| + 1$. Then Theorem 2.2 holds for $(\omega)^{k+1}_A$.*

Proof. Let $(\omega)^{k+1}_A = C_0 \cup \cdots \cup C_{l-1}$ be given where each C_i is Borel.

We begin with an important observation. Suppose that $X \in (\omega)^\omega_A$ and $s \in (X)^*_A$ and $X' \in (X)^\omega_A$ are given such that $X'[lh(s^*)] = X[lh(s^*)]$. There is an obvious canonical homeomorphism of $(s^*, X')^{k+1}_A$ onto $(\omega)^k_{A+1}$. But we are assuming that Theorem 2.2 holds for $(\omega)^k_{A+1}$. Hence there exists $X'' \in (X')^\omega_A$ such that $X''[lh(s^*)] = X[lh(s^*)]$ and $(s^*, X'')^{k+1}_A \subseteq C_i$ for some i. This observation will be applied repeatedly in what follows.

Let $X_0 \in (\omega)^\omega_A$ be arbitrary. Suppose we have constructed $X_n \in (\omega)^\omega_A$. Let t_n be the unique A-segment such that $t^*_n < X_n$ and $\#(t_n) = n$. We claim that there exists $X_{n+1} \in (t^*_n, X_n)^\omega_A$ such that, for each $s \leqslant t_n$ with $\#(s) = 0$, $(s^*, X_{n+1})^{k+1}_A \subseteq C_i$ for some i (depending on s). To see this, let $\{s_{nj}: j < m_n\}$ be an enumeration of all $s \leqslant t_n$ with $\#(s) = 0$. Put $X^0_n = X_n$. By the observation in the previous paragraph, let $X_n^{j+1} \in (X_n^j)_A^\omega$ be such that $t_n^* < X_n^{j+1}$ and $(s_{nj}^*, X_n^{j+1})_A^{k+1} \subseteq C_i$ for some i (depending on j). Finally put $X_{n+1} = X_n^{m_n}$. This proves the claim.

Finally put $Y = \lim_n X_n = \lim_n t_n^* =$ the unique $Y \in (\omega)_A^\omega$ such that $t_n^* < Y$ for all $n \in \omega$. For each $s \in (Y)_A^*$ we have $s \leqslant t_n$ for some n. Hence by construction $(s^*, Y)_A^{k+1} \subseteq C_i$ for some i (depending on s). For $s \in (Y)_A^*$ put $s \in C_i^*$ if and only if $(s, Y)_A^{k+1} \subseteq C_i$. Thus $(Y)_A^* = C_0^* \cup \cdots \cup C_{l-1}^*$. By Lemma 2.4 there exists $Z \in (Y)_A^\omega$ such that $(Z)_A^* \subseteq C_i^*$ for some i. Hence $(s^*, Z)_A^{k+1} \subseteq C_i$ for all $s \in (Z)_A^*$. Hence $(Z)_A^{k+1} \subseteq C_i$. This completes the proof of Lemma 2.5.

Theorem 2.2 is an immediate consequence of Lemmas 2.3 and 2.5.

2.6. *Remark.* Pierre Matet has made the following interesting observation. Let $\mathscr{C}$ be any class of subsets of $(\omega)_A^k$ which is closed under continuous preimages. Suppose that each $C \in \mathscr{C}$ has the property of Baire. Then the proof of Theorem 2.2 goes through unchanged if the hypothesis "C_i Borel" is replaced by "$C_i \in \mathscr{C}$." For example, if all projective sets have the property of Baire, then Theorem 2.2 remains true with "C_i Borel" weakened to "C_i projective."

3. Some Corollaries

In this section we show that several known combinatorial theorems may be derived as corollaries of the Dual Ramsey Theorem 1.2, or of its generalization for A-partitions, Theorem 2.2.

We begin with Ramsey's Theorem itself [32].

3.1. Ramsey's Theorem. *If* $[\omega]^k = C_0 \cup \cdots \cup C_{l-1}$ *then there exists* $Y \in [\omega]^\omega$ *such that* $[Y]^k \subseteq C_i$ *for some* i.

Proof. If X is any partition of ω put

$$X' = \{\min(x) \colon x \text{ is a block of } X\} \setminus \{0\}.$$

Given $[\omega]^k = C_0 \cup \cdots \cup C_{l-1}$ let C_i' be the set of all $X \in (\omega)^{k+1}$ such that $X' \in C_i$. Clearly $(\omega)^{k+1} = C_0' \cup \cdots \cup C_{l-1}'$ and each C_i' is Borel (in fact clopen). Hence by the Dual Ramsey Theorem 1.2 there exists $Y \in (\omega)^\omega$ such that $(Y)^{k+1} \subseteq C_i'$ for some i. Then $Y' \in [\omega]^\omega$ and clearly $[Y']^k \subseteq C_i$. This completes the proof.

We now turn to the Graham–Rothschild Theorem [11]. Following Leeb [22] we view the Graham–Rothschild Theorem as dealing with A-partitions of a finite set. Thus Theorem 2.2, which deals with A-partitions of an infinite set, is an infinitary generalization of the Graham–Rothschild Theorem.

Indeed, part of our original motivation for proving Theorem 2.2 was to strengthen the Graham–Rothschild Theorem by means of a detour through the infinite. (See Theorem 7.1 below.)

Let A be a finite alphabet. We use the A-segment notation of Section 2. For all k, $n \in \omega$ let $(n)_A^k$ be the set of all A-segments s such that $lh(s) = n$ and $\#(s) = k$. If $t \in (n)_A^m$ we write $(t)_A^k = \{s \in (n)_A^k : s \leqslant t\}$.

3.2. GRAHAM–ROTHSCHILD THEOREM. *For all k, l, $m \in \omega$ there exists $n \in \omega$ so large that the following holds. If $(n)_A^k = C_0 \cup \cdots \cup C_{l-1}$ then there exists $t \in (n)_A^m$ such that $(t)_A^k \subseteq C_i$ for some i.*

Proof. Fix A, k, l, m and suppose that the conclusion of the theorem fails. For each n choose a coloring $(n)_A^k = C_0^n \cup \cdots \cup C_{l-1}^n$ which is a counterexample to the conclusion of the theorem. Define a coloring $(\omega)_A^{k+1} = C_0 \cup \cdots \cup C_{l-1}$ as follows. Given $X \in (\omega)_A^{k+1}$, let s be the unique A-segment such that $s^* < X$ and $\#(s) = k$. Put $X \in C_i$ if and only if $s \in C_i^{lh(s)}$. Clearly each C_i is Borel (in fact clopen). By Theorem 2.2 let $Y \in (\omega)_A^\omega$ be such that $(Y)_A^{k+1} \subseteq C_i$ for some i. Let t be the unique A-segment such that $t^* < Y$ and $\#(t) = m$. Put $n = lh(t)$. Then clearly $t \in (n)_A^m$ and $(t)_A^k \subseteq C_i^n$. This contradiction completes the proof.

We now discuss the Halpern–Läuchli Theorem [14, 27, 28, 21]. Let 2^n be the set of all functions from $n = \{0, 1, ..., n-1\}$ into $2 = \{0, 1\}$. We write $2^{<\omega} = \bigcup_{n \in \omega} 2^n$. A *tree* is a set $T \subseteq 2^{<\omega}$ such that $\sigma \in 2^{<\omega}$, $\sigma \subseteq \tau$, and $\tau \in T$ imply $\sigma \in T$. If T is a tree we write $T(n) = T \cap 2^n$. A tree T is said to be *perfect* if it is nonempty and for all $\sigma \in T$ there exist τ_1, $\tau_2 \in T$ such that $\sigma \subseteq \tau_1$ and $\sigma \subseteq \tau_2$ but neither $\tau_1 \subseteq \tau_2$ nor $\tau_2 \subseteq \tau_1$. Let d be a positive integer. We write $(2^n)^d$ for the set of all functions from $d = \{0, 1, ..., d-1\}$ into 2^n. For any sets S_i, $i < d$, let $\prod_{i<d} S_i$ be the d-fold Cartesian product, i.e., the set of all functions f from d into $\bigcup_{i<d} S_i$ such that $f(i) \in S_i$ for all $i < d$. We can now state the Halpern–Läuchli Theorem for finite dimension d.

