Cone avoidance and randomness preservation

Stephen G. Simpson
Department of Mathematics
Pennsylvania State University
University Park, PA 16802, USA
http://www.math.psu.edu/simpson
simpson@math.psu.edu

Frank Stephan
Department of Mathematics
National University of Singapore
Block S17, 10 Lower Kent Ridge Road
Singapore 119076
http://www.comp.nus.edu.sg/∼fstephan
fstephan@comp.nus.edu.sg

First draft: August 27, 2012
This draft: April 21, 2015

Abstract

Let X be an infinite sequence of 0's and 1's. Let f be a computable function. Recall that X is strongly f-random if and only if the a priori Kolmogorov complexity of each finite initial segment τ of X is bounded below by $f(\tau)$ minus a constant. We study the problem of finding a PA-complete Turing oracle which preserves the strong f-randomness of X while avoiding a Turing cone. In the context of this problem, we prove that the cones which cannot always be avoided are precisely the K-trivial ones. We also prove: (1) If f is convex and X is strongly f-random and Y is Martin-Löf random relative to X, then X is strongly f-random relative to Y. (2) X is complex relative to some oracle if and only if X is random with respect to some continuous probability measure.

A version of this paper is in *Annals of Pure and Applied Logic*, 166(3):713–728, 2015.

Simpson's research was partially supported by a Simons Foundation Collaboration Grant. Stephan's research was partially supported by NUS grant R252-000-420-112.

Keywords: algorithmic randomness, Kolmogorov complexity, basis theorems, PA-completeness, Turing degrees, K-triviality, partial randomness.

2010 MSC: Primary 03D32, Secondary 68Q30, 03D28, 03F30.

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1 Introduction

In this paper we prove several new results concerning Martin-Löf randomness and partial randomness. A theme of our results is *randomness preservation*, i.e., the phenomenon that if X is (partially) random then X is (partially) random relative to certain Turing oracles.

The purpose of this introductory section is to provide some additional context and motivation for our results. In §1.1 we review basis theorems in general and the Randomness Preservation Basis Theorem in particular. In §1.2 we discuss the problem of combining two or more basis theorems into one basis theorem. We present a new result to the effect that K-triviality is the only obstacle to combining the Randomness Preservation Basis Theorem with the Cone Avoidance Basis Theorem. In §1.3 we present some other new results concerning partial randomness relative to a Turing oracle. Among the partial randomness notions which we consider are μ-randomness, strong f-randomness, autocomplexity, and complexity.

1.1 Basis theorems

Remark 1.1. A basis theorem is a theorem of the following form:

> Let P be a nonempty, effectively closed set in Euclidean space.
> Then, at least one point of P is "close to being computable."

Or, instead of assuming that P is an effectively closed set in Euclidean space, it suffices to assume that P is an effectively closed set in an effectively compact metric space. See Definition 4.2 below.

Remark 1.2. It is well known that basis theorems play an important role in the foundations of mathematics. The foundational idea underlying these applications is that, even though it is not always possible to find a *computable* point with a desired property, it is nevertheless often possible to find a point which is "close to being computable," in various senses.

Remark 1.3. Several well known basis theorems may be summarized as follows. Let P be a nonempty, effectively closed set in Euclidean space. Then, for each of the following properties, there exists $Z \in P$ such that the property holds.

1. Z is low, i.e., $Z' \leq_{\mathrm{T}} 0'$. This is the *Low Basis Theorem*, 1972 [16].

2. Z is of recursively enumerable Turing degree. This is the *R.E. Basis Theorem*, 1972 [15].

3. Z is hyperimmune-free, i.e., $(\forall f \leq_{\mathrm{T}} Z)\,(\exists g \leq_{\mathrm{T}} 0)\,\forall n\,(f(n) < g(n))$. This is the *Hyperimmune-Free Basis Theorem*, 1972 [16].

4. $X \not\leq_{\mathrm{T}} Z$, where $X \not\leq_{\mathrm{T}} 0$ is given. This is the *Cone Avoidance Basis Theorem*, 1960 [12].

5. $Y \in \mathrm{MLR}^Z$, where $Y \in \mathrm{MLR}$ is given. This is the *Randomness Preservation Basis Theorem*, 2005 [9, 26].

Here $\leq_{\mathrm{T}}$ denotes Turing reducibility, $'$ denotes the Turing jump operator, $\mathrm{MLR} = \{Y \mid Y \text{ is Martin-Löf random}\}$, and $\mathrm{MLR}^Z = \{Y \mid Y \text{ is Martin-Löf random relative to } Z\}$.

Remark 1.4. The Cone Avoidance Basis Theorem is so named because Z avoids the Turing cone above X. This theorem has been applied in foundational studies touching on set existence [12], Turing degrees of complete theories [16, 27, 28], nonstandard models of arithmetic [17], Scott sets [1, Chapter XIX], and models of WKL_0 (see [30, §§VIII.2,IX.2] and [32, §§9,10]).

Remark 1.5. The Randomness Preservation Basis Theorem is so named because Z preserves the randomness of Y. This theorem has been applied to prove several interesting results in the foundations of probability theory [3, 26, 36]. There is also a recent, less well known, basis theorem concerning preservation of strong f-randomness [13, Theorem 4.6].

1.2 Combining basis theorems

Remark 1.6. The question arises:

> Which basis theorems can be combined with each other?

Regarding this question, a large amount of information is known.

For instance, it is known that the Low Basis Theorem and the R.E. Basis Theorem are *incompatible* in the sense that they cannot be combined into one basis theorem. In other words, we can find P as above such that no point of P

is both low and of recursively enumerable Turing degree. (In fact, we can find P as above such that every point of P which is of recursively enumerable Turing degree is *Turing complete*, i.e., $\geq_{\mathrm{T}} 0'$ where $0'$ = the halting problem. This is a consequence of the Arslanov Completeness Criterion [37, Theorem V.5.1].)

Similarly, it is known that the Hyperimmune-Free Basis Theorem is incompatible with the Low Basis Theorem and with the R.E. Basis Theorem. (In fact, any hyperimmune-free Z which is $\leq_{\mathrm{T}} 0'$ is $\leq_{\mathrm{T}} 0$, hence $\notin P$ for a suitably chosen P as above.) Also, the Cone Avoidance Basis Theorem is compatible with the Low Basis Theorem, and with the Hyperimmune-Free Basis Theorem, but not with the R.E. Basis Theorem. (See for instance [10, §2.19.3].)

In addition, the Randomness Preservation Basis Theorem is incompatible with the Hyperimmune-Free Basis Theorem, and with the Low Basis Theorem, and with the R.E. Basis Theorem. To see this, let $Y \in \mathrm{MLR}$ be such that $Y \geq_{\mathrm{T}} 0'$. (The existence of such Y's is a consequence of a famous theorem known as the *Kučera/Gács Theorem*; see [10, Theorems 8.3.2 and 8.5.1] or [22, 3.3.2] or [33, Theorem 3.8].) Then, any hyperimmune-free Z such that $Y \in \mathrm{MLR}^Z$ is $\leq_{\mathrm{T}} 0$ (see [22, Theorem 8.1.18]). And, any $Z \leq_{\mathrm{T}} 0'$ such that $Y \in \mathrm{MLR}^Z$ is K-trivial (see [10, Chapter 11] or [22, Chapter 5]), hence again $\notin P$ for a suitably chosen P as above.

Summarizing these known results, we have Table 1. In this table, 0 denotes incompatibility and 1 denotes compatibility.

Basis Theorems	Low	R.E.	H.I.F.	C.A.	R.P.
Low	1	0	0	1	0
Recursively Enumerable	0	1	0	0	0
Hyperimmune-Free	0	0	1	1	0
Cone Avoidance	1	0	1	1	**???**
Randomness Preservation	0	0	0	**???**	1

Table 1: Combining basis theorems

Remark 1.7. Note that Table 1 has two missing entries. One of our accomplishments in this paper is to fill in the missing entries. We prove that, although the Randomness Preservation Basis Theorem is incompatible with the Cone Avoidance Basis Theorem, the only Turing cones which cannot be avoided in this context are the K-trivial ones. In other words, X is K-trivial if and only if there exist P as above and $Y \in \mathrm{MLR}$ such that $(\forall Z \in P)\,(Y \in \mathrm{MLR}^Z \Rightarrow X \leq_{\mathrm{T}} Z)$. Indeed, we find a fixed P which works for all K-trivial X and all $Y \in \mathrm{MLR}$ such that $Y \geq_{\mathrm{T}} 0'$. See Theorems 2.3 and 3.13 below.

