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THE BAIRE CATEGORY THEOREM IN WEAK SUBSYSTEMS OF SECOND-ORDER ARITHMETIC

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Abstract. Working within weak subsystems of second-order arithmetic $\mathbf{Z}_2$ we consider two versions of the Baire Category theorem which are not equivalent over the base system RCA_0. We show that one version (B.C.T.I) is provable in RCA_0 while the second version (B.C.T.II) requires a stronger system. We introduce two new subsystems of $\mathbf{Z}_2$, which we call RCA_0^+ and WKL_0^+, and show that RCA_0^+ suffices to prove B.C.T.II. Some model theory of WKL_0^+ and its importance in view of Hilbert's program is discussed, as well as applications of our results to functional analysis.

§0. Introduction. This paper consists of some of the material contained in [2], which is concerned with the development of the basic definitions and theorems of functional analysis within second-order arithmetic, $\mathbf{Z}_2$. Such studies take place within a broader program initiated by Friedman and carried forward by Friedman, Simpson, and others. The goal of this program is to examine the *Main Question*: *Which set existence axioms are needed to prove the theorems of "ordinary mathematics*?" An exposition of the meaning of "ordinary mathematics" can be found in [22, 21, 2]—for the purposes of this paper it suffices to note that the theory of complete separable metric spaces is an example of ordinary mathematics.

The language of second-order arithmetic is a two sorted language with *number variables* $i, j, k, m, n, \ldots$ and *set variables* $X, Y, Z, \ldots$. *Numerical terms* are built up as usual from number variables, constant symbols 0 and 1, and the binary operations of addition $(+)$ and multiplication $(\cdot)$. *Atomic formulas* are $t_1 = t_2$, $t_1 < t_2$, and $t_1 \in X$ where t_1 and t_2 are numerical terms. *Formulas* are built up as usual from atomic formulas by means of propositional connectives $\wedge, \vee, \sim, \rightarrow, \leftrightarrow$, number quantifiers $\forall n$ and $\exists n$, and set quantifiers $\forall X$ and $\exists X$. The formal system $\mathbf{Z}_2$ includes the *ordered semiring axioms* for $\mathbf{N}, +, \cdot, 0, 1, <$ as well as the *induction axiom*

$$(0 \in X \wedge \forall n(n \in X \rightarrow n + 1 \in X)) \rightarrow \forall n(n \in X)$$

and the comprehension scheme

$$\exists X \forall n(n \in X \leftrightarrow \varphi(n)),$$

where $\varphi(n)$ is any formula in which X does not occur freely.

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In the course of studying the Main Question it has been noted that a great deal of ordinary mathematics may actually be done in various weak subsystems of $\mathbf{Z}_2$ [1, 2, 8, 22]. In this paper we are primarily concerned with the following three subsystems:

RCA_0. Here the acronym RCA stands for recursive comprehension axiom. Roughly speaking, the axioms of RCA_0 are only strong enough to prove the existence of recursive sets (though they do not rule out the existence of nonrecursive sets). As weak as this system is, it is strong enough to prove some of the elementary facts about countable algebraic structures [8] and continuous functions of a real variable [21, 22]. The axioms of RCA_0 consist of the ordered semiring axioms together with the schemes of Σ^0_1 induction and Δ^0_1 comprehension.

WKL_0. This system consists of RCA_0 plus a further axiom known as *Weak König's lemma* which states that every infinite $\{0,1\}$-tree has a path. This system is very weak from the viewpoint of mathematical logic in that the first-order part of WKL_0 is the same as that of RCA_0, viz., Σ^0_1 induction (this result is due to Harrington; for a proof see [21]). Furthermore, WKL_0 is conservative over Primitive Recursive Arithmetic (PRA) with respect to Π^0_2 sentences [21]. On the other hand, from the mathematical point of view, WKL_0 is very powerful. It is strong enough to prove a great many theorems of ordinary mathematics which are not recursively true and hence not provable in RCA_0. Included in this category are the Heine-Borel covering lemma [2, 6, 21], the prime ideal theorem for countable commutative rings [8], the maximum principle for continuous functions on a closed bounded interval [21], and the local existence theorem for solutions of ordinary differential equations [22].

The above remarks have important implications in the foundations of mathematics vis á vis Hilbert's program. Tait [24] has made a strong case for the identification of Hilbert's notion of finitism with the formal system PRA and pointed out that the primary concern of this finitism is the provability of certain Π^0_1 sentences. Thus a full realization of Hilbert's desire to justify the use of infinitistic mathematics would consist of developing a formal infinitistic system whose consistency was provable in PRA, as it would then follow that any Π^0_1 sentence provable in this infinitistic system was in fact provable in PRA, i.e., finitistically. The infinite objects of the larger system would then be justifiable as devices to be used to prove theorems about noninfinite objects and have these results be finitistically acceptable. Of course Gödel's work dashed any hopes for such a full realization of Hilbert's program, but *partial* realizations are made possible by considering, not systems whose consistency is provable in PRA, but ones which are conservative over PRA with respect to Π^0_1 sentences. The fact noted above that WKL_0 is conservative over PRA with respect to Π^0_2 sentences thus gives a slightly stronger result and indicates that the theorems of ordinary mathematics mentioned above provide partial realizations of Hilbert's Program in this sense. For a fuller exposition of this theme see [19].

ACA_0. The axioms of ACA_0 are the same as those of $\mathbf{Z}_2$ expect that the comprehension scheme is restricted to arithmetical formulas $\varphi(n)$ in which X does not occur freely. ACA_0 permits a smooth theory of sequential convergence [5, 6, 22] and isolates the same portion of mathematical practice which was identified as "predicative analysis" by Weyl in *Das Kontinuum* [25]. For more details on all of these systems see Simpson [21].

Investigations into the Main Question have also revealed the following *Main Theme*: *very often, if a theorem of ordinary mathematics is proved from the "right" set existence axioms, the statement of that theorem will be provably equivalent to those axioms over some weak base system* (for us this is RCA_0). This theme is known as *Reverse Mathematics* and is exhibited in such works as [2, 3, 7, 8, 9, 21, 26]. This type of "reversal", proving that a set of axioms follows from the statement of the theorem, together with the more usual proof of the theorem from the same set of axioms provides the precise knowledge that these axioms are in some sense necessary to prove a theorem of ordinary mathematics. In such a case we have a very complete answer to the Main Question.