3.3. HALPERN–LÄUCHLI THEOREM. *Let d be a positive integer. If $\bigcup_{n \in \omega} (2^n)^d = C_0 \cup \cdots \cup C_{l-1}$ then there exist perfect trees T_i, $i < d$, and an infinite set $Z \subseteq \omega$ such that $\bigcup_{n \in Z} \prod_{i<d} T_i(n) \subseteq C_j$ for some j.*

This theorem is perhaps not very widely known, but it has a number of interesting applications in set theory and mathematical logic [15, 2, 34, 29a, 21].

We shall show that the Halpern–Läuchli Theorem is an easily derived corollary of Theorem 2.2. Instead of working with Theorem 2.2 directly, we shall work with the following special case of it, which is actually equivalent to Lemma 2.4. For any finite alphabet A, let $(\omega)_A^*$ be the set of all A-segments s such that $\#(s) = 0$.

3.4. LEMMA. *If* $(\omega)_A^* = C_0 \cup \cdots \cup C_{l-1}$ *then there exists* $X \in (\omega)_A^\omega$ *such that* $(X)_A^* \subseteq C_j$ *for some* j.

Proof. Given $Y \in (\omega)_A^1$ let Y^1 be the unique $s \in (\omega)_A^*$ such that $s^* \prec Y$. Let C_j^1 be the set of all $Y \in (\omega)_A^1$ such that $Y^1 \in C_j$. Then clearly $(\omega)_A^1 = C_0^1 \cup \cdots \cup C_{l-1}^1$ and each C_j^1 is Borel (in fact clopen). Applying Theorem 2.2 with $k=1$ we obtain $X \in (\omega)_A^\omega$ such that $(X)_A^1 \subseteq C_j^1$ for some $j<l$. Then clearly $(X)_A^* \subseteq C_j$. This completes the proof of Lemma 3.4.

We now show to deduce the Halpern–Läuchli Theorem 3.3 from Lemma 3.4.

Let d be a positive integer, and let $\bigcup_{n\in\omega} (2^n)^d = C_0 \cup \cdots \cup C_{l-1}$ be a coloring as in the hypothesis of the Halpern–Läuchli Theorem. We shall apply Lemma 3.4 with the alphabet $A = 2^d$. For each $s \in (\omega)_A^*$ define $s' \in (2^{lh(s)})^d$ by putting $s'(i)(m) = a(i)$, where $a \in A$ is such that a and m lie in the same block of s. This defines a one-to-one correspondence between $(\omega)_A^*$ and $\bigcup_{n\in\omega} (2^n)^d$. For each $j<l$ let C_j' be the set of all $s \in (\omega)_A^*$ such that $s' \in C_j$. Thus $(\omega)_A^* = C_0' \cup \cdots \cup C_{l-1}'$. By Lemma 3.4 let $X \in (\omega)_A^\omega$ and $j<l$ be such that $(X)_A^* \subseteq C_j'$. For each $i<d$ let T_i be the set of all $\sigma \in 2^{<\omega}$ such that $\sigma \subseteq s'(i)$ for some $s \in (X)_A^*$. It is straightforward to verify that T_i is a perfect tree. Put $Z = \{lh(s): s \in (X)_A^*\}$. It is straightforward to verify that $\bigcup_{n\in Z} \prod_{i<d} T_i(n) \subseteq C_j$. This completes the proof of the Halpern–Läuchli Theorem 3.3.

Recently Laver [21] has obtained an infinite dimensional generalization of the Halpern–Läuchli Theorem. We now show that Laver's Theorem can also be obtained as a consequence of the ideas in this paper.

In order to state Laver's Theorem, let $(2^n)^\omega$ be the set of all functions from ω into 2^n. For any sets S_i, $i \in \omega$, let $\prod_{i\in\omega} S_i$ be the set of all functions f from ω into $\bigcup_{i\in\omega} S_i$ such that $f(i) \in S_i$ for all $i \in \omega$. The infinite dimensional generalization of the Halpern–Läuchli Theorem reads as follows.

3.5. LAVER'S THEOREM. *If* $\bigcup_{n\in\omega} (2^n)^\omega = C_0 \cup \cdots \cup C_{l-1}$ *then there exist perfect trees* T_i, $i \in \omega$, *and an infinite set* $Z \subseteq \omega$ *such that* $\bigcup_{n\in Z} \prod_{i\in\omega} T_i(n) \subseteq C_j$ *for some* j.

Laver's Theorem can be derived from a certain generalization of Lemma 3.4 involving an *infinite alphabet*. Fix $A = \bigcup_{n\in\omega} A_n$, where A_n is a finite alphabet and $A_n \subseteq A_{n+1}$ for all $n \in \omega$. A *restricted A-partition* of ω is an A-partition of ω such that for all $n \in \omega$, if n lies in the same block as $a \in A$, then $a \in A_n$. If X and Y are restricted A-partitions of ω, we say that Y is *coarser than* X if each block of X is a subset of some block of Y and in addition, for each $n \in \omega$, if n lies in a free block x of X but n lies in the same block of Y as $a \in A$, then $a \in A_k$, where $k = \#(X[\min(x)])$. Let $(\omega)_A^\omega$ (respectively $(\omega)_A^k$ for $k \in \omega$) be the set of all restricted A-partitions having infinitely many (respectively exactly k) free blocks. For $X \in (\omega)_A^\omega$ let $(X)_A^\omega$

(respectively $(X)_A^k$ for $k \in \omega$) be the set of all $Y \in (\omega)_A^\omega$ (respectively $(\omega)_A^k$) such that Y is coarser than X. Let $(\omega)_A^*$ be the set of all restricted A-segments s such that $\#(s) = 0$. For $X \in (\omega)_A^\omega$ let $(X)_A^*$ be the set of all $s \in (\omega)_A^*$ such that s^* is coarser than $X[lh(s^*)]$. With these definitions, Theorem 2.2 and Lemma 3.4 make sense in the infinite alphabet setting, and their proofs go through with little change. Then, just as the Halpern–Läuchli Theorem 3.3 was derived from Lemma 3.4, Laver's Theorem 3.5 can be derived from the infinite alphabet generalization of Lemma 3.4. For the alphabet one uses $A = \bigcup_{n \in \omega} A_n$ where A_n is the set of all $a\colon \omega \to 2$ such that $a(i) = 0$ for all $i \geqslant n$. We omit the details.

The observation that the Halpern–Läuchli Theorem 3.3 can be derived as a corollary of the Dual Ramsey Theorem 2.2 is due to Carlson [4]. Carlson was also the first to observe that Laver's Theorem 3.5 can be derived from similar considerations (see Prikry [31]). The above formulation, in terms of an infinite alphabet, is due to Miller and Prikry [25]. Miller and Prikry have also used infinite alphabets to derive an interesting Ellentuck-type theorem for a certain space of infinite trees. Their results [25] appear to be closely related to Milliken's topological generalization of the Halpern–Läuchli Theorem [27, 28].

Another previously known result which is related to the results of this paper is Hindman's Theorem [16]. For precise details concerning the relationship between this paper and Hindman's Theorem, see Remark 6.10 below.

4. A Dual Form of Ellentuck's Theorem

There is a well known topological generalization of Ramsey's Theorem known as the Galvin–Prikry–Ellentuck Theorem or simply Ellentuck's Theorem [10, 7]. The purpose of this section is to prove a dual form of Ellentuck's Theorem. At the end of the section we shall state Ellentuck's Theorem itself and show how to derive it as an easy corollary of our dual form.

Let $(\omega)^\omega$ be the set of all infinite partitions of ω. For $X \in (\omega)^\omega$ and $n \in \omega$ we write

$$X[n] = \{x \cap n \colon x \in X\} \backslash \{\varnothing\}.$$

Here $n = \{0, 1, \ldots, n-1\}$ and so $X[n]$ is a partition of n. We write $lh(X[n]) = n$ and $|X[n]| = \#(X[n]) =$ the number of blocks in $X[n]$. We write $s \prec X$ to mean that s is a *segment* of X, i.e., $s = X[n]$ for some $n = lh(s) \in \omega$.

Let s and t be segments. We write $s \prec t$ to mean that $lh(s) < lh(t)$ and $s = t[lh(s)]$. We write $s \preccurlyeq t$ to mean that $s \prec t$ or $s = t$. We write $s \leqslant t$ to mean that $lh(s) = lh(t)$ and s is coarser than or equal to t. Finally we write $s \leqslant X$ to mean that $s \leqslant X[lh(s)]$.