1.3 Partial randomness relative to a Turing oracle

In addition to our new results mentioned in Remark 1.7, we also prove some other new results concerning partial randomness relative to a Turing oracle.

The following definition from [13, §2] plays a key role.

Definition 1.8. We write $\{0,1\}^* = \bigcup_{n=0}^\infty \{0,1\}^n =$ the set of finite sequences of 0's and 1's. Let $f : \{0,1\}^* \to (-\infty, \infty)$ be a computable function. For $S \subseteq \{0,1\}^*$ we write $\mathrm{pwt}_f(S) = \sup \left\{ \sum_{\tau \in F} 2^{-f(\tau)} \mid F \subseteq S \text{ prefix-free} \right\}$ and $[\![S]\!] = \{X \in \{0,1\}^\mathbb{N} \mid \exists n\, (X{\upharpoonright}n \in S)\}$. We say that X is strongly f-random if $X \notin \bigcap_i [\![S_i]\!]$ for all uniformly recursively enumerable sequences $S_i \subseteq \{0,1\}^*$, $i = 1, 2, \ldots$ such that $\forall i\, (\mathrm{pwt}_f(S_i) \le 2^{-i})$.

Remark 1.9. Let KA denote *a priori Kolmogorov complexity* (see [10, §3.16] or [38]). The following characterization from [13, §2] is a straightforward generalization of [4, Corollary 4.10]:

X is strongly f-random if and only if $\exists c\, \forall n\, (\mathrm{KA}(X{\upharpoonright}n) \ge f(X{\upharpoonright}n) - c)$.

Remark 1.10. Martin-Löf randomness is just strong f-randomness with $f(\tau) = |\tau| =$ the length of τ. In this case we have $\mathrm{pwt}_f(A) = \lambda([\![A]\!])$ where λ is the fair coin probability measure on $\{0,1\}^\mathbb{N}$, also known as the uniform measure or Lebesgue measure, given by $\lambda([\![\tau]\!]) = 2^{-|\tau|}$ for all $\tau \in \{0,1\}^*$. The corresponding special case of Remark 1.9 is a famous theorem known as *Schnorr's Theorem* or *the Schnorr/Levin Theorem*. See [10, Theorem 6.2.3] or [22, page 105] or [33, Theorem 10.7].

Remark 1.11. A famous theorem known as *Van Lambalgen's Theorem* (see [10, §6.9] or [22, Theorem 3.4.6] or [33, Theorem 3.6]) implies the following.

> If X is Martin-Löf random, and if Y is Martin-Löf random relative to X, then X is Martin-Löf random relative to Y.

One of our new results in this paper is a generalization of this, replacing Martin-Löf randomness by strong f-randomness. Namely, under a convexity assumption on f, we prove the following.

> If X is strongly f-random, and if Y is Martin-Löf random relative to X, then X is strongly f-random relative to Y.

See Theorem 5.8 below. This result appears to be new, even in well studied special cases such as $f(\tau) = |\tau|/2$.

Remark 1.12. Recall from [6, 7, 25, 26] the notion of μ-randomness where μ is a Borel probability measure. Namely, X is said to be μ-random if $X \notin \bigcap_i U_i$ whenever U_i is uniformly Σ^0_1 relative to μ and $\mu(U_i) \le 2^{-i}$ for all i. (For a fuller explanation, see §4 below.) Recall also that μ is said to be continuous if $\mu(\{X\}) = 0$ for all X. Note that λ is continuous, and λ-randomness is the same as Martin-Löf randomness. One of our new results in this paper is a coding-free version of Schnorr's Theorem for μ-randomness. See Theorem 4.10 below. We also obtain a version of Van Lambalgen's Theorem for μ-randomness. See Theorem 4.14 below.

Remark 1.13. Recall from [18] the notions of autocomplexity and complexity for $X \in \{0,1\}^{\mathbb{N}}$. By [13, §7] we have the following characterizations in terms of strong f-randomness.

1. X is autocomplex if and only if X is strongly f-random for some computable f such that $\{f(X{\upharpoonright}n) \mid n \in \mathbb{N}\}$ is unbounded.

2. X is complex if and only if X is strongly f-random for some computable, length-invariant f such that $\{f(X{\upharpoonright}n) \mid n \in \mathbb{N}\}$ is unbounded.

Here f is said to be length-invariant if $\forall \sigma\, \forall \tau\, (|\sigma| = |\tau| \Rightarrow f(\sigma) = f(\tau))$.

Remark 1.14. One may also consider autocomplexity and complexity relative to Turing oracles. By [18, 24, 26] (see also Theorem 6.4 below) we have:

> X is autocomplex relative to some Turing oracle if and only if X is μ-random for some μ with $\mu(\{X\}) = 0$, if and only if $X \not\leq_{\mathrm{T}} 0$.

One of our new results in this paper is as follows.

> X is complex relative to some Turing oracle if and only if X is μ-random for some continuous μ.

See Theorem 6.6 below. The class $\{X \mid X$ is μ-random for some continuous $\mu\}$ has been studied extensively [26].

2 Combining two basis theorems

In this section we prove that, except for K-triviality, the Cone Avoidance Basis Theorem and the Randomness Preservation Basis Theorem are compatible. See Theorem 2.3 below.

Remark 2.1. The concept of K-triviality will be defined and used later, in §3. Our results in this section are more conveniently formulated in terms of a related concept, LR-reducibility. Recall from [10, 22, 33, 34] that, by definition, $X \leq_{\mathrm{LR}} Z$ if and only if $\mathrm{MLR}^Z \subseteq \mathrm{MLR}^X$. Obviously $X \leq_{\mathrm{LR}} Z$ if $X \leq_{\mathrm{T}} Z$, but the converse is known to fail. (In fact, there are recursively enumerable sets $A \leq_{\mathrm{LR}} 0$ such that $A \not\leq_{\mathrm{T}} 0$.) Two major theorems concerning K-triviality (see [10, Chapter 11] or [22, Chapter 5]) are as follows.

1. X is K-trivial if and only if $X \leq_{\mathrm{LR}} 0$.

2. X is K-trivial if and only if $X \leq_{\mathrm{T}} Z$ for some $Z \in \mathrm{MLR}^X$.

It is also known that $X \leq_{\mathrm{LR}} 0$ implies $X' \leq_{\mathrm{T}} 0'$.

The following theorem implies that, given countably many non-K-trivial instances of our two basis theorems, we can simultaneously satisfy all of them. Note that part 2 of the theorem was already implicit in [13].

Theorem 2.2. Let P be a nonempty effectively closed set in Euclidean space.

1. Assume that $(\forall i \in \mathbb{N})\,(X_i \not\leq_{\mathrm{LR}} 0$ or $X_i \not\leq_{\mathrm{T}} Y)$ where $Y \in \mathrm{MLR}$. Then, there exists $Z \in P$ such that $\forall i\,(X_i \not\leq_{\mathrm{T}} Z)$ and $Y \in \mathrm{MLR}^Z$.

2. Assume that $(\forall i \in \mathbb{N})\,(X_i$ has one of the following properties):

   (a) X_i is Martin-Löf random.

   (b) X_i is strongly f_i-random, where $f_i : \{0,1\}^* \to (-\infty, \infty)$ is a specific computable function.

   (c) X_i is autocomplex.

   (d) X_i is complex.

   (e) $X_i \not\leq_{\mathrm{LR}} 0$.

   Then $\exists Z\,(Z \in P$ and $\forall i\,(X_i$ has the same property relative to $Z))$.

Proof. To prove 1, let $Q = \{Z \in P \mid Y \in \mathrm{MLR}^Z\}$. By the Randomness Preservation Basis Theorem, Q is nonempty. Since Q is $\Sigma^{0,Y}_2$, we may apply the relativization to Y of a variant of the Cone Avoidance Basis Theorem (see for instance [16, Theorem 2.5]) to find $Z \in Q$ such that $\forall i\,(X_i \not\leq_{\mathrm{T}} Y \Rightarrow X_i \not\leq_{\mathrm{T}} Z)$. On the other hand, for all i such that $X_i \leq_{\mathrm{T}} Y$, we have $X_i \not\leq_{\mathrm{LR}} 0$, hence $Y \notin \mathrm{MLR}^{X_i}$ by Remark 2.1, hence $X_i \not\leq_{\mathrm{LR}} Z$, hence again $X_i \not\leq_{\mathrm{T}} Z$.