In [2] our primary aim was to examine some of the fundamental theorems of functional analysis on separable spaces in the context of Reverse Mathematics. Among these theorems were the Banach-Steinhaus theorem and the Open Mapping and Closed Graph theorems. In standard texts on real analysis these theorems are usually proved using the Baire Category theorem. In the setting of weak subsystems of $\mathbf{Z}_2$, however, the version of the Baire Category theorem needed to prove the Banach-Steinhaus theorem and the version needed to prove the Open Mapping and Closed Graph theorems are not the same. This result is due to two notions of a closed subset of a complete separable metric space which are not equivalent in weak subsystems. These notions are discussed in detail in [2] and [3] and are summarized in §1 below, along with the necessary technical definitions and results about complete separable metric spaces and their topology in weak subsystems of $\mathbf{Z}_2$. In §2 we present the two versions of the Baire Category theorem referred to above and show that the first version is easily proved in RCA_0. In §3 we consider the axiomatic strength needed to prove the second version of the theorem. In §4 we introduce two new subsystems of $\mathbf{Z}_2$, which we call RCA_0^+ and WKL_0^+, and show that the system RCA_0^+ suffices to prove the second version of the Baire Category theorem. In §5 we consider the Open Mapping and Closed Graph theorems. Finally, in §6, we consider some model theory of WKL_0^+ and show that WKL_0^+ is conservative over PRA with respect to Π_2^0 sentences.

§1. Metric spaces. Within RCA_0 we define a (code for a) *complete separable metric space* to consist of a set $A \subset \mathbf{N}$ together with a function $d: A \times A \to \mathbf{R}$ such that for all $a, b, c \in A$:

(i) $d(a,a) = 0$,
(ii) $d(a,b) = d(b,a)$,
(iii) $d(a,c) \leq d(a,b) + d(b,c)$.

Now let (A,d) be a code for a complete separable metric space, as above. We define, again within RCA_0, a *point in the completion* $\hat{A}$ to be a function $f: \mathbf{N} \to A$ such that

$$\forall n \forall i [d(f(n), f(n+i)) < 2^{-n}].$$

The idea here is that (A,d) is a code for the complete separable metric space $\hat{A}$ consisting of all such points. For example, $\mathbf{R} = \hat{\mathbf{Q}}$ under the usual psuedometric. Of course $\hat{A}$ does not formally exist within RCA_0. A point $f: \mathbf{N} \to A$ will be denoted by $x = \langle a_n : n \in \mathbf{N} \rangle$ where $a_n = f(n)$. Two points $x = \langle a_n : n \in \mathbf{N} \rangle$ and $y = \langle b_n : n \in \mathbf{N} \rangle$ are said to be *equal* if $\forall n [d(a_n, b_n) < 2^{-n+1}]$. A *sequence* of points of $\hat{A}$ is a function $f: \mathbf{N} \to \hat{A}$ and is denoted by $\langle x_n : n \in \mathbf{N} \rangle$ where $x_n = f(n)$. We extend the psuedometric d on A to a psuedometric $\hat{d}$ on $\hat{A}$ by defining

$$\hat{d}(\langle a_n : n \in \mathbf{N} \rangle, \langle b_n : n \in \mathbf{N} \rangle) = \langle c_{n,n} : n \in \mathbf{N} \rangle,$$

where

$$\langle c_{n,k} : k \in \mathbf{N} \rangle = d(a_{n+3}, b_{n+3}).$$

We will sometimes use $d(x, y)_n$ to denote $c_{n,n}$. Where no confusion will result, d will be used to denote both d and $\hat{d}$. We embed A into $\hat{A}$ by identifying the element $a \in A$ with the point $x_a \in \hat{A}$ defined by $x_a = \langle a : n \in \mathbf{N} \rangle$. Thus, under this embedding, A is a countable dense subset of $\hat{A}$.

Two examples of complete separable metric spaces we will need later are:

EXAMPLE 1.1. Infinite product spaces. Given an infinite sequence of (codes for) complete separable metric spaces $\hat{A}_i$, $i \in \mathbf{N}$, we can form the infinite product space $\hat{A} = \prod_{i=0}^{\infty} \hat{A}_i$ as follows. For each $i \in \mathbf{N}$, we let c_i be the smallest element of $A_i \subset \mathbf{N}$ (in the usual ordering of $\mathbf{N}$). We define

$$A = \bigcup_{m=0}^{\infty} (A_0 \times \cdots \times A_m) = \{\langle a_i : i \leq m \rangle : m \in \mathbf{N}, a_i \in A_i\}$$

and $d: A \times A \to \mathbf{R}$ by

$$d(\langle a_i : i \leq m \rangle, \langle b_i : i < n \rangle) = \sum_{i=0}^{\infty} \frac{d_i(a'_i, b'_i)}{1 + d_i(a'_i, b'_i)} \cdot \frac{1}{2^i},$$

where

$$a'_i = \begin{cases} a_i & \text{if } i \leq m, \\ c_i & \text{otherwise} \end{cases}$$

and

$$b'_i = \begin{cases} b_i & \text{if } i \leq n, \\ c_i & \text{otherwise.} \end{cases}$$

We can then prove within RCA_0 the following facts: (i) $\hat{A}$ is a complete separable metric space; (ii) the points of $\hat{A}$ can be identified with the sequences $\langle x_i : i \in \mathbf{N} \rangle$, where $x_i \in \hat{A}_i$ for all $i \in \mathbf{N}$; and (iii) under this identification, the metric on $\hat{A}$ is given by

$$d(\langle x_i : i \in \mathbf{N} \rangle, \langle y_i : i \in \mathbf{N} \rangle) = \sum_{i=0}^{\infty} \frac{d_i(x_i, y_i)}{1 + d_i(x_i, y_i)} \cdot \frac{1}{2^i}.$$

These three conditions define the usual textbook construction of the product of a sequence of complete separable metric spaces. Thus we are justified in writing

$$\hat{A} = \prod_{i=0}^{\infty} \hat{A}_i.$$

In particular, we have within RCA_0 the *Cantor space*

$$2^{\mathbf{N}} = \{0,1\}^{\mathbf{N}} = \prod_{i=0}^{\infty} \{0,1\}$$

and the *Baire space*

$$\mathbf{N}^{\mathbf{N}} = \prod_{i=0}^{\infty} \mathbf{N}.$$

The points of the Cantor space and the Baire space can be identified with functions $f\colon \mathbf{N} \to \{0,1\}$ and $f\colon \mathbf{N} \to \mathbf{N}$ respectively.