For $X \in (\omega)^\omega$ and $s \leqslant X$, let (s, X) be the set of all $Y \in (\omega)^\omega$ such that $s < Y$ and Y is coarser than X. We refer to (s, X) as a *dual Ellentuck neighborhood.* A set $C \subseteq (\omega)^\omega$ is said to be *Ramsey* if for each dual Ellentuck neighborhood (s, X) there exists $Y \in (s, X)$ such that $(s, Y) \subseteq C$ or $(s, Y) \cap C = \varnothing$. A set $C \subseteq (\omega)^\omega$ is said to be *Ramsey null* if for each dual Ellentuck neighborhood (s, X) there exists $Y \in (s, X)$ such that $(s, Y) \cap C = \varnothing$.

The *dual Ellentuck topology* on $(\omega)^\omega$ is the topology whose basic open sets are the dual Ellentuck neighborhoods. Note that the dual Ellentuck topology is finer than the "classical" topology on $(\omega)^\omega$ which was considered in Section 1.

In any topological space, a set is said to be *meager* if it is disjoint from the intersection of a countable collection of dense open sets. A set is said to have the *property of Baire* if it is equal to an open set modulo a meager set. It is well known that in any topological space, the collection of all sets with the property of Baire is closed under countable Boolean operations and the Souslin operation [19] as well as many other countable set operations [33]. Therefore, the following theorem tells us that a great many subsets of $(\omega)^\omega$ are Ramsey.

4.1. Dual Ellentuck Theorem. *A set $C \subseteq (\omega)^\omega$ is Ramsey if and only if it has the property of Baire with respect to the dual Ellentuck topology. A set $C \subseteq (\omega)^\omega$ is Ramsey null if and only if it is meager with respect to the dual Ellentuck topology.*

Following Galvin and Prikry [10] and Ellentuck [7], we shall present the proof of Theorem 4.1 as a sequence of lemmas. Since the following lemma is very easy, we shall leave its proof to the reader.

4.2. Lemma. *Suppose $s \leqslant s' \leqslant X \in (\omega)^\omega$. For any $Y \in (s, X)$ there exists $Y' \in (s', X)$ such that $(s, Y') = (s, Y)$.*

Until Lemma 4.8 let O be a fixed subset of $(\omega)^\omega$. For any dual Ellentuck neighborhood (s, X), we say that X *accepts* s if $(s, X) \subseteq O$. We say that X *rejects* s if there is no $Y \in (s, X)$ such that Y accepts s.

4.3. Lemma. *Suppose $s \leqslant s' \leqslant X \in (\omega)^\omega$. Then there exists $Y' \in (s', X)$ such that Y' accepts or rejects s.*

Proof. If X rejects s, let Y' be any element of (s', X). Otherwise, let $Y \in (s, X)$ be such that $(s, Y) \subseteq O$, and by Lemma 4.2 let $Y' \in (s', X)$ be such that $(s, Y') = (s, Y)$. In either case the desired conclusion follows.

4.4. LEMMA. *Let (s, X) be any dual Ellentuck neighborhood. There exists $Y \in (s, X)$ such that Y accepts or rejects all t with $s \preccurlyeq t \leqslant Y$.*

Proof. Let Y_0 be any element of (s, X) and put $t_0 = s$. Suppose that we have constructed $Y_n \in (s, X)$ and $t_n \prec Y_n$ such that $s \preccurlyeq t_n$ and $|t_n| = |s| + n$. Let $\{t_n^j : j < m_n\}$ be an enumeration of all t such that $s \preccurlyeq t \leqslant t_n$. Put $Y_n^0 = Y_n$. By Lemma 4.3 let $Y_n^{j+1} \in (t_n, Y_n^j)$ be such that Y_n^{j+1} accepts or rejects t_n^j. Put $Y_{n+1} = Y_n^{m_n}$. Let t_{n+1} be the smallest t such that $t_n \prec t \prec Y_{n+1}$ and $|t| = |t_n| + 1 = |s| + n + 1$. Finally put $Y = \lim_n Y_n =$ the unique $Y \in (s, X)$ such that $t_n \prec Y$ for all $n \in \omega$. We claim that Y accepts or rejects all t with $s \preccurlyeq t \leqslant Y$. To see this, let t be given with $s \preccurlyeq t \leqslant Y$. Let n be such that $lh(t_n) \leqslant lh(t) < lh(t_{n+1})$ and let $j < m_n$ be such that $t_n^j \preccurlyeq t$. Clearly $|t_n^j| = |t|$. Since by construction Y accepts or rejects t_n^j, it follows that Y accepts or rejects t. This completes the proof of Lemma 4.4.

Let (s, X) be any dual Ellentuck neighborhood. We say that X *strongly rejects* s if X rejects s and X rejects all t with $s \prec t \leqslant X$ and $|t| = |s| + 1$.

4.5. LEMMA. *If X rejects s, there exists $Y \in (s, X)$ such that Y strongly rejects s.*

Proof. By Lemma 4.4 we may safely assume that X accepts or rejects all t with $s \preccurlyeq t \leqslant X$. If u is any segment, define $u^* = u \cup \{\{lh(u)\}\} =$ the unique segment v such that $u \prec v$ and $lh(v) = lh(u) + 1$ and $|v| = |u| + 1$. Let $(s, X)^*$ be the set of all u such that $s \preccurlyeq u$ and $|s| = |u|$ and $u^* \leqslant X$. Let C_0^* (respectively C_1^*) be the set of all $u \in (s, X)^*$ such that X accepts (respectively rejects) u^*. Thus $(s, X)^* = C_0^* \cup C_1^*$. By Lemma 2.4 there exists $Y \in (s, X)$ such that either $(s, Y)^* \subseteq C_0^*$ or $(s, Y)^* \subseteq C_1^*$. (Here we are applying Lemma 2.4 to an alphabet of cardinality $|s|$.) We claim that $(s, Y)^* \subseteq C_1^*$. Otherwise $(s, Y)^* \subseteq C_0^*$ which would imply that Y accepts all $u \in (s, Y)^*$. From this it would follows that Y accepts s, a contradiction since X rejects s. This proves the claim. Thus Y rejects all $u \in (s, Y)^*$. Now given t such that $s \prec t \leqslant Y$ and $|t| = |s| + 1$, there is a unique $u \in (s, Y)^*$ with $u^* \preccurlyeq t$. Since $|u^*| = |t|$ and Y rejects u^*, it follows that Y rejects t. This completes the proof.

4.6. LEMMA. *Suppose $s \leqslant s' \leqslant X \in (\omega)^\omega$. If X rejects s then there exists $Y' \in (s', X)$ such that Y' strongly rejects s.*

Proof. By Lemma 4.5 let $Y \in (s, X)$ be such that Y strongly rejects s. By Lemma 4.2 let $Y' \in (s', X)$ be such that $(s, Y') = (s, Y)$. Then clearly Y' strongly rejects s.

4.7. LEMMA. *If X rejects s then there exists $Y \in (s, X)$ such that Y rejects all t with $s \preccurlyeq t \leqslant Y$.*

Proof. Let Y_0 be any element of (s, X) and put $t_0 = s$. Suppose that we have constructed $Y_n \in (s, X)$ and $t_n \prec Y_n$ such that $s \preccurlyeq t_n$ and $|t_n| = |s| + n$ and Y_n rejects all t with $s \preccurlyeq t \preccurlyeq t_n$. Let $\{t_n^j : j < m_n\}$ be an enumeration of all t such that $s \preccurlyeq t \preccurlyeq t_n$. Put $Y_n^0 = Y_n$. By Lemma 4.6 let $Y_n^{j+1} \in (t_n, Y_n^j)$ be such that Y_n^{j+1} strongly rejects t_n^j. Put $Y_{n+1} = Y_n^{m_n}$. Let t_{n+1} be the smallest t such that $t_n \prec t \prec Y_{n+1}$ and $|t| = |t_n| + 1 = |s| + n + 1$. By construction Y_{n+1} rejects all t with $s \preccurlyeq t \preccurlyeq t_{n+1}$. Finally put $Y = \lim_n Y_n =$ the unique $Y \in (s, X)$ such that $t_n \prec Y$ for all $n \in \omega$. We claim that Y rejects all t with $s \preccurlyeq t \preccurlyeq Y$. To see this, let t be given with $s \preccurlyeq t \preccurlyeq Y$. Let n be such that $lh(t_n) \leqslant lh(t) < lh(t_{n+1})$ and let $j < m_n$ be such that $t_n^j \preccurlyeq t$. Since $t_n^j \preccurlyeq t$ and $|t_n^j| = |t|$ and Y rejects t_n^j it follows that Y rejects t. This completes the proof.