To prove 2, apply the Kučera/Gács Theorem to find $Y \in \mathrm{MLR}$ such that $\forall i\,(X_i \leq_{\mathrm{T}} Y)$. By the Randomness Preservation Basis Theorem, let $Z \in P$ be such that $Y \in \mathrm{MLR}^Z$. Our conclusion is now immediate by [13, Theorems 1.1, 4.4, 5.1, 7.5.1, 7.5.2]. □

Theorem 2.3. Let P be a nonempty effectively closed set in Euclidean space. If X is non-K-trivial and Y is Martin-Löf random, there exists $Z \in P$ such that $X \not\leq_{\mathrm{T}} Z$ and $Y \in \mathrm{MLR}^Z$.

Proof. This is immediate from Theorem 2.2 and Remark 2.1. □

3 Low-for-Ω PA-completeness

In this section we prove that the results of the previous section are sharp. In particular, by Theorem 3.13 below, K-triviality is indeed an obstacle to combining the Cone Avoidance Basis Theorem with the Randomness Preservation Basis Theorem.

Definition 3.1. Recall from [10, Chapter 11] or [22, Chapter 5] that $X \in \{0,1\}^{\mathbb{N}}$ is said to be <u>K-trivial</u> if $\mathrm{KP}(X{\upharpoonright}n) \leq^+ \mathrm{KP}(n)$ for all n. Here KP denotes *prefix-free Kolmogorov complexity*, and $\leq^+$ denotes $\leq$ modulo an additive constant.

Definition 3.2. Recall from [10, 22] that Chaitin's Ω is $\equiv_{\mathrm{T}} 0'$ and Martin-Löf random. A Turing oracle Z is said to be low-for-Ω (see [10, Chapter 15] or [22, §8.1]) if Ω is Martin-Löf random relative to Z. An equivalent condition is that some $Y \equiv_{\mathrm{T}} 0'$ is Martin-Löf random relative to Z. Another equivalent condition is that every Martin-Löf random $Y \leq_{\mathrm{T}} 0'$ is Martin-Löf random relative to Z. (These equivalences follow from [21, Theorem 4.3], a.k.a., the *XYZ Theorem* [13, Theorem 1.1].) It is also known [2] that Z is low-for-Ω if and only if $\{X \mid X \leq_{\mathrm{LR}} Z\}$ is countable.

Definition 3.3. A Turing oracle Z is said to be PA-complete if it is Turing equivalent to some complete, consistent extension of Peano Arithmetic. Equivalently, every nonempty effectively closed set P in Euclidean space contains at least one point which is Turing reducible to Z. (See [16] or [31, §6].)

Remark 3.4. The Randomness Preservation Basis Theorem may be restated as follows: $(\forall Y \in \mathrm{MLR})\, \exists Z\, (Y \in \mathrm{MLR}^Z$ and Z is PA-complete). The special case $Y = \Omega$ is known as the *Low-for-Ω Basis Theorem* (see [10, Chapter 15] or [22, §8.1]). In other words, $\exists Z\, (Z$ is low-for-Ω and PA-complete). We are going to prove $\forall X\, (X$ K-trivial $\Rightarrow X \leq_{\mathrm{T}} Z)$ for all such Z. See Theorem 3.11 below.

Lemma 3.5. If Z is low-for-Ω, then $\mathrm{KP}(n) \leq^+ \mathrm{KP}^Z(n)$ for infinitely many n.

Proof. This result is due to Miller [20, Theorem 3.3]. See also [10, Theorem 15.6.2] or [22, Theorem 8.1.9]. □

Lemma 3.6. If Z is PA-complete, there is a Z-recursive function $\widetilde{\mathrm{KP}}\colon \mathbb{N} \to \mathbb{N}$ such that $\mathrm{KP}^Z(n) \leq^+ \widetilde{\mathrm{KP}}(n) \leq \mathrm{KP}(n)$ for all n.

Proof. Let $U\colon\subseteq \{0,1\}^* \to \mathbb{N}$ be a prefix-free partial recursive function such that $\mathrm{KP}(n) = \mathrm{KP}_U(n) = \min\{|\sigma| \mid U(\sigma) = n\}$ for all n. Let G_U be the set of characteristic functions of graphs of prefix-free partial functions from $\{0,1\}^*$ to $\mathbb{N}$ which extend U. Clearly G_U is a nonempty effectively closed subset of $\{f \mid f\colon\{0,1\}^* \times \mathbb{N} \to \{0,1\}\}$ so let $g \in G_U$ be such that $g \leq_T Z$. Let $\widetilde{\mathrm{KP}} = \mathrm{KP}_{\widetilde{U}}$ where g is the characteristic function of the graph of $\widetilde{U}$. For all n we have $\widetilde{\mathrm{KP}}(n) \leq \mathrm{KP}(n)$ because $\widetilde{U}$ extends U, and $\mathrm{KP}^Z(n) \leq^+ \widetilde{\mathrm{KP}}(n)$ because $\widetilde{U}$ is prefix-free and partial Z-recursive. It remains to show that $\widetilde{\mathrm{KP}} \leq_{\mathrm{T}} Z$, but this is clear because $\widetilde{\mathrm{KP}}(n) = |\sigma|$ for the least σ such that $g(\sigma, n) = 1$. □

Lemma 3.7. If Z is low-for-Ω and PA-complete, there is an infinite Z-recursive set A such that $\widetilde{\mathrm{KP}}(n) \leq \mathrm{KP}(n) \leq^+ \widetilde{\mathrm{KP}}(n)$ for all $n \in A$.

Proof. By Lemma 3.6 we have $\widetilde{\mathrm{KP}}(n) \leq \mathrm{KP}(n)$ for all n. By Lemmas 3.5 and 3.6 let a be a constant such that $\mathrm{KP}(n) \leq \widetilde{\mathrm{KP}}(n) + a$ for infinitely many n. Since $\widetilde{\mathrm{KP}}$ is Z-recursive, the set $S = \{n \mid \mathrm{KP}(n) \leq \widetilde{\mathrm{KP}}(n) + a\}$ is Z-recursively enumerable. Let A be an infinite Z-recursive subset of S. □

The next two lemmas are essentially due to Chaitin [5].

Lemma 3.8. $\forall c\, \exists d\, \forall n\, (|\{\tau \in \{0,1\}^n \mid \mathrm{KP}(\tau) \leq \mathrm{KP}(n) + c\}| \leq d)$.

Proof. See [10, Theorem 11.1.3] or [22, Theorem 2.2.26(ii)]. □

Definition 3.9. A <u>tree</u> is a set $T \subseteq \{0,1\}^*$ which is closed under initial segments, i.e., $(\forall \tau \in T)\,(\forall n < |\tau|)\,(\tau{\upharpoonright}n \in T)$. We write $[T] = \{X \in \{0,1\}^{\mathbb{N}} \mid \forall n\,(X{\upharpoonright}n \in T)\} = \{\text{paths through } T\}$.

Lemma 3.10. Let T be a recursively enumerable tree such that $\exists d\,\forall n\,(n \in A \Rightarrow |T \cap \{0,1\}^n| \leq d)$ for some infinite recursively enumerable set A. Then $[T]$ is finite and $\forall X\,(X \in [T] \Rightarrow X$ is recursive$)$.

Proof. Clearly $|[T]| \leq d$. Let $e \leq d$ be as large as possible such that $|T \cap \{0,1\}^n| = e$ for infinitely many $n \in A$. Let m be such that $|T \cap \{0,1\}^n| \leq e$ for all $n > m$ such that $n \in A$. Let $B = \{n > m \mid n \in A \text{ and } |T \cap \{0,1\}^n| = e\}$. Clearly B is infinite and recursively enumerable and the function $n \mapsto T \cap \{0,1\}^n$ for $n \in B$ is partial recursive. Let $P = \{X \in \{0,1\}^{\mathbb{N}} \mid \forall n\,(n \in B \Rightarrow X{\upharpoonright}n \in T \cap \{0,1\}^n)\}$. Clearly $P = [T]$ and P is a Π^0_1 subset of $\{0,1\}^{\mathbb{N}}$. Thus $[T]$ is a finite Π^0_1 subset of $\{0,1\}^{\mathbb{N}}$. It follows that every path through T is recursive. □

Theorem 3.11. If Z is low-for-Ω and PA-complete, then $\forall X\,(X$ K-trivial $\Rightarrow X \leq_{\mathrm{T}} Z)$.