Within RCA_0 we define a code for an *open ball* in $\hat{A}$, a complete separable metric space with code (A, d), to be an ordered pair (x, ε) where $x \in \hat{A}$ and $\varepsilon \in \mathbf{R}^+$ (the positive reals). We say that a point $y \in \hat{A}$ is an *element* of the ball (x, ε) if $d(x, y) < \varepsilon$. An open ball is a *basic open set* if it is of the form (a, r) where $a \in A$ and $r \in \mathbf{Q}^+$ (the positive rationals). A code for an *open set* U is a sequence of basic open sets $\langle (a_n, r_n)\colon n \in \mathbf{N} \rangle$. We say that a point $x \in \hat{A}$ *belongs to* U if there is a basic open set $(a, r) \in U$ such that x is an element of (a, r). We will denote an element x of an open ball (y, ε) by writing $x \in (y, \varepsilon)$ and a point x belonging to an open set U by writing $x \in U$.

One natural definition of a closed set is that it is the complement of an open set. Thus we define a (code for a) *closed set* C to be a sequence of basic open sets $\langle (a_n, r_n)\colon n \in \mathbf{N} \rangle$ and say that a point x in a complete separable metric space $\hat{A}$ *belongs to* C if $d(a_n, x) \geq r_n$ for all $n \in \mathbf{N}$. Note that a code for a closed set may also be regarded as a code for the open set which is its complement. It is then easy to prove, over RCA_0, such standard results as the countable union of open sets is open and the countable intersection of closed sets is closed [3]. However, since every closed subset of a complete separable metric space is itself a complete separable metric space, a second natural definition of a closed set is that it is the closure of a countable set of points. We therefore define a (code for a) *separably closed set* $\bar{S}$ to consist of a sequence $S = \langle x_n\colon n \in \mathbf{N} \rangle$ of points from a complete separable metric space $\hat{A}$. We say a point $x \in \hat{A}$ *belongs to* $\bar{S}$ if $\forall r \in \mathbf{Q}^+ \exists n[d(x, x_n) < r]$. We will occasionally write $\overline{\{x_n\colon n \in \mathbf{N}\}}$ for $\bar{S}$. Note that the definition of a separably closed set is equivalent to that of a closed subspace of $\hat{A}$ given in [4]. We define a *separably open set* in a complete separable metric space to be the complement of a separably closed set. Thus a code for a separably open set O is a sequence $\langle x_n\colon n \in \mathbf{N} \rangle$ of points in $\hat{A}$. We say that a point $x \in \hat{A}$ *belongs to* O, written $x \in O$, if $x \notin \overline{\{x_n\colon n \in \mathbf{N}\}}$.

In the context of weak subsystems of second-order arithmetic there is an important distinction between these two definitions of closed set: relatively strong axioms are required to prove their equivalence. Specifically we have the following:

THEOREM 1.2 (RCA_0). *The following are equivalent*:

(i) ACA_0;

(ii) *If $\bar{S}$ is a separably closed subset of a complete separable metric space then $\bar{S}$ is a closed set.*

PROOF. See [3].

THEOREM 1.3 (RCA_0). *The following are equivalent*:

(i) Π^1_1-CA_0;

(ii) *If C is a closed subset of a complete separable metric space then C is a separably closed set.*

PROOF. See [3]. The system Π^1_1-CA$_0$ is Π^1_1 comprehension (i.e., the comprehension scheme is restricted to Π^1_1 formulas) and is much stronger than the three systems mentioned above.

Thus for an arbitrary complete separable metric space the equivalence of the two notions of closed set requires, and is equivalent to, Π^1_1-CA$_0$. In the setting of compact spaces this equivalence can be proved in the weaker system ACA$_0$ [3]. It also follows that, in terms of separably open and closed sets, the standard results on countable unions and intersections referred to above require, and again are equivalent to, Π^1_1-CA$_0$ over RCA$_0$ [3].

Let $\hat{A}$ and $\hat{B}$ be complete separable metric spaces with codes A and B, respectively. Within RCA$_0$ we define a (code for a) *continuous partial function from $\hat{A}$ to $\hat{B}$* to be a function $\Phi: \mathbf{N} \to A \times \mathbf{Q}^+ \times B \times \mathbf{Q}^+$ such that for all $m, n \in \mathbf{N}$, $a, a' \in A$, $b, b' \in B$ and $r, r', s, s' \in \mathbf{Q}^+$:

(i) $\Phi(m) = (a, r, b, s) \wedge \Phi(n) = (a', r', b', s') \to d(b, b') < s + s'$;

(ii) $\Phi(m) = (a, r, b, s) \wedge (b, s) \leq (b', s') \to \exists k[\Phi(k) = (a, r, b', s')]$;

(iii) $\Phi(m) = (a, r, b, s) \wedge (a', r') \leq (a, r) \to \exists k[\Phi(k) = (a', r', b, s)]$.