4.8. Lemma. *Suppose that $O \subseteq (\omega)^\omega$ is open with respect to the dual Ellentuck topology. Then O is Ramsey.*

Proof. Let (s, X) be any dual Ellentuck neighborhood. If X does not reject s, let $Y \in (s, X)$ be such that $(s, Y) \subseteq O$. If X rejects s, then by Lemma 4.7 let $Y \in (s, X)$ be such that Y rejects all t with $s \preccurlyeq t \preccurlyeq Y$. We claim that $(s, Y) \cap O = \varnothing$. If not, let $Z \in (s, Y) \cap O$. Since O is open with respect to the dual Ellentuck topology, there exists $u \prec Z$ such that $(u, Z) \subseteq (s, Y) \cap O$. We may safely assume that $lh(u) \geqslant lh(s)$. Since $u \prec Z \in (s, Y)$ it follows that $s \preccurlyeq u \preccurlyeq Y$ so Y rejects u. On the other hand, $Z \in (u, Y)$ and Z accepts u. This contradiction proves the claim. Thus in either case $(s, Y) \subseteq O$ or $(s, Y) \cap O = \varnothing$. This completes the proof.

4.9. Lemma. *Let $M \subseteq (\omega)^\omega$ be meager with respect to the dual Ellentuck topology. Then M is Ramsey null.*

Proof. Let (s, X) be a given dual Ellentuck neighborhood. We shall find a $Y \in (s, X)$ such that $(s, Y) \cap M = \varnothing$. Let $M \subseteq (\omega)^\omega \setminus \bigcap_{n \in \omega} O_n$, where each O_n is dense open with respect to the dual Ellentuck topology. Let Y_0 be any element of (s, X) and put $t_0 = s$. Suppose that we have already constructed $Y_n \in (s, X)$ and $t_n \prec Y_n$ such that $s \preccurlyeq t_n$ and $|t_n| = |s| + n$. Let $\{t_n^j : j < m_n\}$ be an enumeration of all t such that $s \preccurlyeq t \preccurlyeq t_n$. Put $Y_n^0 = Y_n$. By Lemma 4.8 and 4.2 let $Y_n^{j+1} \in (t_n, Y_n^j)$ be such that $(t_n^j, Y_n^{j+1}) \subseteq O_n$ or $(t_n^j, Y_n^{j+1}) \cap O_n = \varnothing$. The latter alternative is impossible since O_n is dense. Put $Y_{n+1} = Y_n^{m_n}$ and let t_{n+1} be the smallest t such that $t_n \prec t \prec Y_{n+1}$ and $|t| = |t_n| + 1 = |s| + n + 1$. Finally put $Y = \lim_n Y_n =$ the unique $Y \in (s, X)$ such that $t_n \prec Y$ for all n. We claim that $(s, Y) \cap M = \varnothing$. To see this, let $Z \in (s, Y)$ be given. Then for each n there exists $j < m_n$ such that $t_n^j \prec Z$. Hence $Z \in (t_n^j, Y_n^{j+1}) \subseteq O_n$. Thus $(s, Y) \subseteq \bigcap_{n \in \omega} O_n$ so $(s, Y) \cap M = \varnothing$. This completes the proof.

4.10. Lemma. *Suppose that $C \subseteq (\omega)^\omega$ has the property of Baire with respect to the dual Ellentuck topology. Then C is Ramsey.*

Proof. By hypothesis we have

$$(C\backslash O) \cup (O\backslash C) \subseteq M$$

where O (respectively M) is open (respectively meager) with respect to the dual Ellentuck topology. Let (s, X) be a given dual Ellentuck neighborhood. By Lemma 4.9 let $Y \in (s, X)$ be such that $(s, Y) \cap M = \varnothing$. By Lemma 4.8 let $Z \in (s, Y)$ be such that $(s, Z) \subseteq O$ or $(s, Z) \cap O = \varnothing$. Then clearly $(s, Z) \subseteq C$ or $(s, Z) \cap C = \varnothing$. This completes the proof.

4.11. LEMMA. *If $C \subseteq (\omega)^{\omega}$ is Ramsey null (respectively Ramsey), then C is meager (respectively has the property of Baire) with respect to the dual Ellentuck topology.*

Proof. If C is Ramsey null, it follows by definition that $(\omega)^{\omega}\backslash C$ contains a dense open set. Hence C is meager. If C is Ramsey, it follows by definition that $C\backslash$(interior of C) is Ramsey null, hence meager, so C has the property of Baire. This completes the proof.

Theorem 4.1 is an immediate consequence of Lemmas 4.9, 4.10 and 4.11.

We now present some corollaries of Theorem 4.1. The following corollary is the Dual Galvin–Prikry Theorem 1.3, generalized to A-partitions.

4.12. COROLLARY. *Let A be a finite alphabet. If $(\omega)^{\omega}_A = C_0 \cup \cdots \cup C_{l-1}$ where each C_i is Borel (in the "classical" topology of Section 1), then there exists $Z \in (\omega)^{\omega}_A$ such that $(Z)^{\omega}_A \subseteq C_i$ for some i.*

Proof. Pick any $X \in (\omega)^{\omega}_A$ and let $s \leqslant X$ be such that $|s| = |A|$. Then there is an obvious canonical homeomorphism $h\colon (s, X) \cong (\omega)^{\omega}_A$, where both spaces have the classical topology. Since the dual Ellentuck topology is finer than the classical topology, each classical Borel set in (s, X) is also Borel with respect to the dual Ellentuck topology and hence has the property of Baire with respect to that topology. Thus repeated application of Lemma 4.10 gives $Y \in (s, X)$ such that $(s, Y) \subseteq h^{-1}(C_i)$ for some i. Put $Z = h(Y)$. Then $h^{-1}((Z)^{\omega}_A) = (s, Y) \subseteq h^{-1}(C_i)$ so $(Z)^{\omega}_A \subseteq C_i$. This completes the proof.

The next corollary is the dual form of a theorem of Louveau and Simpson [23].

4.13. THEOREM. *Let (s, X) be a dual Ellentuck neighborhood. Suppose $f\colon (s, X) \to M$, where M is a possibly nonseparable metric space. Suppose that $f^{-1}(O)$ is Ramsey for each open set $O \subseteq M$. Then there exists $Y \in (s, X)$ such that the image of f on (s, Y) is separable.*

Proof. The proof which is given in Louveau–Simpson [23] for Ellentuck neighborhoods carries over to the dual situation, using the ideas of this section and those of Section 5.

We end this section by showing that the original Galvin–Prikry–Ellentuck Theorem [10, 7] can be derived from our Dual Ellentuck Theorem 4.1. Let $[\omega]^\omega$ be the set of all infinite subsets of ω. If s is a finite subset of $X \in [\omega]^\omega$, let $[s, X]$ be the set of all $Y \in [\omega]^\omega$ such that $s \subseteq Y \subseteq X$. We refer to $[s, X]$ as an *Ellentuck neighborhood*. We say that $C \subseteq [\omega]^\omega$ is *Ramsey* if for each Ellentuck neighborhood $[s, X]$ there exists $Y \in [s, X]$ such that $[s, Y] \subseteq C$ or $[s, Y] \cap C = \varnothing$. We say that $C \subseteq [\omega]^\omega$ is *Ramsey null* if for each Ellentuck neighborhood $[s, X]$ there exists $Y \in [s, X]$ such that $[s, Y] \cap C = \varnothing$. The *Ellentuck topology* on $[\omega]^\omega$ is the topology whose basic open sets are the Ellentuck neighborhoods.

4.14. ELLENTUCK'S THEOREM. *A subset of $[\omega]^\omega$ is Ramsey (respectively Ramsey null) if and only if it has the property of Baire (respectively is meager) with respect to the Ellentuck topology.*

Proof. Define a continuous onto function $\varphi: (\omega)^\omega \to [\omega]^\omega$ be

$$\varphi(X) = \{\min(x) - 1 : 0 \notin x \in X\}.$$

It is easy to verify that under φ the image of an open set is open. From this it follows that the inverse image of a dense open set is dense open, hence the inverse image of a meager set is meager. It is also easy to check that the image of a Ramsey set is Ramsey. Thus the Dual Ellentuck Theorem 4.1 easily implies the nontrivial part of Ellentuck's Theorem 4.14.