Proof. Let $\widetilde{\mathrm{KP}}$ and A be as in Lemmas 3.6 and 3.7. Let a be a constant such that $\mathrm{KP}(n) \leq \widetilde{\mathrm{KP}}(n) + a$ for all $n \in A$. Suppose X is K-trivial. Let b be a constant such that $\mathrm{KP}(X{\upharpoonright}n) \leq \mathrm{KP}(n) + b$ for all n. Let

$$T = \{\tau \in \{0,1\}^* \mid (\forall n \leq |\tau|)\,(n \in A \Rightarrow \mathrm{KP}(\tau{\upharpoonright}n) \leq \widetilde{\mathrm{KP}}(n) + a + b)\}.$$

Clearly T is a tree. Since $\mathrm{KP}(n) \leq \widetilde{\mathrm{KP}}(n) + a$ for all $n \in A$, we have $X \in [T]$. Since $\widetilde{\mathrm{KP}}$ and A are Z-recursive, T is Z-recursively enumerable. Since $\widetilde{\mathrm{KP}}(n) \leq \mathrm{KP}(n)$ for all n, we may apply Lemma 3.8 with $a + b = c$ to obtain a constant d such that $|T \cap \{0,1\}^n| \leq d$ for all $n \in A$. Since A is infinite and Z-recursive, we may apply the Z-relativization of Lemma 3.10 to conclude that $\forall X\,(X \in [T] \Rightarrow X \leq_{\mathrm{T}} Z)$. This completes the proof. □

Theorem 3.12.

1. If Z is low-for-Ω and PA-complete, then $\forall X\,(X$ is K-trivial $\Leftrightarrow (X \leq_{\mathrm{T}} Z$ and $X \leq_{\mathrm{T}} 0'))$.

2. $\forall X\,(X$ is K-trivial $\Leftrightarrow \forall Z\,((Z$ is low-for-Ω and PA-complete$) \Rightarrow X \leq_{\mathrm{T}} Z))$.

Proof. Combine Theorems 2.3 and 3.11. □

The next result is a strong converse to Theorem 2.3.

Theorem 3.13. We can find $Y \in \mathrm{MLR}$ and P a nonempty effectively closed set in Euclidean space such that the following holds. For all $Z \in P$ and all K-trivial X, $Y \in \mathrm{MLR}^Z$ implies $X \leq_{\mathrm{T}} Z$.

Proof. Let $Y = \Omega$ and let $P =$ the set of complete, consistent extensions of Peano Arithmetic. Our conclusion is then a restatement of Theorem 3.11. □

We end this section with a counterpoint to Theorems 3.11 and 3.12.

Theorem 3.14. If Z is low-for-Ω and PA-complete, then $\forall A\,(A$ recursively enumerable $\Rightarrow (A \leq_{\mathrm{T}} Z$ or $Z' \leq_{\mathrm{T}} A \oplus Z))$.

Proof. Let A be recursively enumerable such that $A \not\leq_{\mathrm{T}} Z$. Since Z is PA-complete, we have $0' \leq_{\mathrm{T}} A \oplus Z$ by Day/Reimann [7, Corollary 2.1]. Since Z is low-for-Ω, we have $Z' \leq_{\mathrm{T}} Z \oplus 0'$ by Nies [22, Fact 3.6.19(ii)]. Combining these two observations, we have $Z' \leq_{\mathrm{T}} A \oplus Z$. ☐

4 An approach to μ-randomness

Let μ be a Borel probability measure on $\{0,1\}^{\mathbb{N}}$. We present a coding-free approach to μ-randomness. In this context we prove generalizations of Schnorr's Theorem and Van Lambalgen's Theorem. See Theorems 4.10 and 4.14 below.

Remark 4.1. Our approach is derived from the one in [7, 26] augmented by ideas from [23, §3.3]. A more elaborate and general coding-free approach may be found in [11].

Definition 4.2. An effectively presented complete separable metric space consists of a complete separable metric space D with metric ρ together with a sequence a of points $a_n \in D$, $n \in \mathbb{N}$ such that $\{a_n \mid n \in \mathbb{N}\}$ is dense in D and the function $(m,n) \mapsto \rho(a_m, a_n) : \mathbb{N} \times \mathbb{N} \to [0,\infty)$ is computable. An effectively compact metric space is an effectively presented complete separable metric space D, ρ, a such that $(\forall z \in D)\,\forall i\,(\exists n < c_i)\,(\rho(z, a_n) < 2^{-i})$ for some computable function $c : \mathbb{N} \to \mathbb{N}$. In this situation, a code for z is defined to be a function $f \in \prod_i \{n \mid n < c_i\}$ such that $\forall i\,(\rho(z, a_{f(i)}) \leq 2^{-i})$. Note that $\prod_i \{n \mid n < c_i\}$ is effectively homeomorphic to $\{0,1\}^{\mathbb{N}}$, so we may identify codes as points in $\{0,1\}^{\mathbb{N}}$. It is then straightforward to show that $C_z = \{Z \in \{0,1\}^{\mathbb{N}} \mid Z \text{ is a code for } z\}$ is uniformly Π^0_1 relative to any $Z \in C_z$.

Remark 4.3. The concepts in the previous definition are well known. See for instance [30, Definitions II.5.1 and III.2.3].

Remark 4.4. Let z be a point in an effectively compact metric space. From [6, 19] we know that z is not always equivalent to a Turing oracle. Nevertheless, using the ideas of [23, §3.3], it is possible to relativize many familiar recursion-theoretic concepts to z, just as if z were a Turing oracle. This theme is illustrated in the following definition and lemma.

Definition 4.5. Let z be a point in an effectively compact metric space. We define $m : \{0,1\}^* \to [0,\infty)$ to be left-r.e. relative to z if $m(\tau)$ is left-r.e. relative to Z uniformly for all $Z \in C_z$ and all $\tau \in \{0,1\}^*$. A semimeasure m which is left-r.e. relative to z is said to be universal if for all such semimeasures $\overline{m}$ we have $\exists c\,\forall \tau\,(\overline{m}(\tau) \leq 2^c m(\tau))$. A sequence of sets $U_i \subseteq \{0,1\}^{\mathbb{N}}$, $i = 1, 2, \ldots$ is said to be uniformly Σ^0_1 relative to z if $\{(X, i) \mid X \in U_i\} \subseteq \{0,1\}^{\mathbb{N}} \times \mathbb{N}$ is uniformly Σ^0_1 relative to any $Z \in C_z$. Equivalently, $U_i = \bigcap_{Z \in C_z} U_i^Z$ where U_i^Z

is Σ^0_1 relative to Z uniformly for all $Z \in C_z$ and all i. We define $\mathrm{KA}^z(\tau) = \sup\{\mathrm{KA}^Z(\tau) \mid Z \in C_z\}$.

Lemma 4.6. Let z be a point in an effectively compact metric space. Then $\mathrm{KA}^z = -\log_2 m^z$ where m^z is a universal left-r.e. semimeasure relative to z.

Proof. For all $Z \in \{0,1\}^{\mathbb{N}}$ let $m^Z = \sum_i 2^{-i} m_i^Z$ where m_i^Z, $i = 1, 2, \ldots$ is a uniform enumeration of all left r.e. semimeasures relative to Z. Clearly m^Z is a universal left-r.e. semimeasure relative to Z, so we may safely assume that $\mathrm{KA}^Z = -\log_2 m^Z$. Define $m^z = \inf\{m^Z \mid Z \in C_z\}$ and $m_i^z = \inf\{m_i^Z \mid Z \in C_z\}$. Clearly $\mathrm{KA}^z = -\log_2 m^z$. Using compactness of C_z, it is straightforward to show that m^z and m_i^z are left-r.e. semimeasures relative to z. Note that m_i^z, $i = 1, 2, \ldots$ is a uniform enumeration of all left-r.e. semimeasures relative to z, hence $\overline{m}^z = \sum_i 2^{-i} m_i^z$ is a universal left-r.e. semimeasure relative to z. Moreover, we clearly have $m^z \geq \overline{m}^z$, hence m^z is likewise universal. □

Definition 4.7. As is well known, the set of Borel probability measures on $\{0,1\}^{\mathbb{N}}$ with the Prokhorov metric is an effectively compact metric space. (See for instance [23, Lemma 3.3.8] or [26, §2.4].) Let μ be such a measure, and let z be a point in an effectively compact metric space. Then, the ordered pair μ, z is again a point in such a space. A test for μ-randomness relative to z is sequence of sets $U_i \subseteq \{0,1\}^{\mathbb{N}}$, $i = 1, 2, \ldots$ which is uniformly Σ^0_1 relative to the pair μ, z and such that $\mu(U_i) \leq 2^{-i}$ for all i. We define $X \in \{0,1\}^{\mathbb{N}}$ to be μ-random relative to z if $X \notin \bigcap_i U_i$ for all such tests.