Here $(a, r) \leq (b, s)$ means $r \leq s - d(a, b)$. We write $(a, r, b, s) \in \Phi$ if $\Phi(n) = (a, r, b, s)$ for some $n \in \mathbf{N}$. The idea here is that Φ encodes a continuous partial function Φ from $\hat{A}$ to $\hat{B}$. Intuitively, $(a, r, b, s) \in \Phi$ is a piece of information to the effect that $d(\phi(x), b) \leq s$ whenever $d(x, a) < r$. A point $x \in \hat{A}$ is said to *belong to the domain* of ϕ if, for all $\varepsilon > 0$, there exists a $(a, r, b, s) \in \Phi$ such that $d(x, a) < r$ and $s < \varepsilon$. If $x \in \hat{A}$ is in the domain of ϕ, we define $\phi(x)$ to be the point $y \in \hat{B}$ such that $d(y, b) \leq s$ for all $(a, r, b, s) \in \Phi$ with $d(x, a) < r$. We can prove, within RCA$_0$, that y exists by using the code Φ and the μ-operator. Note that $y = \phi(x)$ is unique up to the equality of points in a complete separable metric space as defined above.

We define a (code for a) *separable Banach space* to be a set $A \subset \mathbf{N}$ together with operations $+: A \times A \to A$, $-: A \times A \to A$, and $\cdot: \mathbf{Q} \times A \to A$ and a distinguished element $0 \in A$ such that $\langle A, +, -, \cdot, 0 \rangle$ forms a vector space over the rational field $\mathbf{Q}$. In addition we require a function $\|\cdot\|: A \to \mathbf{R}$ satisfying:

(i) $\|qa\| = |q| \, \|a\|$ for all $a \in A$ and $q \in \mathbf{Q}$;

(ii) $\|a + b\| \leq \|a\| + \|b\|$ for all $a, b \in A$.

Thus a code for a separable Banach space is a countable pseudonormed vector space over the rationals. As usual we define a pseudometric on A by setting $d(a, b) = \|a - b\|$ for $a, b \in A$. We define a *point of the separable Banach space $\hat{A}$* to be a point of the completion $\hat{A}$ of A under this metric. Thus points of $\hat{A}$ are sequences $\langle a_n : n \in \mathbf{N} \rangle$ such that $\forall n \forall i (\|a_n - a_{n+i}\| < 2^{-n})$.

If $x = \langle a_n : n \in \mathbf{N} \rangle$ is a point in $\hat{A}$, we define

$$\|x\| = \langle c_{n,n} : n \in \mathbf{N} \rangle,$$

where $\langle c_{n,k} : n \in \mathbf{N} \rangle = \|a_{n+1}\|$. We also define the sum of two elements of $\hat{A}$ by

$$\langle a_n : n \in \mathbf{N} \rangle + \langle b_n : n \in \mathbf{N} \rangle = \langle a_{n+1} + b_{n+1} : n \in \mathbf{N} \rangle,$$

and the scalar multiple of an element of $\hat{A}$ by a real by

$$\langle q_n : n \in \mathbf{N} \rangle \langle a_n : n \in \mathbf{N} \rangle = \langle q_{n+m} a_{n+m} : n \in \mathbf{N} \rangle,$$

where $m \in \mathbf{N}$ is the least such that $((\|a_0\|)_0 + |q_0| + 2)2^{-m} \leq 1$. Thus $\hat{A}$ enjoys the usual properties of a normed vector space over $\mathbf{R}$.

Let $\hat{A}$ and $\hat{B}$ be separable Banach spaces. We define a *continuous linear operator* from $\hat{A}$ to $\hat{B}$ to be a totally defined continuous function $\phi: \hat{A} \to \hat{B}$ such that $\phi(\alpha x + \beta y) = \alpha\phi(x) + \beta\phi(y)$ for all $x, y \in \hat{A}$ and $\alpha, \beta \in \mathbf{R}$. We define a (code for a) *bounded linear operator* from $\hat{A}$ to $\hat{B}$ to be a function $F: A \to \hat{B}$ such that:

(i) $F(q_1 a_1 + q_2 a_2) = q_1 F(a_1) + q_2 F(a_2)$ for all $q_1, q_2 \in \mathbf{Q}$ and $a_1, a_2 \in A$;

(ii) there exists a real number α such that $\|F(a)\| \leq \alpha\|a\|$ for all $a \in A$.

For F and α as above and $x = \langle a_n : n \in \mathbf{N} \rangle \in \hat{A}$, we define $F(x)$ to be the unique $y \in \hat{B}$ such that for all $n \in \mathbf{N}$

$$\|y - F(a_n)\| \leq 2^{-n}\alpha.$$

Thus $\|F(x)\| \leq \alpha\|x\|$ for all $x \in \hat{A}$. We write $F: \hat{A} \to \hat{B}$ to denote this state of affairs. If $\alpha \in \mathbf{R}$ is such that $\|F(x)\| \leq \alpha\|x\|$ for all $x \in \hat{A}$, we write $\|F\| \leq \alpha$. Specializing to the case $\hat{B} = \mathbf{R}$ we obtain a *bounded linear functional* on $\hat{A}$. We have the following standard result from Banach space theory relating the two types of operators defined above.

THEOREM 1.4 (RCA_0). *Given a continuous linear operator $\phi: \hat{A} \to \hat{B}$, there exists a bounded linear operator $F: \hat{A} \to \hat{B}$ such that $F(x) = \phi(x)$ for all $x \in \hat{A}$. The converse also holds.*

PROOF. See [4].

Specializing the definitions above to the context of separable Banach spaces, we define a *closed linear subspace C* of a separable Banach space $\hat{A}$ to be a closed subset C of $\hat{A}$ which is also a linear subset of $\hat{A}$. On the other hand we define a *separably closed linear subspace* $\bar{S}$ of $\hat{A}$ to be a separably closed subset of $\hat{A}$ which is also linear.

THEOREM 1.5 (RCA_0). *Suppose $\hat{A}$ is a separable Banach space and $S = \langle x_n : n \in \mathbf{N} \rangle$ is a code for a separably closed linear subspace of $\hat{A}$. Then there exists a separable Banach space $\hat{B}$ and a norm preserving map $\Psi: \hat{B} \xrightarrow[\text{onto}]{1\text{-}1} \bar{S}$.*

PROOF. See [2].

Thus each separably closed linear subspace $\bar{S}$ of a separable Banach space $\hat{A}$ is isometrically isomorphic to a separable Banach space $\hat{B}$. We will consider $\bar{S}$ to be a separable Banach space by identifying it with $\hat{B}$.