5. DUAL MATHIAS FORCING

In this section we assume familiarity with the rudiments of forcing [17]. The purpose of this section is to study the dual form of a well known forcing notion due to Mathias [24]. Mathias forcing and dual Mathias forcing are alike in that they both add a new real to the universe. The difference between the two kinds of forcing is as follows: while Mathias forcing adds a very thin (but infinite) subset of ω, dual Mathias forcing adds a very coarse (but infinite) partition of ω.

5.1. DEFINITION. Let M be a transitive model of set theory containing all the ordinals. We define P^M to be the set of all (codes for) dual Ellentuck neighborhoods (s, X) such that $X \in (\omega)^\omega \cap M$. P^M is ordered by inclusion: $(s, X) \leqslant (t, Y)$ if and only if $(s, X) \subseteq (t, Y)$. We regard P^M as a notion of forcing over M. This is *dual Mathias forcing*. A set $D \subseteq P^M$ is called *dense* if for all $p \in P^M$ there exists $q \leqslant p$ such that $q \in D$. A partition $X \in (\omega)^\omega$ is called *M-generic* if $X \in \bigcup D$ for all dense $D \subseteq P^M$ such that $D \in M$. Clearly there is a canonical one-to-one correspondence between M-generic filters $G \subseteq P^M$ and M-generic partitions.

A dense set $D \subseteq P^M$ is said to be *strongly dense* if for all $(s, X) \in P^M$ there exists $Y \in (s, X) \cap M$ such that $(s, Y) \in D$.

5.2. LEMMA. *Let σ be a sentence of the forcing language for P^M. Then the set of all $p \in P^M$ such that $p \Vdash \sigma$ or $p \Vdash \sim\sigma$ is strongly dense.*

Proof. Within M define $O_1 = \{Y: (t, Y) \Vdash \sigma$ for some $t \prec Y\}$ and $O_2 = \{Y: (t, Y) \Vdash \sim\sigma$ for some $t \prec Y\}$. Clearly $O_1 \cup O_2$ is a dense open subset of $(\omega)^\omega$ with respect to the Ellentuck topology. Applying Lemma 4.8 within M we see that for any $(s, X) \in P^M$ there exists $Y \in (s, X) \cap M$ such that either $(s, X) \cap M \subseteq O_1$ or $(s, X) \cap M \subseteq O_2$. In the former case $(s, Y) \Vdash \sigma$ and in the latter case $(s, Y) \Vdash \sim\sigma$. This completes the proof.

5.3. LEMMA. *Let $D \in M$ be a dense subset of P^M. Then there exists $D^* \in M$ such that D^* is a strongly dense subset of P^M and $\bigcup D^* = \bigcup D$.*

Proof. We may safely assume that D is *dense open*, i.e., D is dense and for all $p \in D$ and $q \leqslant p$, $q \in D$. If $(s, X) \in P^M$, let us say that (s, X) *captures* D if for all $Y \in (s, X) \cap M$ there exists $t \prec Y$ such that $s \leqslant t$ and $(t, X) \in D$. Let D^* be the set of all $(s, X) \in P^M$ such that (s, X) captures D.

Clearly $\bigcup D \subseteq \bigcup D^*$. We claim that $\bigcup D^* \subseteq \bigcup D$. To see this, let $Y \in (s, X) \in D^*$ be given. Let T be the tree of all t such that $s \leqslant t \leqslant X$ and $(t, X) \notin D$. Here T is ordered by $\leqslant$. Thus $(s, X) \in D^*$ means precisely that T is well founded, i.e., contains no infinite path in M. By absoluteness, T remains well founded in $M[Y]$. Hence there exists $t \prec Y$ such that $t \notin T$. Thus $Y \in (t, X) \in D$ so $Y \in \bigcup D$. This proves the claim.

It remains to show that D^* is strongly dense. Let $(s, X) \in P^M$ be given. As in the proof of Lemma 4.4 we can find $Y \in (s, X)$ such that for all t with $s \leqslant t \leqslant Y$ either $(t, Y) \in D$ or there is no $Z \in (t, Y)$ such that $(t, Z) \in D$. Let O be the set of all $Z \in (s, Y)$ such that $(t, Y) \in D$ for some t such that $s \leqslant t \prec Z$. By Lemma 4.8 there exists $W \in (s, Y)$ such that either $(s, W) \subseteq O$ or $(s, W) \cap O = \varnothing$. By density of D there exists $(t, Z) \in D$ such that $s \leqslant t \prec Z$ and $(t, Z) \subseteq (s, W)$. Since $Z \in (t, Y)$ and $(t, Z) \in D$, it follows by construction of Y that $(t, Y) \in D$. Hence $Z \in O$. Hence $(s, W) \subseteq O$. Hence (s, W) captures D, i.e., $(s, W) \in D^*$. This completes the proof.

5.4. LEMMA. *A partition $X \in (\omega)^\omega$ is M-generic if and only if $X \in \bigcup D$ for all strongly dense $D \subseteq P^M$ such that $D \in M$.*

Proof. Immediate from Lemma 5.3.

5.5. LEMMA. *If* $X \in (\omega)^\omega$ *is* M*-generic and* $Y \in (X)^\omega$, *then* Y *is* M*-generic.*

Proof. By Lemma 5.4 it suffices to show that $Y \in \bigcup D$ for every strongly dense $D \in M$. Given such a D, let D' be the set of all $(s, Z) \in P^M$ such that $(t, Z) \in D$ for all $t \leqslant s$. We claim that D' is (strongly) dense. To see this, let $(s, W) \in P^M$ be given and let $\{t_j : j < m\}$ be an enumeration of all $t \leqslant s$. Put $W_0 = W$ and by Lemma 4.2 let $W_{j+1} \in (s, W_j)$ be such that $(t_j, W_{j+1}) \in D$. Finally put $Z = W_m$. Then clearly $(s, Z) \in D'$. This proves the claim. Since D' is dense and X is M-generic, we have $X \in \bigcup D'$, i.e., there exists $(s, Z) \in D'$ such that $X \in (s, Z)$. Since $Y \in (X)^\omega$ it follows that $Y \in (t, Z)$ for some $t \leqslant s$. Thus $Y \in \bigcup D$. This completes the proof of Lemma 5.5.

We now present a theorem which gives a rather general sufficient condition for a set $C \subseteq (\omega)^\omega$ to be Ramsey. The condition applies in many models of set theory which are constructed by forcing.

5.6. DEFINITION. Let M be a transitive model of set theory containing all the ordinals. Let L_M be the language of set theory augmented by a 1-place predicate $\mathbf{M}$ intended to stand for M. A class C is said to be *locally* M*-definable* if there exists an L_M-formula φ and parameters $a_1, \ldots, a_k \in M$ such that

$$C = \{Z : M[Z] \models \varphi(Z, a_1, \ldots, a_k)\}.$$

5.7. THEOREM. *Let* $C \subseteq (\omega)^\omega$ *be locally* M*-definable. Assume that for every real* X *there exists an* $M[X]$*-generic partition of* ω. *Then* C *is Ramsey.*

Proof. Let (s, X) be a given dual Ellentuck neighborhood. We want to find a $Z \in (s, X)$ such that either $(s, Z) \subseteq C$ or $(s, Z) \cap C = \varnothing$. We may safely assume that $X \in M$. As in Definition 5.6 let $\varphi, a_1, \ldots, a_k \in M$ be such that $C = \{Z : M[Z] \models \varphi(Z, a_1, \ldots, a_k)\}$. Put $\sigma \equiv \varphi(\mathbf{Z}, a_1, \ldots, a_k)$, where $\mathbf{Z}$ is a forcing term which denotes an M-generic partition of ω. By Lemma 5.2 let $Y \in (s, X) \cap M$ be such that either $(s, Y) \Vdash \sigma$ or $(s, Y) \Vdash {\sim}\sigma$. For definiteness assume $(s, Y) \Vdash \sigma$. Let Z' be any M-generic partition of ω. Let $Y' \in (\omega)^\omega \cap M$ be such that $Z' \in (Y')^\omega$. Let $s' \prec Z'$ be such that $|s'| = |s|$ and let $f\colon (s', Y') \cong (s, Y)$ be the obvious canonical M-coded homeomorphism. Put $Z = f(Z')$. Then clearly Z is another M-generic partition of ω. Since $Z \in (s, Y)$ and $(s, Y) \Vdash \sigma$ it follows that $Z \in C$. Furthermore, for any $W \in (s, Z)$, we have by Lemma 5.5 that W is also M-generic, hence $W \in C$. Thus $(s, Z) \subseteq C$. This completes the proof.