Theorem 4.8. Let z be a point in an effectively compact metric space. Let μ be a Borel probability measure on $\{0,1\}^{\mathbb{N}}$. Then $X \in \{0,1\}^{\mathbb{N}}$ is μ-random relative to z if and only if X is μ-random relative to some $Z \in C_z$.

Proof. We follow the idea of [23, Lemma 3.3.31, Theorem 3.3.33]. Let $U_i^{M,Z}$, $i = 1, 2, \ldots$ be a universal test for μ-randomness relative to $M \oplus Z$ uniformly for all $M \in C_\mu$ and $Z \in \{0,1\}^{\mathbb{N}}$. Then $U_i^{\mu,Z} = \bigcap_{M \in C_\mu} U_i^{M,Z}$, $i = 1, 2, \ldots$ is a universal test for μ-randomness relative to Z uniformly for all $Z \in \{0,1\}^{\mathbb{N}}$. And in turn, $U_i^{\mu,z} = \bigcap_{Z \in C_z} U_i^{\mu,Z}$, $i = 1, 2, \ldots$ is a universal test for μ-randomness relative to z. The latter statement easily implies our theorem. □

Corollary 4.9. Let μ be a Borel probability measure on $\{0,1\}^{\mathbb{N}}$. Then X is μ-random if and only if X is μ-random relative to M for some code M of μ.

Proof. As a special case of Theorem 4.8 we have: X is μ-random relative to μ if and only if X is μ-random relative to M for some $M \in C_\mu$. This statement is equivalent to our corollary. □

Theorem 4.10. Let z be a point in an effectively compact metric space. Let μ be a Borel probability measure on $\{0,1\}^{\mathbb{N}}$. Then $X \in \{0,1\}^{\mathbb{N}}$ is μ-random relative to z if and only if $\exists c \forall n\, (\mathrm{KA}^{\mu,z}(X{\restriction}n) \geq -\log_2 \mu([\![X{\restriction}n]\!]) - c)$.

Proof. For the "if" direction, suppose X is not μ-random relative to z, say $X \in \bigcap_i U_i$ where U_i is a test for μ-randomness relative to z. Define $\overline{m}^{\mu,z}(\tau) = \sum_i 2^i \mu(U_{2i} \cap [\![\tau]\!])$ and note that $\overline{m}^{\mu,z}$ is a left-r.e. semimeasure relative to the pair μ, z. By Lemma 4.6 with the pair μ, z in place of z, let $m^{\mu,z}$ be a universal left-r.e. semimeasure relative to μ, z. Let c be a constant such that $\forall \tau\, (\overline{m}^{\mu,z}(\tau) \leq 2^c m^{\mu,z}(\tau))$. It follows that $\forall i\, \forall \tau\, (2^i \mu(U_{2i} \cap [\![\tau]\!]) \leq 2^c m^{\mu,z}(\tau))$. But $X \in \bigcap_i U_{2i}$, hence $\forall i\, \exists n\, ([\![X{\upharpoonright}n]\!] \subseteq U_{2i})$, hence $\forall i\, \exists n\, (2^i \mu([\![X{\upharpoonright}n]\!]) \leq 2^c m^{\mu,z}(X{\upharpoonright}n))$, hence $\forall i\, \exists n\, (\mathrm{KA}^{\mu,z}(X{\upharpoonright}n) \leq -\log_2 \mu([\![X{\upharpoonright}n]\!]) + c - i)$.

For the "only if" direction, suppose X is μ-random relative to z. For each i let $S_i = \{\tau \mid \mathrm{KA}^{\mu,z}(\tau) < -\log_2 \mu([\![\tau]\!]) - i\}$. Clearly S_i is uniformly μ-recursively enumerable relative to μ, z. As in [13, Definition 2.3] let $\widehat{S}_i$ be the set of minimal elements of S_i. Since $[\![S_i]\!] = [\![\widehat{S}_i]\!]$ and $\widehat{S}_i$ is prefix-free, we have $\mu([\![S_i]\!]) = \mu([\![\widehat{S}_i]\!]) = \sum_{\tau \in \widehat{S}_i} \mu([\![\tau]\!]) = \sum_{\tau \in \widehat{S}_i} 2^{\log_2 \mu([\![\tau]\!])} \leq \sum_{\tau \in \widehat{S}_i} 2^{-\mathrm{KA}^{\mu,z}(\tau) - i} \leq 2^{-i}$. Since X is μ-random relative to z, it follows that $X \notin \bigcap_i [\![S_i]\!]$, hence $\exists i\, \forall n\, (\mathrm{KA}^{\mu,z}(X{\upharpoonright}n) \geq -\log_2 \mu([\![X{\upharpoonright}n]\!]) - i)$. ☐

Remark 4.11. Theorem 4.10 may be viewed as a coding-free variant of [14, Theorem 6.2.1]. Note that Theorem 4.10 involves a priori complexity, KA, rather than prefix-free complexity, KP. We do not know whether Theorem 4.10 holds for KP instead of KA. The original, non-coding-free theorem in [14, §6.2] was stated for KP and it also holds for KA.

In order to prove a version of Van Lambalgen's Theorem for μ-randomness, we first generalize a well known lemma concerning Σ^0_1 subsets of $\{0,1\}^{\mathbb{N}}$.

Lemma 4.12. Let z be a point in an effectively compact metric space. Let $U \subseteq \{0,1\}^{\mathbb{N}}$ be Σ^0_1 relative to z. Given a Borel probability measure μ on $\{0,1\}^{\mathbb{N}}$ and a real number r, we can effectively find a set $U[\mu, r] \subseteq \{0,1\}^{\mathbb{N}}$ with the following properties.

1. $U[\mu, r] \subseteq U$ and $\mu(U[\mu, r]) \leq r$.
2. $U[\mu, r] = U$ provided $\mu(U) < r$.
3. $U[\mu, r]$ is uniformly Σ^0_1 relative to z, μ, r.

Proof. By Definition 4.5 we have $U = \bigcap_{Z \in C_z} U^Z$ where U^Z is Σ^0_1 relative to Z uniformly for all $Z \in \{0,1\}^{\mathbb{N}}$. For each $s \in \mathbb{N}$ let U^Z_s be the part of U^Z which is enumerated prior to stage s. Let $U^Z[\mu, r]$ be the union of U^Z_s for all s such that $\mu(U^Z_s) < r$. Clearly $U^Z[\mu, r] \subseteq U^Z$, and $\mu(U^Z[\mu, r]) \leq r$, and $U^Z[\mu, r] = U^Z$ provided $\mu(U^Z) < r$, and $U^Z[\mu, r]$ is uniformly Σ^0_1 relative to Z, μ, r. It follows that $U[\mu, r] = \bigcap_{Z \in C_z} U^Z[\mu, r]$ has the desired properties. ☐

Definition 4.13. If μ and ν are Borel probability measures on $\{0,1\}^{\mathbb{N}}$, let $\mu \times \nu$ be the product measure on $\{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$. We identify $\{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$ with $\{0,1\}^{\mathbb{N}}$ via the standard pairing function $(X, Y) \mapsto X \oplus Y$ where $(X \oplus Y)(2n) = X(n)$ and $(X \oplus Y)(2n+1) = Y(n)$ for all $n \in \mathbb{N}$. Thus $\mu \times \nu$ is again a Borel probability measure on $\{0,1\}^{\mathbb{N}}$.

Theorem 4.14. Let μ and ν be Borel probability measures on $\{0,1\}^{\mathbb{N}}$. For $X, Y \in \{0,1\}^{\mathbb{N}}$ the following are equivalent.

1. $X \oplus Y$ is $\mu \times \nu$-random.