We conclude this section with some defintions needed in the statements of the Baire Category theorem to follow. We say that an open set U is *dense* if for every basic open set (a, r) there exists a point $x \in \hat{A}$ belonging to U which is also an element of (a, r). A closed set C is then said to be *nowhere dense* if the open set which is the complement of C is dense and equivalently (over RCA_0), if C contains no open ball. Similarly for separably open and closed sets.

§2. The Baire category theorem. In standard metric space theory the Baire Category theorem can be stated as follows: *Given a complete metric space X and a sequence $\langle O_n : n \in \mathbf{N} \rangle$ of open dense subsets of X, $\bigcap_{\mathbf{N}} O_n$ is dense.* Due to the nontrivial distinction between open and separably open sets noted above we obtain two versions of the Baire Category theorem which are not equivalent over RCA_0:

B.C.T.I. Let $\hat{A}$ be a complete separable metric space, and let $\langle U_n : n \in \mathbf{N} \rangle$ be a sequence of (codes for) open dense subsets of $\hat{A}$. If U is a (code for a) nonempty open set in $\hat{A}$, then there is a point $x \in \hat{A}$ such that $x \in U$ and $x \in U_n$ for all $n \in \mathbf{N}$.

B.C.T.II. Let $\hat{A}$ be a complete separable metric space, and let

$$\langle\langle x_{n,k} \colon k \in \mathbf{N}\rangle \colon n \in \mathbf{N}\rangle$$

be a sequence of (codes for the compliments of) dense separably open subsets O_n of $\hat{A}$. If U is a (code for a) nonempty open set in $\hat{A}$, then there is a point $x \in \hat{A}$ such that $x \in U$ and $x \in O_n$ for all $n \in \mathbf{N}$ (i.e., $x \notin \overline{\{x_{n,k} \colon k \in \mathbf{N}\}}$ for any $n \in \mathbf{N}$).

THEOREM 2.1 (RCA_0). *B.C.T.I holds for any complete separable metric space.*

PROOF. Reasoning within RCA_0 we define a point $x = \langle a_n \colon n \in \mathbf{N}\rangle$ by recursion on n so that $x \in \hat{A}$ and $x \in U_n$ for all $n \in \mathbf{N}$. Since U_0 is dense, we can find $(a_0, r_0) \in A \times \mathbf{Q}^+$ such that $(a_0, r_0) \subset U_0$ (note that this is Σ^0_1) and $r_0 \leq 1/2$. Let $\varphi(n, a, r, b, s)$ be a Σ^0_1 formula which expresses the following: $(a, r) \in A \times \mathbf{Q}^+$, $(b, s) \in A \times \mathbf{Q}^+$, $(b, s) < (a, r)$, $(b, s) \subset U_n$, and $s \leq 2^{-n-1}$. From the density of U_n, it follows that for each $(n, a, r) \in \mathbf{N} \times A \times \mathbf{Q}^+$ there exists (b, s) such that $\varphi(n, a, r, b, s)$. Write

$$\varphi(n, a, r, b, s) \equiv \exists k \theta(n, a, r, b, s, k),$$

where θ is Σ^0_0. By minimization [21], there exists a function

$$f \colon \mathbf{N} \times A \times \mathbf{Q}^+ \to \mathbf{N} \times A \times \mathbf{Q}^+$$

such that $f(n, a, r)$ is the least (k, b, s) such that $\theta(n, a, r, b, s, k)$ holds. By primitive recursion [21], there exists a function $g \colon \mathbf{N} \to A \times \mathbf{Q}^+$ such that $g(0) = (a_0, r_0)$ and, for all $n \in \mathbf{N}$, $g(k+1) = (a_{k+1}, r_{k+1})$, where $f(n, a_n, r_n) = (k_n, a_{n+1}, r_{n+1})$. Hence $\varphi(n, a_n, r_n, a_{n+1}, r_{n+1})$ holds for all n. It is not hard to check that $x = \langle a_n \colon n \in \mathbf{N}\rangle$ is a point of $\hat{A}$ and that $x \in U_n$ for all n. See also [2] and [21].

Thus B.C.T.I holds in RCA_0. We immediately have the following:

COROLLARY 2.2 (RCA_0). *If $\langle E_n; n \in \mathbf{N}\rangle$ is a sequence of closed nowhere dense subsets of a complete separable metric space $\hat{A}$ and U is any nonempty open subset of $\hat{A}$, then there exists $x \in \hat{A}$ such that $x \notin \bigcup_{\mathbf{N}} E_n$.*

PROOF. By definition, the code for each E_n is also a code for a dense open set $U_n = {\sim} E_n$. Applying Theorem 2.1 we obtain a point $x \in U$ such that $x \in \bigcap_{\mathbf{N}} U_n = \bigcap_{\mathbf{N}} {\sim} E_n$; hence $x \notin \bigcup_{\mathbf{N}} E_n$.

With this corollary in hand we can then prove the Banach-Steinhaus theorem:

THEOREM 2.3 (RCA_0). *Let $\langle T_n \colon n \in \mathbf{N}\rangle$ be a sequence of bounded linear operators on a separable Banach space $\hat{A}$ such that for all $x \in \hat{A}$, $\sup_{\mathbf{N}} \|T_n(x)\| < \infty$. Then there exists $M \in \mathbf{N}$ such that for all $x \in \hat{A}$ and $n \in \mathbf{N}$, $\|T_n(x)\| \leq M\|x\|$.*

PROOF. For each $m, n \in \mathbf{N}$, let

$$C_{m,n} = \{x \mid \|T_m(x)\| \leq n\}.$$

Since $\|T_n(x)\|$ is a continuous function from $\hat{A}$ to $\mathbf{R}$ [2], it follows that $C_{m,n}$ exists within RCA_0 and is a closed set [2]. Let

$$C_n = \bigcap_{m \in \mathbf{N}} C_{m,n} = \{x \mid \forall m(\|T_m(x)\| \leq n)\}.$$