For the proof of the next theorem, we assume familiarity with Solovay [38]. Let M be a transitive model of ZFC (Zermelo–Fraenkel set theory with the Axiom of Choice) containing all the ordinals. Assume that M contains an inaccessible cardinal, κ. Let Q^M be the weak direct product of the partial orderings $\alpha^{<\omega}$, $\alpha < \kappa$, where each $\alpha^{<\omega}$ is ordered by reverse inclusion. Thus Q^M is the notion of forcing due to Lévy in which each cardinal less than κ is collapsed to ω. Let $G \subseteq Q^M$ be an M-generic filter.

5.8. THEOREM. *In $M[G]$, every $C \subseteq (\omega)^\omega$ which is definable in terms of real parameters and parameters from M is Ramsey.*

Proof. By Solovay [38], any such C is locally $M[X]$-definable for some real X. Furthermore $(2^{2^{\aleph_0}})^{M[X]}$ is countable in $M[G]$, so $M[G]$ contains an $M[X]$-generic partition of ω. It follows by Theorem 5.7 that C is Ramsey.

5.9. COROLLARY. *If ZFC plus "there exists an inaccessible cardinal" is consistent, then so is ZFC plus "every subset of $(\omega)^\omega$ which is ordinal definable from a real is Ramsey."*

Proof. This follows from Theorem 5.8 by standard forcing techniques.

5.10. COROLLARY. *If ZFC plus "there exists an inaccessible cardinal" is consistent, then so is ZF plus DC (the Axiom of Dependent Choices) plus "every subset of $(\omega)^\omega$ is Ramsey."*

Proof. Let $M[G]$ be as in Theorem 5.8. Let $N \subseteq M[G]$ be the inner model consisting of all sets which are hereditarily definable over $M[G]$ in terms of parameters from M and real parameters from $M[G]$. By standard techniques it follows that N is a model of ZF plus DC plus "every subset of $(\omega)^\omega$ is Ramsey." The corollary follows by standard forcing techniques.

5.11. *Remark.* All of the results in this section were inspired by analogous known results for Mathias forcing [24].

6. PROOF OF LEMMA 2.4

The purpose of this section is to prove Lemma 2.4, which has already played a key role in the proofs of Theorems 2.2 and 4.1. At the end of the section we comment on some strengthenings of Lemma 2.4 which are due to Carlson [4, 5].

We restate Lemma 2.4 in the following equivalent form:

6.1. THEOREM. *Let A be a finite alphabet. If $(\omega)_A^* = C_0 \cup \cdots \cup C_{l-1}$ then there exists $X \in (\omega)_A^\omega$ such that $(X)_A^* \subseteq C_i$ for some i.*

Here $(\omega)_A^*$ denotes the set of all A-segments such that $\#(s) = 0$.

On the face of it, Theorem 6.1 may appear to be only a special case of Lemma 2.4. To see that Theorem 6.1 is actually equivalent to Lemma 2.4, let

$Y \in (\omega)_A^\omega$ be given and note that there is a canonical homeomorpism h: $(\omega)_A^\omega \cong (Y)_A^\omega$. If $(Y)_A^* = C_0^* \cup \cdots \cup C_{l-1}^*$ is a given coloring of $(Y)_A^*$, let $(\omega)_A^* = C_0 \cup \cdots \cup C_{l-1}$ be the corresponding coloring of $(\omega)_A^*$. Theorem 6.1 implies the existence of an $X \in (\omega)_A^\omega$ such that $(X)_A^* \subseteq C_i$ for some i. Put $Z = h(X)$. Then $(Z)_A^* \subseteq C_i^*$. Thus Lemma 2.4 follows from Theorem 6.1.

Instead of proving Theorem 6.1 directly, we shall prove a somewhat stronger result:

6.2. Definition. We say that $X \in (\omega)_A^\omega$ is *special* if for all free blocks x_1 and x_2 of X, either $\max(x_1) < \min(x_2)$ or $x_1 = x_2$ or $\min(x_1) > \max(x_2)$. (This condition implies, but is not equivalent to, the condition that all free blocks of X are finite.)

We shall prove Theorem 6.1 with the conclusion strengthened to say that X is special.

Before proceeding we introduce some notation. Let $\langle\omega\rangle_A^\omega$ be the set of all special $X \in (\omega)_A^\omega$. For $X \in \langle\omega\rangle_A^\omega$ let $\langle X\rangle_A^\omega$ be the set of all $Y \in \langle\omega\rangle_A^\omega$ such that Y is coarser than X. Thus $\langle X\rangle_A^\omega = (X)_A^\omega \cap \langle\omega\rangle_A^\omega$. By a *special A-segment* we mean any A-segment s such that $s \prec X$ for some special $X \in (\omega)_A^\omega$. For $m \in \omega$ let $\langle\omega\rangle_A^m$ be the set of all special A-segments s such that $\#(s) = m$. In particular $\langle\omega\rangle_A^0 = (\omega)_A^*$. For $X \in \langle\omega\rangle_A^\omega$ and $m \in \omega$ let $\langle X\rangle_A^m$ be the set of all $s \in \langle\omega\rangle_A^m$ such that $s^* \leqslant X$. In particular $\langle X\rangle_A^0 = (X)_A^*$. We also write $\langle X\rangle_A^{<\omega} = \bigcup_{m\in\omega} \langle X\rangle_A^m$ and $\langle\omega\rangle_A^{<\omega} = \bigcup_{m\in\omega} \langle\omega\rangle_A^m$.

Our strengthened version of Theorem 6.1 reads as follows:

6.3. Theorem. *Let A be a finite alphabet. If $\langle\omega\rangle_A^0 = C_0 \cup \cdots \cup C_{l-1}$ then there exists $X \in \langle\omega\rangle_A^\omega$ such that $\langle X\rangle_A^0 \subseteq C_i$ for some i.*

For the proof of Theorem 6.3 the following notation will be convenient. Given $s \in \langle\omega\rangle_A^m$ and $t \in \langle\omega\rangle_A^n$ let $s \oplus t \in \langle\omega\rangle_A^{m+n}$ be the *concatenation* of s and t. Thus z is a block of $s \oplus t$ if and only if either (i) z is a free block of s; or (ii) $z = \{lh(s) + j: j \in y\}$, where y is a free block of t; or (iii) $z = x \cup \{lh(s) + j: j \in y \setminus \{a\}\}$, where x is a block of s, y is a block of t, and $x \cap A = y \cap A = \{a\}$. Thus $lh(s \oplus t) = lh(s) + lh(t)$.

Our proof of Theorem 6.3 will depend on a finite combinatorial theorem of Hales and Jewett [13] which generalizes van der Waerden's Theorem on arithmetic progressions [40]. To state the Hales–Jewett Theorem, we use the notation which we have already introduced in Section 3 for Theorem 3.2.

6.4. Hales–Jewett Theorem. *Let A be a finite alphabet. For all l there exists n so large that if $(n)_A^0 = C_0 \cup \cdots \cup C_{l-1}$ there exists $t \in (n)_A^1$ such that $(t)_A^0 \subseteq C_i$ for some i.*

Proof. The Hales–Jewett Theorem is just the special case $k = 0$, $m = 1$ of the Graham–Rothschild Theorem [11] which we have already presented as Theorem 3.2. However, our proof of Theorem 3.2 used Theorem 2.2, whose proof used Lemma 2.4, whose proof will use Theorem 6.4. It would therefore be inappropriate for us now to cite Theorem 3.2 in proving Theorem 6.4. Instead, for a proof of Theorem 6.4, we refer the reader to the original paper of Hales and Jewett [13] or to the expository monograph of Graham, Rothschild and Spencer [12].

We now proceed to the proof of Theorem 6.3. Until further notice A is a fixed finite alphabet. For $X \in \langle\omega\rangle_A^\omega$ and any set D, let us say that D is *dense in* $\langle X\rangle_A^0$ if $\langle Y\rangle_A^0 \cap D \neq \varnothing$ for all $Y \in \langle X\rangle_A^\omega$.