2. X is μ-random relative to ν and Y is ν-random relative to the pair μ, X.

Proof. We imitate the standard proof of Van Lambalgen's Theorem. See for instance [33, Theorem 3.6]. To prove $1 \Rightarrow 2$, suppose Y is not ν-random relative to the pair μ, X, say $Y \in \bigcap_i V_i$ where $\nu(V_i) < 2^{-i}$ and V_i is uniformly Σ^0_1 relative to μ, X. Using the notation of Lemma 4.12, let $W_i = \{\overline{X} \oplus \overline{Y} \mid \overline{Y} \in V_i[\nu, 2^{-i}]\}$. Then $X \oplus Y \in \bigcap_i W_i$ and $(\mu \times \nu)(W_i) \leq 2^{-i}$ and W_i is uniformly Σ^0_1 relative to $\mu \times \nu$, contradicting 1. To prove $2 \Rightarrow 1$, suppose $X \oplus Y$ is not $\mu \times \nu$-random, say $X \oplus Y \in \bigcap_i W_i$ where $(\mu \times \nu)(W_i) \leq 2^{-i}$ and W_i is uniformly Σ^0_1 relative to $\mu \times \nu$. Let $U_i = \{\overline{X} \mid \nu(V_i^{\overline{X}}) > 2^{-i}\}$ where $V_i^{\overline{X}} = \{\overline{Y} \mid \overline{X} \oplus \overline{Y} \in W_{2i}\}$. Note that U_i is uniformly Σ^0_1 relative to μ, ν. Moreover $\mu(U_i) \leq 2^{-i}$, because otherwise we would have $(\mu \times \nu)(W_{2i}) \geq \mu(U_i) \cdot 2^{-i} > 2^{-i} \cdot 2^{-i} = 2^{-2i}$, a contradiction. Let $\widetilde{U}_j = \bigcup_{i=j}^{\infty} U_i$ and note that $\mu(\widetilde{U}_j) \leq 2^{-j-1}$ and $\widetilde{U}_j$ is again uniformly Σ^0_1 relative to μ, ν. Hence $X \notin \bigcap_j \widetilde{U}_j$, so for all but finitely many i we have $X \notin U_i$, i.e., $\nu(V_i^X) \leq 2^{-i}$. Moreover $Y \in \bigcap_i V_i^X$ and V_i^X is uniformly Σ^0_1 relative to the triple X, μ, ν, contradicting 2. This completes the proof. □

Corollary 4.15. Let μ and ν be Borel probability measures on $\{0,1\}^{\mathbb{N}}$. Suppose X is μ-random relative to ν and Y is ν-random relative to the pair μ, X. Then X is μ-random relative to the pair ν, Y.

Proof. Apply Theorem 4.14 twice. □

Remark 4.16. Theorem 4.14 seems to be "folklore," i.e., well known but not in the literature. The special case $\mu = \nu$ has appeared as [7, Theorem 1.7].

5 A product theorem for strong f-randomness

The purpose of this section is to prove a product theorem for strong f-randomness. See Theorem 5.8 below. We first prove a generalization of the Effective Capacitability Theorem from [25].

Definition 5.1 (compare [25, §3.3])**.** For $f : \{0,1\}^* \to (-\infty, \infty)$ we say that $X \in \{0,1\}^{\mathbb{N}}$ is <u>effectively f-capacitable</u> if X is μ-random for some Borel probability measure μ on $\{0,1\}^{\mathbb{N}}$ such that $\exists c\, \forall \tau\, (\mu([\![\tau]\!]) \leq 2^{c - f(\tau)})$.

Definition 5.2 ([13, §8], compare [25, §2.3])**.** For $f : \{0,1\}^* \to (-\infty, \infty)$ we say that f is <u>convex</u> if $\mathrm{wt}_f(\tau) \leq \mathrm{wt}_f(\tau^\frown\langle 0\rangle) + \mathrm{wt}_f(\tau^\frown\langle 1\rangle)$ for all $\tau \in \{0,1\}^*$. Here we are writing $\mathrm{wt}_f(\tau) = 2^{-f(\tau)}$.

Theorem 5.3 (Effective Capacitability Theorem, compare [25, Theorem 14, Corollary 23])**.** Let $f : \{0,1\}^* \to (-\infty, \infty)$ be computable and convex. For $X \in \{0,1\}^{\mathbb{N}}$ the following are equivalent.

1. X is strongly f-random.

2. X is effectively f-capacitable.

In order to prove Theorem 5.3, we use the following definitions and lemma.

Definition 5.4. Recall from [25, §3.3] that a semimeasure is a function $m : \{0,1\}^* \to [0,1]$ such that $m(\tau) \geq m(\tau^\frown\langle 0\rangle) + m(\tau^\frown\langle 1\rangle)$ for all $\tau \in \{0,1\}^*$. Dually, we define a submeasure to be a function $m : \{0,1\}^* \to [0,\infty]$ such that $m(\tau) \leq m(\tau^\frown\langle 0\rangle) + m(\tau^\frown\langle 1\rangle)$ for all $\tau \in \{0,1\}^*$. Note that f is convex if and only if wt_f is a submeasure.

Definition 5.5. A function $m : \{0,1\}^* \to [0,\infty)$ is said to be left-r.e. (respectively right-r.e.) if the real numbers $m(\tau)$ are uniformly left recursively enumerable (respectively right recursively enumerable) for all $\tau \in \{0,1\}^*$.

Lemma 5.6. Let m_1 be a semimeasure and let m_2 be a submeasure.

1. If $m_1(\langle\rangle) \leq 1 \leq m_2(\langle\rangle)$ and $\forall\tau\,(m_1(\tau) \leq m_2(\tau))$, then we can find a Borel probability measure μ on $\{0,1\}^{\mathbb{N}}$ such that $\forall\tau\,(m_1(\tau) \leq \mu([\![\tau]\!]) \leq m_2(\tau))$.

2. If m_1 is left-r.e. and m_2 is right-r.e., then the set of all such Borel probability measures is effectively closed.

Proof. The proof of part 1 is based on the following observation:

Given $a_0 + a_1 \leq b \leq c_0 + c_1$ where $a_0 \leq c_0$ and $a_1 \leq c_1$, we can find b_0 and b_1 such that $b = b_0 + b_1$ and $a_0 \leq b_0 \leq c_0$ and $a_1 \leq b_1 \leq c_1$. (1)

To prove (1), define $h(t) = a_0 + t(c_0 - a_0) + a_1 + t(c_1 - a_1)$ and note that $h(0) = a_0 + a_1$ and $h(1) = c_0 + c_1$. By the Intermediate Value Theorem, there exists t such that $0 \leq t \leq 1$ and $h(t) = b$. Then $b_0 = a_0 + t(c_0 - a_0)$ and $b_1 = a_1 + t(c_1 - a_1)$ are as desired.

To prove part 1, define $\mu([\![\tau]\!])$ by induction on the length of τ beginning with $\mu([\![\langle\rangle]\!]) = 1$. Assume inductively that $\mu([\![\tau]\!])$ has been defined such that $m_1(\tau) \leq \mu([\![\tau]\!]) \leq m_2(\tau)$. Then

$$m_1(\tau^\frown\langle 0\rangle) + m_1(\tau^\frown\langle 1\rangle) \leq \mu([\![\tau]\!]) \leq m_2(\tau^\frown\langle 0\rangle) + m_2(\tau^\frown\langle 1\rangle)$$

so by (1) we can find $\mu([\![\tau^\frown\langle 0\rangle]\!])$ and $\mu([\![\tau^\frown\langle 1\rangle]\!])$ such that

$$\begin{aligned} \mu([\![\tau]\!]) &= \mu([\![\tau^\frown\langle 0\rangle]\!]) + \mu([\![\tau^\frown\langle 1\rangle]\!]) \\ \text{and } m_1(\tau^\frown\langle 0\rangle) &\leq \mu([\![\tau^\frown\langle 0\rangle]\!]) \leq m_2(\tau^\frown\langle 0\rangle) \\ \text{and } m_1(\tau^\frown\langle 1\rangle) &\leq \mu([\![\tau^\frown\langle 1\rangle]\!]) \leq m_2(\tau^\frown\langle 1\rangle). \end{aligned}$$

This proves part 1 of our lemma, and part 2 is obvious. □

We are now ready to prove Theorem 5.3.