Then C_n is closed [2]. Claim $\hat{A} = \bigcup_{\mathbf{N}} C_n$. Indeed, fix $x \in \hat{A}$. By hypothesis we have $\sup_{\mathbf{N}} \|T_n(x)\| < \infty$, say $\sup_{\mathbf{N}} \|T_n(x)\| \leq m$. Then for all $n \in \mathbf{N}$, $x \in C_{n,m}$ and hence

$x \in \bigcap_{n \in \mathbf{N}} C_{n,m} = C_m$. Since x was arbitrary, it follows that $\hat{A} = \bigcup_{\mathbf{N}} C_n$. Now, from Corollary 2.2 it follows that not all of the C_n's are nowhere dense; i.e., for some $n_0 \in \mathbf{N}$, C_{n_0} contains a basic open set (a_0, r_0). Fix $x \in \hat{A}$ such that $\|x\| \neq 0$, and consider $a_0 + r_0 x/2\|x\|$. We have

$$\left\| a_0 - \left(a_0 + \frac{r_0}{2\|x\|} x \right) \right\| = \left\| \frac{r_0}{2\|x\|} x \right\| = r_0/2 < r_0,$$

so $(a_0 + r_0 x/2\|x\|) \in (a_0, r_0)$. Also, for any $n \in \mathbf{N}$,

$$\frac{r_0}{2\|x\|} \|T_n(x)\| = \left\| T_n\left(\frac{r_0}{2\|x\|} x \right) \right\| \le \left\| T_n\left(a_0 + \frac{r_0}{2\|x\|} x \right) \right\| + \|T_n(a_0)\|.$$

Then, since a_0 and $a_0 + r_0 x/2\|x\|$ both belong to (a_0, r_0) which in turn is a member of $C_{n_0} = \{x \mid \forall m \|T_m(x)\| \le r_0\}$, it follows that $(r_0/2\|x\|)\|T_n(x)\| \le 2n_0$. Thus, for all $n \in \mathbf{N}$ and $x \in \hat{A}$, $\|T_n(x)\| \le (4n_0/r_0)\|x\|$ (we assumed $\|x\| \neq 0$, but clearly this also holds if $\|x\| = 0$). Taking any $M \in \mathbf{N}$ with $M \ge 4n_0/r_0$ completes the proof of the theorem.

Thus we see that another important theorem of ordinary mathematics can be proved in the weak base system RCA_0. However, other interesting consequences of the Baire Category theorem require the stronger version B.C.T.II, which we turn to now.

§3. B.C.T.II. We now consider the axiomatic strength needed to prove B.C.T.II. As an initial approximation we have the following.

THEOREM 3.1 (ACA_0). *B.C.T.II holds for any complete separable metric space* $\hat{A}$.

PROOF. Let $\langle\langle x_{n,k}: n \in \mathbf{N}\rangle: n \in \mathbf{N}\rangle$ be a sequence of nowhere dense sets in $\hat{A}$. By Theorem 1.3 we can find, within ACA_0, a code for a closed set E_n such that $E_n = \overline{\{x_{n,k}: k \in \mathbf{N}\}}$. Then, by Corollary 2.2, given any open set U there exists $x \in U$ such that $x \notin \bigcup_{\mathbf{N}} E_n$.

Thus it takes a system no stronger than ACA_0 to prove B.C.T.II. From the following we see that WKL_0 does not suffice to prove B.C.T.II.

THEOREM 3.2. WKL_0 *cannot prove B.C.T.II.*

PROOF. By Theorem 38 of Kleene [12] there is a complete extension T of Peano arithmetic which is Δ^0_2. By Scott [15] and Scott-Tennenbaum [16] the subsets X of ω which are binumerable in T (i.e., such that there is a formula φ such that $n \in X$ iff $\sim\varphi(n) \in T$) form a countable ω-model M of WKL_0 and, furthermore, each set X in M is uniformly Δ^0_2; i.e., there is a Δ^0_2 function $f: \omega \times \omega \to \{0,1\}$ such that $X \in M$ iff $X = \{n \mid f(m,n) = 0\}$ for some $m \in \omega$. Let $x_m \in 2^\omega$ be the characteristic function of $\{n \mid f(n,m) = 0\}$. Note that x_m is contained in our model M. There is a sequence $\langle x_{m,k}: k \in \omega\rangle$ of recursive $\{0.1\}$-functions such that, for all $n \in \omega$, $\lim_{n\to\infty} x_{m,k}(n) = x_m(n)$ and there is a $k_n \in \omega$ such that $x_{m,k}(n) = x_m(n)$ for all $k \ge k_n$ (Theorem 2 [17], extended in [13]). Now consider 2^ω_M, the Cantor space in M. Given any $x \in 2^\omega_M$, x can be considered as the characteristic function of some set X in M and hence $x = x_m$ for some $m \in \omega$. Since $x_m \in \overline{\{x_{m,k}: k \in \omega\}}$ we have

$$2^\omega_M = \bigcup_{m \in \omega} \overline{\{x_{m,k}: k \in \omega\}}.$$

Fix $m \in \omega$. We claim $\langle x_{m,k}: k \in \omega \rangle$ is nowhere dense in 2^ω_M. Indeed, fix a basic open set (σ, r) in 2^ω_M. Let $lh(\sigma) = n$, and let $k_n \in \omega$ be such that $x_{m,k}(n) = x_m(n)$ for all $k \geq k_n$. Then at most $k_n - 1$ points in the sequence $\langle x_{m,k}: k \in \omega \rangle$ extend $\sigma^\frown(1 - x_m(n))$. Therefore there is a $\tau \in 2^{<\omega}_M$ such that $\tau \supset \sigma^\frown(1 - x_m(n))$ and τ is incompatible with $x_{m,0}, \ldots, x_{m,k_n - 1}$. It follows that τ is incompatible with $x_{m,k}$ for all $k \in \omega$ and hence that $(\tau, 2^{-lh(\tau)}) \cap \overline{\{x_{m,k}: k \in \omega\}} = \varnothing$. Since $(\tau, 2^{-lh(\tau)}) < (\sigma, r)$, it follows that $\langle x_{m,k}: k \in \omega \rangle$ is nowhere dense, as desired. Thus B.C.T.II fails in M and so, by soundness, WKL_0 cannot prove B.C.T.II.