6.5. LEMMA. *If D is dense in $\langle X\rangle_A^0$ then there exists $t \in \langle X\rangle_A^1$ such that $(t)_A^0 \subseteq D$.*

Proof. First we claim that there exists $s \in \langle X\rangle_A^{<\omega}$ such that $(t)_A^0 \cap D \neq \varnothing$ for all $t \in \langle X\rangle_A^{<\omega}$ such that $s \leqslant t$. Suppose not. Let $t_0 \in \langle X\rangle_A^0$ be such that $t_0 \notin D$. Given $t_m \in \langle X\rangle_A^m$ let $s_{m+1} \in \langle X\rangle_A^{m+1}$ be such that $t_m^* \leqslant s_{m+1}$. Then, since the claim fails with $s = s_{m+1}$, let $t_{m+1} \in \langle X\rangle_A^{m+1}$ be such that $s_{m+1} \leqslant t_{m+1}$ and $(t_{m+1})_A^0 \cap D = \varnothing$. Finally let $Y = \lim_m t_m =$ the unique $Y \in \langle\omega\rangle_A^\omega$ such that $t_m < Y$ for all $m \in \omega$. Then $\langle Y\rangle_A^0 = \bigcup_{m\in\omega} (t_m)_A^0$ is disjoint from D. Thus D is not dense in $\langle X\rangle_A^0$. This contradiction proves the claim.

Now let s be as in the above claim. Let l be the cardinality of $(s)_A^0$. By the Hales–Jewett Theorem 6.4, let n be so large that for any A-segment $u \in \langle\omega\rangle_A^n$ and any coloring $(u)_A^0 = C_0 \cup \cdots \cup C_{l-1}$ there exists $v \in (u)_A^1$ such that $(v)_A^0 \subseteq C_i$ for some $i < l$. Pick $u \in \langle\omega\rangle_A^n$ such that $s \oplus u \in \langle X\rangle_A^{<\omega}$. Color $(u)_A^0$ by letting $\{s_i : i < l\}$ be an enumeration of $(s)_A^0$ and defining C_i to be the set of all $w \in (u)_A^0$ such that $s_i \oplus w \in D$. By choice of s we have $(u)_A^0 = C_0 \cup \cdots \cup C_{l-1}$. Hence by choice of n there exist $v \in (u)_A^1$ and $i < l$ such that $(v)_A^0 \subseteq C_i$. In other words $(s_i \oplus v)_A^0 \subseteq D$. Put $t = s_i \oplus v$. Then $t \in \langle X\rangle_A^1$ and $(t)_A^0 \subseteq D$. This completes the proof of Lemma 6.5.

We find it convenient to extend the $\oplus$ notation to infinite A-partitions. Given $s \in \langle\omega\rangle_A^{<\omega}$ and $Y \in \langle\omega\rangle_A^\omega$ let $s \oplus Y$ be the *concatenation* of s and Y, i.e., $s \oplus Y$ is the unique $Z \in \langle\omega\rangle_A^\omega$ such that $s \oplus t < Z$ for all $t < Y$.

6.6. LEMMA. *Suppose D is dense in $\langle X\rangle_A^0$. Then there exists $s \in \langle\omega\rangle_A^1$ and $Y \in \langle\omega\rangle_A^\omega$ such that $s \oplus Y \in \langle X\rangle_A^\omega$ and $\{t : (s \oplus t)_A^0 \subseteq D\}$ is dense in $\langle Y\rangle_A^0$.*

Proof. Assume not. Then for all $s \oplus Y \in \langle X\rangle_A^\omega$ with $s \in \langle\omega\rangle_A^1$ there exists $Z \in \langle Y\rangle_A^\omega$ such that $(s \oplus t)_A^0 \nsubseteq D$ for all $t \in \langle Z\rangle_A^0$. We apply this assumption repeatedly as follows. Let $X = s_0 \oplus Y_0$, where $s_0 \in \langle\omega\rangle_A^1$. Given $s_0 \oplus \cdots \oplus s_n \oplus Y_n \in \langle X\rangle_A^\omega$, let $Z_n \in \langle Y_n\rangle_A^\omega$ be such that $(s \oplus t)_A^0 \nsubseteq D$ for all $s \in (s_0 \oplus \cdots \oplus s_n)_A^1$ and all $t \in \langle Z_n\rangle_A^0$. Then let $Z_n = s_{n+1} \oplus Y_{n+1}$ where $s_{n+1} \in \langle\omega\rangle_A^1$. Finally put $W = s_0 \oplus \cdots \oplus s_n \oplus \cdots =$ the unique $W \in \langle X\rangle_A^\omega$

such that $s_0 \oplus \cdots \oplus s_n \prec W$ for all n. Given $u \in \langle W \rangle_A^1$ we have $u \in (s_0 \oplus \cdots \oplus s_n \oplus t)_A^1$ for some $n \in \omega$ and $t \in \langle Z_n \rangle_A^0$. Hence by construction $(u)_A^0 \not\subseteq D$. But this contradicts Lemma 6.5. Lemma 6.6 is proved.

6.7. LEMMA. *In Lemma* 6.6, *we may strengthen the conclusion to say that* $r \in D$ *where* $r^* \leqslant s$.

Proof. Let $s_0 \in \langle \omega \rangle_A^1$ and $Y_0 \in \langle \omega \rangle_A^\omega$ be as in the conclusion of Lemma 6.6. Thus $s_0 \oplus Y_0 \in \langle X \rangle_A^\omega$ and $D_0 = \{t: (s_0 \oplus t)_A^0 \subseteq D\}$ is dense in $\langle Y_0 \rangle_A^0$. Given Y_n and D_n such that D_n is dense in $\langle Y_n \rangle_A^0$, apply Lemma 6.6 to obtain $s_{n+1} \in \langle \omega \rangle_A^1$ and $Y_{n+1} \in \langle \omega \rangle_A^\omega$ such that $s_{n+1} \oplus Y_{n+1} \in \langle Y_n \rangle_A^\omega$ and $D_{n+1} = \{t: (s_{n+1} \oplus t)_A^0 \subseteq D_n\}$ is dense in $\langle Y_{n+1} \rangle_A^0$. Note that $D_{n+1} = \{t: (s_0 \oplus \cdots \oplus s_{n+1} \oplus t)_A^0 \subseteq D\}$ by induction on n. Finally put $W = s_0 \oplus \cdots \oplus s_n \oplus \cdots =$ the unique $W \in \langle X \rangle_A^\omega$ such that $s_0 \oplus \cdots \oplus s_n \prec W$ for all n. Since D is dense in X there exists $r \in \langle W \rangle_A^0$ such that $r \in D$. Let n be such that $lh(r) < lh(s_0 \oplus \cdots \oplus s_n)$. Let $s \in (s_0 \oplus \cdots \oplus s_n)_A^1$ be such that $r^* \leqslant s$, and put $Y = Y_n$. Since $(s)_A^0 \subseteq (s_0 \oplus \cdots \oplus s_n)_A^0$, we have that $D_{n+1} \subseteq \{t: (s \oplus t)_A^0 \subseteq D\}$ is dense in $\langle Y_n \rangle_A^0 = \langle Y \rangle_A^0$. Thus s and Y satisfy the conclusion of Lemma 6.6 and in addition $r \in D$ where $r^* \leqslant s$. This proves Lemma 6.7.

6.8. LEMMA. *If D is dense in* $\langle X \rangle_A^0$ *then there exists* $W \in \langle X \rangle_A^\omega$ *such that* $\langle W \rangle_A^0 \subseteq D$.

Proof. Let $X \in \langle \omega \rangle_A^\omega$ and D be given such that D is dense in $\langle X \rangle_A^0$. Repeat the proof of Lemma 6.7 but applying Lemma 6.7 instead of Lemma 6.6. In particular $r_0 \in D$ and $r_{n+1} \in D_n$ where $r_0^* \leqslant s_0$ and $r_{n+1}^* \leqslant s_{n+1}$. We claim that $\langle W \rangle_A^0 \subseteq D$. Let $r \in \langle W \rangle_A^0$ be given. If $lh(r) < lh(s_0)$ we have $r = r_0 \in D$. If $lh(s_0 \oplus \cdots \oplus s_n) \leqslant lh(r) < lh(s_0 \oplus \cdots \oplus s_n \oplus s_{n+1})$, we have $r \in (s_0 \oplus \cdots \oplus s_n \oplus r_{n+1})_A^0$. Since $r_{n+1} \in D_n$ it follows that $r \in D$. This completes the proof of Lemma 6.8.

Theorem 6.3 follows easily from Lemma 6.8 by induction on l. The proof of Theorem 6.3, and hence of all the other major theorems of this paper, is now complete.