Proof of Theorem 5.3. To prove $1 \Rightarrow 2$, assume that X is strongly f-random. By the Kučera/Gács Theorem, let Y be Martin-Löf random such that $X \leq_{\mathrm{T}} Y$. Let Φ be a partial recursive functional such that $X = \Phi^Y$. For each $\tau \in \{0,1\}^*$ let $V_\tau = \{\overline{Y} \mid \tau \subset \Phi^{\overline{Y}}\}$. Note that the indexed family V_τ, $\tau \in \{0,1\}^*$ is a *Levin system* in the sense of [13, Definition 4.1]. By [13, Lemma 4.2] let c be a rational number such that $\forall n\,(\lambda(V_{X\restriction n}) < 2^{c-f(X\restriction n)})$. By [13, Lemma 4.3] let $\widetilde{V}_\tau \subseteq V_\tau$, $\tau \in \{0,1\}^*$ be a Levin system such that $\forall \tau\,(\lambda(\widetilde{V}_\tau) \leq 2^{c-f(\tau)})$ and $\forall n\,(\widetilde{V}_{X\restriction n} = V_{X\restriction n})$. Note that $\lambda(\widetilde{V}_{\langle\rangle}) = 1$ and $\tau \mapsto \lambda(\widetilde{V}_\tau)$ is a left-r.e. semimeasure, and by convexity $\tau \mapsto 2^{c-f(\tau)}$ is a right-r.e. submeasure. Let P be the set of Borel probability measures μ such that $\lambda(\widetilde{V}_\tau) \leq \mu(\llbracket\tau\rrbracket) \leq 2^{c-f(\tau)}$ for all τ. By Lemma 5.6 P is nonempty and effectively closed, so by the Randomness Preservation Basis Theorem plus Corollary 4.9, let $\mu \in P$ be such that Y is Martin-Löf random relative to μ, i.e., λ-random relative to μ. We claim that X is μ-random. Otherwise, suppose $X \in \bigcap_i U_i$ where U_i is uniformly Σ^0_1 relative to μ and $\mu(U_i) \leq 2^{-i}$ for all i. Let $W_i = \bigcup_{\tau \in S_i} \widetilde{V}_\tau$ where $S_i = \{\tau \mid \llbracket\tau\rrbracket \subseteq U_i\}$. As in [13, Definition 2.3] let $\widehat{S}_i$ be the set of minimal elements of S_i and note that $\widehat{S}_i$ is prefix-free and $\llbracket\widehat{S}_i\rrbracket = \llbracket S_i\rrbracket = U_i$. Note also that S_i and consequently W_i are uniformly Σ^0_1 relative to μ. For each i we have $\exists n\,(X\restriction n \in S_i)$ and for this n we have $Y \in V_{X\restriction n} = \widetilde{V}_{X\restriction n}$, hence $Y \in W_i$, so $Y \in \bigcap_i W_i$. But for each i we also have $\lambda(W_i) = \sum_{\tau\in\widehat{S}_i} \lambda(\widetilde{V}_\tau) \leq \sum_{\tau\in\widehat{S}_i} \mu(\llbracket\tau\rrbracket) = \mu(U_i) \leq 2^{-i}$ contradicting the fact that Y is λ-random relative to μ. This proves our claim. Our claim implies that X is effectively f-capacitable, and this proves $1 \Rightarrow 2$.

To prove $2 \Rightarrow 1$, assume that X is not strongly f-random. By [13, Theorem 8.16] it follows that X is not vehemently f-random, say $X \in \bigcap_i U_i$ where U_i is uniformly Σ^0_1 and $\mathrm{vwt}_f(U_i) \leq 2^{-i}$ for all i. By [13, Lemma 8.15] let $S_i \subseteq \{0,1\}^*$ be uniformly r.e. such that $U_i \subseteq \llbracket S_i\rrbracket$ and $\mathrm{pwt}_f(S_i) \leq 2^{-i+1}$ for all i. To show that X is not effectively f-capacitable, let μ be a Borel probability measure on $\{0,1\}^{\mathbb{N}}$ such that $\exists c\,\forall\tau\,(\mu(\llbracket\tau\rrbracket) \leq 2^{c-f(\tau)})$. Then for all i we have $\mu(U_i) \leq \mu(\llbracket S_i\rrbracket) = \mu(\llbracket\widehat{S}_i\rrbracket) = \sum_{\tau\in\widehat{S}_i} \mu(\llbracket\tau\rrbracket) \leq \sum_{\tau\in\widehat{S}_i} 2^{c-f(\tau)} \leq 2^c\mathrm{pwt}_f(S_i) \leq 2^{c-i+1}$ so X is not μ-random. This proves $2 \Rightarrow 1$ and thus Theorem 5.3. ∎

Remark 5.7. The length-invariant case of Theorem 5.3 is due to Reimann and Kjos-Hanssen (see [25, Theorem 14, Corollary 23]). Our proof of Theorem 5.3 is similar to Reimann's proof [25] of the length-invariant case.

Our new result is as follows.

Theorem 5.8. Let $f : \{0,1\}^* \to (-\infty,\infty)$ be computable and convex. If X is strongly f-random, and if Y is Martin-Löf random relative to X, then X is strongly f-random relative to Y.

Proof. Let Q be the set of Borel probability measures μ on $\{0,1\}^{\mathbb{N}}$ such that X is μ-random and $\exists c\,\forall\tau\,(\mu(\llbracket\tau\rrbracket) \leq 2^{c-f(\tau)})$. By Theorem 5.3 Q is nonempty. Clearly Q is Σ^0_2 relative to X, so by the Randomness Preservation Basis Theorem relative to X, let $\mu \in Q$ be such that Y is Martin-Löf random relative to μ, X. Since λ is computable, it follows that Y is λ-random relative to μ, X. But then, by Corollary 4.15, X is μ-random relative to Y. Hence, by Theorem 5.3 relativized to Y, X is strongly f-random relative to Y. $\square$

Remark 5.9 (compare [13, Remark 4.5])**.** In Theorem 5.8 the assumption "Y is Martin-Löf random" cannot be weakened to "Y is strongly f-random." For example, define $X(n) = Y(2n)$ where Y is Martin-Löf random. Then X is strongly $1/2$-random (indeed Martin-Löf random), and Y is strongly $1/2$-random relative to X, but of course X is not strongly $1/2$-random relative to Y.

Corollary 5.10 (compare [33, Corollary 3.9])**.** Let $f : \{0,1\}^* \to (-\infty, \infty)$ be computable and convex. Assume that X is strongly f-random and $X \leq_{\mathrm{T}} Y$ and Y is Martin-Löf random relative to Z and $Z \geq_{\mathrm{T}} 0'$. Then X is strongly f-random relative to Z.

Proof. Since $Z \geq_{\mathrm{T}} 0'$, we may assume by the Kučera/Gács Theorem that Z is Martin-Löf random. Since Y is Martin-Löf random relative to Z, it follows by Van Lambalgen's Theorem that Z is Martin-Löf random relative to Y. Since $X \leq_{\mathrm{T}} Y$, it follows that Z is Martin-Löf random relative to X. It then follows by Theorem 5.8 that X is strongly f-random relative to Z. $\square$

Remark 5.11. It is known [13, Theorem 4.4] that Corollary 5.10 holds even if we drop two of the assumptions, namely, Turing completeness of Z and convexity of f. We conjecture that Theorem 5.8 holds without the convexity assumption.

6 Complexity relative to a Turing oracle

In this section we obtain a new characterization of complexity relative to a Turing oracle. We also obtain a new proof of some known characterizations of autocomplexity relative to a Turing oracle. See Theorems 6.4 and 6.6 below.

Before proving Theorems 6.4 and 6.6, we note the following alternative characterizations.

Theorem 6.1. Let Z be a Turing oracle, and suppose $X \in \{0,1\}^{\mathbb{N}}$.

1. The following are equivalent.
   (a) X is autocomplex relative to Z.
   (b) There exists $g \in \mathrm{DNR}^Z$ such that $g \leq_{\mathrm{T}}^Z X$, i.e., $g \leq_{\mathrm{T}} X \oplus Z$.
2. The following are pairwise equivalent.
   (a) X is complex relative to Z.
   (b) There exists $g \in \mathrm{DNR}^Z$ such that $g \leq_{\mathrm{tt}}^Z X$, i.e., $g = \Phi^Z(X)$ for some total Z-recursive functional $\Phi^Z : \{0,1\}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$.
   (c) There exists $g \in \mathrm{DNR}^Z$ such that $g \leq_{\mathrm{wtt}}^Z X$, i.e., $g \leq_{\mathrm{T}} X \oplus Z$ with Z-recursively bounded use of X.

Here $\mathrm{DNR}^Z = \{g \in \mathbb{N}^{\mathbb{N}} \mid g \text{ is diagonally nonrecursive relative to } Z\}$.

Proof. This is just [18, Theorem 2.3] relativized to Z. $\square$

We now begin the proofs of Theorems 6.4 and 6.6.

Definition 6.2. A function $f : \{0,1\}^* \to (-\infty,\infty)$ is said to be monotone if $\forall\sigma\,\forall\tau\,(\sigma \subset \tau \Rightarrow f(\sigma) \le f(\tau))$. Note that KA is monotone.

Lemma 6.3. Given a computable function $f : \{0,1\}^* \to (-\infty,\infty)$, we can find a computable, convex, monotone function $\widetilde{f} : \{0,1\}^* \to (-\infty,\infty)$ such that the following holds. For all $X \in \{0,1\}^{\mathbb{N}}$, if X is strongly f-random and $\{f(X{\restriction}n) \mid n \in \mathbb{N}\}$ is unbounded, then the same holds with $\widetilde{f}$ instead of f. Moreover, if f is length-invariant, then so is $\widetilde{f}$.