COROLLARY 3.3. RCA_0 *cannot prove B.C.T.II.*

PROOF. $\mathrm{WKL}_0 \supseteq \mathrm{RCA}_0$.

COROLLARY 3.4. *There is a recursive counterexample to B.C.T.II.*

PROOF. The sequences $\langle x_{m,k}: k \in \omega \rangle$ used in the proof of the theorem are recursive sequences of points in 2^ω and hence exist in Rec, the ω-model of RCA_0 consisting of the recursive subsets of ω. Also, since M is an ω-model of WKL_0 and hence contains Rec, for any recursive point $x \in 2^\omega$ there is an $m \in \omega$ such that $x \in \overline{\{x_{m,k}: k \in \omega\}}$. Thus

$$2^\omega_{\mathrm{Rec}} = \bigcup_{m \in \omega} \overline{\{x_{m,k}: k \in \omega\}},$$

where each $\langle x_{m,k}: k \in \omega \rangle$ is nowhere dense so that B.C.T.II fails in Rec.

§4. The systems RCA_0^+ and WKL_0^+. In this section we introduce a subsystem of $\mathbf{Z}_2$ which suffices to prove B.C.T.II; we call the system RCA_0^+. The axioms of RCA_0^+ are those of RCA_0 plus the following scheme (*): let σ, τ range over $2^{<\mathbf{N}}$, X range over $2^{\mathbf{N}}$, and let φ be any arithmetic formula; then

$$(*) \qquad \forall n \forall \sigma \exists \tau (\tau \supset \sigma \wedge \varphi(n, \tau)) \to \exists X \forall n \exists k (\varphi(n, X[k])).$$

The idea here is that given a sequence of arithmetically defined dense subsets of $2^{<\mathbf{N}}$ there exists, within RCA_0^+, a point in $2^{\mathbf{N}}$ which meets them all. The system WKL_0^+ is the system formed by adding the axiom scheme (*) to the axioms of WKL_0.

We begin by showing that a version of the scheme (*) holds in the Baire space $\mathbf{N}^{\mathbf{N}}$. In order to do so we first define a map from a certain subset of $2^{<\mathbf{N}}$ to $\mathbf{N}^{<\mathbf{N}}$ as follows: let

$$S = \{\sigma \in 2^{<\mathbf{N}} \mid \sigma(lh(\sigma) - 1) = 1\},$$

i.e., $\sigma \in S$ iff σ terminates in a one. Note that S exists within RCA_0. For $\sigma \in S$, define

$t_0 = \mu k[\sigma(k) = 1]$,

$t_{i+1} = (\mu k > (t_0 + \cdots + t_i + i)[\sigma(k) = 1]) - (t_0 + \cdots + t_i + i + 1)$ for i such that $t_0 + \cdots + t_i + i \leq lh(\sigma)$.

Let $\pi(\sigma) = \langle t_0, \ldots, t_k \rangle$ where $t_0 + \cdots + t_k + k = lh(\sigma)$. Then:

(1) π exists within RCA_0;

(2) $\pi: S \xrightarrow[\text{onto}]{\text{1-1}} \mathbf{N}^{<\mathbf{N}}$;

(3) if $\sigma_1, \sigma_2 \in S$ and $\sigma_1 \supset \sigma_2$, then $\pi(\sigma_1) \supset \pi(\sigma_2)$;

(4) if $\tau_1^*, \tau_2^* \in \mathbf{N}^{<\mathbf{N}}$ and $\tau_1^* \supset \tau_2^*$, then $\pi^{-1}(\tau_1^*) \supset \pi^{-1}(\tau_2^*)$.

The idea here is that $\pi(\sigma)$ lists the number of zeros occurring between successive ones and the length of $\pi(\sigma)$ is the number of ones occurring in σ. For example, if $\sigma = \langle 010011 \rangle$ then $\pi(\sigma) = \langle 1,2,0 \rangle$.

THEOREM 4.1 (RCA_0^+). *Let σ^*, τ^* range over $\mathbf{N}^{<\mathbf{N}}$, and suppose that ψ is an arithmetic formula such that*

$$\forall n \forall \sigma^* \exists \tau^* (\tau^* \supset \sigma^* \wedge \psi(n, \tau^*)).$$

Then there exists an $x \in \mathbf{N}^{\mathbf{N}}$ such that $\forall n \exists i (\psi(n, x[i]))$.

PROOF. For $n \in \mathbf{N}$ and $\tau \in 2^{<\mathbf{N}}$, let

$$\varphi(n,\tau) \equiv \tau \in S \wedge \psi(n, \pi(\tau)).$$

We claim that $\forall n \forall \sigma \exists \tau (\tau \supset \sigma \wedge \psi(n, \varphi(n,\tau))$. Indeed, fix $n \in \mathbf{N}$ and $\sigma \in 2^{<\mathbf{N}}$. We consider two cases:

Case 1. $\sigma \in S$. Then $\pi(\sigma)$ is defined and, by hypothesis, there is a $\tau^* \in \mathbf{N}^{<\mathbf{N}}$ such that $\tau^* \supset \tau(\sigma)$ and $\psi(n, \tau^*)$. By note (4) above, $\tau = \tau^{-1}(\tau^*) \supset \sigma$ and, by definition, we have $\varphi(n, \tau)$.

Case 2. $\sigma \notin S$. In this case take $\sigma' = \sigma^\frown\langle 1 \rangle$ and apply case 1 to obtain $\tau \supset \sigma' \supset \sigma$ with $\varphi(n, \tau)$.

Now apply the principal axiom of RCA_0^+ to obtain $y \in 2^{\mathbf{N}}$ such that $\forall n \exists i_n \varphi(n, y[i_n])$. Define

$$x = \langle \pi(y[i]) \colon i \in \mathbf{N} \rangle$$

and note that x exists within RCA_0 (given y). By note (3) above, $x \in \mathbf{N}^{\mathbf{N}}$ and, by definition,

$$\varphi(n, y[i_n]) \quad \Rightarrow \quad \psi(n, \pi(y[i_n])).$$

Therefore $\forall n \exists i \psi(n, x[i])$, as desired.