It is natural to view the Dual Ellentuck Theorem 4.1 as a topological generalization of Theorem 6.1. We may therefore ask whether Theorem 6.3 has an analogous topological generalization. Carlson [4, 5] has answered this question in the affirmative. We now state Carlson's result. Let A be a finite alphabet. Given $X \in \langle \omega \rangle_A^\omega$ and $s \leqslant X$ define $\langle s, X \rangle_A^\omega$ to be the set of all $Y \in \langle X \rangle_A^\omega$ such that $s \prec Y$. We refer to $\langle s, X \rangle_A^\omega$ as an *Ellentuck neighborhood* in $\langle \omega \rangle_A^\omega$. The *Ellentuck topology* on $\langle \omega \rangle_A^\omega$ is the topology whose basic open sets are the Ellentuck neighborhoods. We say that $C \subseteq \langle \omega \rangle_A^\omega$ is *Ramsey* if for each Ellentuck neighborhood $\langle s, X \rangle_A^\omega$ there exists $Y \in \langle s, X \rangle_A^\omega$ such that $\langle s, Y \rangle_A^\omega \subseteq C$ or $\langle s, Y \rangle_A^\omega \cap C = \varnothing$. Carlson's result reads as follows:

6.9. CARLSON'S THEOREM. *A set $C \subseteq \langle\omega\rangle_A^\omega$ is Ramsey if and only if it has the property of Baire with respect to the Ellentuck topology on $\langle\omega\rangle_A^\omega$.*

Proof. See Carlson [5] or Prikry [31]. Surprisingly, the proof of this theorem is considerably more difficult than the proof of the Dual Ellentuck Theorem 4.1.

6.10. *Remark.* When A is the empty set, Carlson's Theorem 6.9 reduces to Ellentuck's Theorem 4.14. See also Ellentuck [7]. When A is a one-element set, Carlson's Theorem 6.9 reduces to Milliken's [26] topological generalization of the well known theorem of Hindman [16].

6.11. *Remark.* Carlson has also obtained results for $\langle\omega\rangle_A^\omega$ analogous to the results of Sections 2 and 5 above. More recently, Carlson has obtained another result which is a common generalization of Theorems 4.1 and 6.9. We omit the statement of this result since it cannot be given conveniently in terms of the notation which is at hand.

7. SUGGESTIONS FOR FURTHER RESEARCH

In this section we describe several possible research projects which are suggested by the results of this paper.

A very interesting recent development in finite combinatorics is the Paris–Harrington Theorem [29, 12]. A finite set $X \subseteq \omega$ is said to be *relatively large* if $|X| \geqslant \min(X)$. Let PH be the statement that for all $k, l, m \in \omega$ there exists $n \in \omega$ so large that the following holds. For any coloring $[n]^k = C_0 \cup \cdots \cup C_{l-1}$ there exists a relatively large set $X \subseteq n$ such that $|X| \geqslant m$ and $[X]^k \subseteq C_i$ for some i. Thus PH is a transparent generalization of the Finite Ramsey Theorem. The truth of PH follows easily from the infinite Ramsey Theorem [32]. Paris and Harrington showed that any proof of PH must involve a detour through the infinite. Namely, they showed that PH is not provable in finite set theory or in first order Peano arithmetic [29].

From the viewpoint of the mathematical logician, it is natural to ask whether there exist finite combinatorial statements which are like PH, but stronger in the sense that they cannot be proved in reasonably strong subsystems of second-order arithmetic. There has been some progress toward finding such statements [8, 9]. Simpson's original motivation for proving Theorem 2.2 was to prove the following finite combinatorial statement. Let A be a finite alphabet. An A-segment t is said to be *relatively large* if $\#(t) \geqslant \mu(t)$ where $\mu(t)$ is the least m such that m belongs to a free block of t. The following theorem is a strengthening of the Graham–Rothschild Theorem 3.2, just as PH is a strengthening of the Finite Ramsey Theorem.

7.1. THEOREM. *Let A be a finite alphabet. For all k, l, $m \in \omega$ there exists n so large that the following holds. If $(n)^k_A = C_0 \cup \cdots \cup C_{l-1}$ there exists a relatively large A-segment t such that $lh(t) = n$, $\#(t) \geqslant m$, and $(t)^k_A \subseteq C_i$ for some i.*

Proof. This follows easily from Theorem 2.2, just as in our proof of Theorem 3.2.

7.2. CONJECTURE. *Theorem 7.1 is not provable in the formal system $\Pi^1_1 - CA_0$ [35].*

The Paris–Harrington Theorem is closely related to Jockusch's [18] recursion-theoretic analysis of Ramsey's Theorem [32]. There is also a recursion-theoretic analysis of the Galvin–Prikry Theorem [10] due to Solovay [39] and Simpson [37]; this analysis was applied to finite combinatorics in [8]. It would therefore be desirable to carry out a recursion-theoretic analysis of the Dual Ramsey Theorem 1.2. The following conjecture is a starting point.

7.3. CONJECTURE. *There exists an arithmetical coloring $(\omega)^3 = C_0 \cup C_1$ such that for any $X \in (\omega)^\omega$ with $(X)^3 \subseteq C_i$ for some $i < 2$, the hyperjump of the empty set is arithmetical in X.*

As mentioned in Section 3, Theorem 2.2 is best viewed as an infinitary generalization of the Graham–Rothschild Theorem 3.2. This suggests the following problem:

7.4. PROBLEM. *Find an appropriate infinitary generalization of the Graham–Leeb–Rothschild Theorem concerning vector spaces over a finite field* [12].

A number of questions arise from the fact that Theorem 1.2 is in a certain precise sense the dual of Ramsey's Theorem. This fact suggests that one should try to dualize other set theoretic concepts and results. For instance, one might try to dualize the concept of an ultrafilter on ω. This suggests the study of maximal filters and/or maximal ideals in the lattice of partitions of ω. One might also try to dualize the theory of large cardinals [6, 17]. This suggests the study of dual Ramsey properties for uncountable cardinals. Pierre Matet hopes to report on these matters in the near future.

We now make some comments on dual Mathias forcing. All of the results of Section 5 were inspired by the known analogous results for Mathias forcing. We may continue by pointing out that, like Mathias forcing, dual Mathias forcing satisfies Baumgartner's Axiom A [1] (see also Shelah [36]). Therefore, like Mathias forcing, dual Mathias forcing can be iterated $\aleph_2$

times with countable support. The resulting model of ZFC is likely to have some interesting properties. In the case of Mathias forcing, the model is known to be interesting in that it satisfies Borel's Conjecture (Laver [20]; see also Baumgartner [1] and Shelah [36]).

There is an open problem connected with the so-called Axiom of Determinancy [3]. Let AD be the assertion that every set $C \subseteq \omega^\omega$ is determined, and let $AD_{\mathbb{R}}$ be the assertion that every set $C \subseteq \mathbb{R}^\omega$ is determined. Pierre Matet has shown that $AD_{\mathbb{R}}$ implies that every $C \subseteq (\omega)^\omega$ is Ramsey. (Matet's proof makes use of the ideas of Prikry [30] who showed that $AD_{\mathbb{R}}$ implies every $C \subseteq [\omega]^\omega$ is Ramsey.) Also, from the results of Section 5 and some unpublished results of the Cabal [3], it follows that AD implies every $C \subseteq (\omega)^\omega$ in $L(\mathbb{R})$ is Ramsey. (The Cabal has proved this result for $[\omega]^\omega$.) It is open whether AD implies that every $C \subseteq (\omega)^\omega$ is Ramsey. (The corresponding problem for $[\omega]^\omega$ is also open.)

A number of other possible research topics suggest themselves. We mention dual Ramsey quantifiers (see [35]) and dual indiscernibles in model theory. The possibilities are endless.

Note added in proof. With respect to problem 7.4, Carlson in March 1983 verified the following conjecture of Simpson: for every finite field F and every finite Borel coloring of the k-dimensional affine subspace of F^ω, there exists a closed infinite-dimensional monochromatic affine subspace. For other recent extensions and applications of the Dual Ramsey Theorem, see R. L. Graham, Recent developments in Ramsey theory (preprint, July 1983, *in* "Proceedings of the International Congress of Mathematicians," to appear); P. Matet, Partitions and filters (preprint, April 1984); H. J. Prömel, S. G. Simpson, and B. Voigt, A dual form of Erdös–Rado's canonization theorem (preprint, April 1984); S. G. Simpson, Recursion-theoretic aspects of the Dual Ramsey Theorem (preprint, May 1984, *in* "Proceedings of the April 1984 Recursion Theory Week in Oberwolfach (H. D. Ebbinghaus, G. Müller, and G. Sacks, Eds.), Lecture Notes in Mathematics, Springer-Verlag, to appear); and B. Voight, Parameter-words, trees, and vector spaces (preprint, July 1983).

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