Proof. Let F be the smallest monotone function which majorizes f, i.e., $F(\tau) = \max\{f(\sigma) \mid \sigma \subseteq \tau\}$. Define $\widetilde{f}(\tau)$ by recursion on $|\tau|$ as follows: $\widetilde{f}(\langle\rangle) = F(\langle\rangle)$, $\widetilde{f}(\tau^\frown\langle i\rangle) = \min(F(\tau^\frown\langle i\rangle), \widetilde{f}(\tau)+1)$ for $i = 0,1$. Obviously $\widetilde{f}$ is computable and monotone and $\widetilde{f}(\tau) \le F(\tau)$ for all τ. Also, $\widetilde{f}$ is convex, because $\widetilde{f}(\tau^\frown\langle i\rangle) \le \widetilde{f}(\tau) + 1$ for $i = 0,1$. If f is length-invariant then so is F, hence so is $\widetilde{f}$. Suppose now that X is strongly f-random. Recall from [13, §2] that strong f-randomness is equivalent to strong f-complexity. Thus X is strongly f-complex, i.e., $\mathrm{KA}(X{\restriction}n) \ge^+ f(X{\restriction}n)$ for all n. Since KA is monotone, it follows by the definition of F that X is strongly F-complex, hence strongly $\widetilde{f}$-complex, hence strongly $\widetilde{f}$-random. If $\{f(X{\restriction}n) \mid n \in \mathbb{N}\}$ is unbounded, so is $\{F(X{\restriction}n) \mid n \in \mathbb{N}\}$, and this together with the monotonicity of F implies that $\{\widetilde{f}(X{\restriction}n) \mid n \in \mathbb{N}\}$ is unbounded. This completes the proof. $\square$

Theorem 6.4. For $X \in \{0,1\}^{\mathbb{N}}$ the following are pairwise equivalent.

1. X is autocomplex relative to some oracle Z.
2. X is autocomplex relative to some PA-complete oracle Z.
3. X is μ-random for some Borel probability measure μ on $\{0,1\}^{\mathbb{N}}$ such that $\mu(\{X\}) = 0$.
4. X is noncomputable.

Proof. The equivalence $1 \Leftrightarrow 2$ is obtained by relativizing [13, Theorem 7.5.3] to Z. To prove $1 \Rightarrow 3$, assume that X is autocomplex relative to Z. By Remark 1.13 and Lemma 6.3 relativized to Z, X is strongly f-random relative to Z for some convex $f \le_{\mathrm{T}} Z$ such that $\{f(X{\restriction}n) \mid n \in \mathbb{N}\}$ is unbounded. But then, by Theorem 5.3 relative to Z, X is μ-random relative to Z for some Borel probability measure μ such that $\exists c\,\forall\tau\,(\mu([\![\tau]\!]) \le 2^{c-f(\tau)})$. For this c we have $\forall n\,(\mu([\![X{\restriction}n]\!]) \le 2^{c-f(X{\restriction}n)})$, hence $\mu(\{X\}) = 0$, thus proving $1 \Rightarrow 3$. The implication $3 \Rightarrow 4$ is trivial because $X \in \bigcap_n [\![X{\restriction}n]\!]$. To prove $4 \Rightarrow 1$, assume that X is noncomputable. By a famous theorem known as the *Posner/Robinson Theorem* (see [24] or [35, Lemma 3.4.1]), let Z be an oracle such that $X \oplus Z \equiv_{\mathrm{T}} Z'$. In particular X is autocomplex relative to Z. $\square$

Lemma 6.5. Let μ be a Borel probability measure μ on $\{0,1\}^{\mathbb{N}}$. Suppose X is μ-random and $\mu(\{X\}) = 0$. Then, we can find a μ-computable, convex, monotone function $f : \{0,1\}^* \to [0,\infty]$ such that $\{f(X{\upharpoonright}n) \mid n \in \mathbb{N}\}$ is unbounded and X is strongly f-random relative to (some code for) μ. If μ is continuous, we can choose f to be length-invariant.

Proof. By Corollary 4.9 let $M \in \{0,1\}^{\mathbb{N}}$ be a code for μ such that X is μ-random relative to M. Let $f(\tau) = -\log_2 \mu(\llbracket\tau\rrbracket)$. Clearly f is μ-computable, convex, and monotone. Since $\mu(\{X\}) = 0$ we have $\lim_n \mu(\llbracket X{\upharpoonright}n\rrbracket) = 0$, hence $\{f(X{\upharpoonright}n) \mid n \in \mathbb{N}\}$ is unbounded. By Theorem 4.10 we have $\mathrm{KA}^M(X{\upharpoonright}n) \geq^+ -\log_2 \mu(\llbracket X{\upharpoonright}n\rrbracket)$ for all n, i.e., X is strongly f-complex relative to M, hence by [13, §2] X is strongly f-random relative to M. If μ is continuous, we can repeat the above argument with a length-invariant f, namely $f(\tau) = \min\{-\log_2 \mu(\llbracket\sigma\rrbracket) \mid |\sigma| = |\tau|\}$. ◻

Theorem 6.6. For $X \in \{0,1\}^{\mathbb{N}}$ the following are pairwise equivalent.

1. X is complex relative to some oracle Z.
2. X is complex relative to some PA-complete oracle Z.
3. X is μ-random for some continuous Borel probability measure μ on $\{0,1\}^{\mathbb{N}}$.

Proof. The equivalence $1 \Leftrightarrow 2$ is obtained by relativizating [13, Theorem 7.5.4] to Z. To prove $1 \Rightarrow 3$, assume that X is complex relative to Z. By Remark 1.13 and Lemma 6.3 relative to Z, X is strongly f-random relative to Z for some length-invariant, convex, unbounded $f \leq_{\mathrm{T}} Z$. But then, by Theorem 5.3 relative to Z, X is μ-random relative to Z for some Borel probability measure μ such that $\exists c\, \forall \tau\, (\mu(\llbracket\tau\rrbracket) \leq 2^{c-f(\tau)})$. Since f is length-invariant and unbounded, μ is continuous, thus proving $1 \Rightarrow 3$. To prove $3 \Rightarrow 1$, assume that X is μ-random for some continuous μ. By Lemma 6.5 there is an oracle Z such that X is strongly f-random relative to Z for some length-invariant, unbounded $f \leq_{\mathrm{T}} Z$. Hence, by Remark 1.13 relative to Z, X is complex relative to Z. ◻

Remark 6.7. Theorem 6.4 was essentially already known, being a combination of known results from [13, 18, 26]. However, Theorem 6.6 appears to be new. In connection with Theorem 6.6, note that the class $\{X \in \{0,1\}^{\mathbb{N}} \mid X \text{ is } \mu\text{-random for some continuous Borel probability measure } \mu \text{ on } \{0,1\}^{\mathbb{N}}\}$ has been studied extensively in [26].

We finish by presenting another product theorem.

Theorem 6.8. Suppose $X, Y \in \{0,1\}^{\mathbb{N}}$.

1. If X is autocomplex, and if Y is Martin-Löf random relative to X, then X is autocomplex relative to Y.
2. If X is complex, and if Y is Martin-Löf random relative to X, then X is complex relative to Y.

Proof. If X is autocomplex, it follows by Remark 1.13 that X is strongly f-random for some computable f such that $\{f(X{\upharpoonright}n) \mid n \in \mathbb{N}\}$ is unbounded. By Lemma 6.3 we may safely assume that f is convex. But then, if Y is Martin-Löf random relative to X, Theorem 5.8 implies that X is strongly f-random relative to Y, hence by Remark 1.13 X is autocomplex relative to Y. This proves part 1 of our theorem. The proof of part 2 is similar. □

Remark 6.9. Theorem 6.8 and similar results such as [13, Theorems 7.5.1, 7.5.2] are of a different flavor than other randomness preservation results. This is because complexity relative to an oracle does not necessarily imply complexity or even autocomplexity. For example, let X be nonrecursive such that no $g \in \mathrm{DNR}$ is $\leq_{\mathrm{T}} X$. By Theorem 6.1 X is neither complex nor autocomplex, but by Theorem 6.4 X is autocomplex relative to some oracle. If X is in addition nonhyperarithmetical, it follows by Theorem 6.6 and [26, Theorem 5.9] that X is complex relative to some oracle.

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