We now show that B.C.T.II holds for any complete separable metric space $\hat{A}$. Given any such space, let A be the code for $\hat{A}$ and $\langle a_n \colon n \in \mathbf{N} \rangle$ an enumeration of A. For each $\sigma^* \in \mathbf{N}^{<\mathbf{N}}$ with $lh(\sigma^*) = m$, define, for $0 \le k < m - 1$,

$b_0 = a_{\sigma^*(0)}$;

$b_{k+1} =$ the $\sigma^*(k+1)$st element in some fixed enumeration of $\{a \mid d(a, b_k) < 2^{-(k+1)}\}$.

Now let $\pi(\sigma^*) = \langle b_0, \ldots, b_{m-1}, b_{m-1}, \ldots \rangle$. Then $\pi \colon \mathbf{N}^{<\mathbf{N}} \to \hat{A}$ and π exists within RCA_0. Note:

(1) if $\sigma_1^*, \sigma_2^* \in \mathbf{N}^{<\mathbf{N}}$ and $\sigma_1^* \supset \sigma_2^*$, then $\pi(\sigma_1^*)(i) = \pi(\sigma_2^*)(i)$ for all $i < lh(\sigma_2^*)$; and conversely,

(2) if $z = \langle b_0, \ldots, b_k, b_k, \ldots \rangle \in \hat{A}$ is such that for all $i \in \mathbf{N}$ $d(b_i, b_{i+1}) < 2^{-(i+1)}$, then for all $i > k$ there is a $\sigma^* \in \mathbf{N}^{<\mathbf{N}}$ such that $lh(\sigma^*) = i$ and $\pi(\sigma^*) = z$.

THEOREM 4.2 (RCA_0^+). *B.C.T.II holds for any complete separable metric space $\hat{A}$.*

PROOF. Let $\hat{A}$ be a complete separable metric space, let π be as above, and let $\langle \langle x_{n,k} \colon k \in \mathbf{N} \rangle \colon n \in \mathbf{N} \rangle$ code a sequence $\langle O_n \colon n \in \mathbf{N} \rangle$ of separably open dense sets in $\hat{A}$. Fix a basic open set (a_0, r_0), and let $\tau_0^* \in \mathbf{N}^{<\mathbf{N}}$ be such that $lh(\tau_0^*) = i_0 + 1$, where $i_0 \ge 2$ is such that $2^{-i_0} < r_0/2$, and $\pi(\tau_0^*) = \langle a_0, a_0, a_0, \ldots \rangle$. For $\tau^* \in \mathbf{N}^{<\mathbf{N}}$ define

$$\varphi(n, \tau^*) \equiv \exists i \forall k \forall \gamma^* [\gamma^* \supset \tau^* \to d(\pi(\tau_0^{*\frown} \gamma^*), x_{n,k}) \ge 2^{-i}].$$

We claim that $\forall n \forall \sigma^* \exists \tau^* [\tau^* \supset \sigma^* \wedge \varphi(n, \tau^*)]$. Indeed, fix n and σ^*. Let

$$lh(\tau_0^{*\frown}\sigma^*) = \langle b_0, \ldots, b_{m-1}, b_{m-1}, b_{m-1}, \ldots \rangle.$$

Note $b_0 = \cdots = b_{i_0} = a_0$ so

$$d(a_0, b_{m-1}) \leq d(b_{i_0}, b_{i_0+1}) + \cdots + d(b_{m-2}, b_{m-1}) < 2^{-i_0}.$$

Since O_n is dense, there is a $(b, r) < (b_{m-1}, 2^{-m})$ with $(b, r) \cap \overline{\{x_{n,k} : k \in \mathbf{N}\}} = \varnothing$ and hence $d(a_0, b) \leq d(a_0, b_{m-1}) + d(b_{m-1}, b) < r_0$ (since $2^{-m} \leq 2^{-i_0}$), and $d(b, x_{n,k}) \geq r$ for all k. Choose $i \geq m$ so that $2^{-i} \leq \min(r/2, r_0 - d(b, a_0))$. Let

$$z = \langle b_0, \ldots, b_{m-1}, b, b, b, \ldots \rangle.$$

By note (2) above there is a $\tau^{*\prime} \in \mathbf{N}^{<\mathbf{N}}$ with $p(\tau^{*\prime}) = z$ and $lh(\tau^{*\prime}) = i + 1$ and, by note (1), $\tau^{*\prime} \supset \tau_0^{*\frown}\sigma^*$. Let $\tau^* \in \mathbf{N}^{<\mathbf{N}}$ be such that $\tau_0^{*\frown}\tau^* = \tau^{*\prime}$. Then $\tau^* \supset \sigma^*$. Suppose $\gamma^* \supset \tau^*$, and let

$$\pi(\tau_0^{*\frown}\gamma^*) = \langle b_0, \ldots, b_{m-1}, b, \ldots, b, b_{i+1}, \ldots \rangle,$$

where the terms $b_{i+1}, \ldots$ depend upon γ^*. Then for any k,

$$\begin{aligned} r &\leq d(b, x_{n,k}) \\ &\leq d(b, b_{i+1}) + d(b_{i+1}, \pi(\tau_0^{*\frown}\gamma^*)) + d(\pi(\tau_0^{*\frown}\gamma^*), x_{n,k}) \\ &< 2^{-(i+1)} + 2^{-(i+1)} + d(\pi(\tau_0^{*\frown}\gamma^*), x_{n,k}). \end{aligned}$$

Therefore

$$d(\pi(\tau_0^{*\frown}\gamma^*), x_{n,k}) \geq r - 2^{-i} \geq r/2 \geq 2^{-i}$$

and so $\varphi(n, \tau^*)$ holds and the claim follows.

Now apply Theorem 4.1 to obtain $y \in \mathbf{N}^{\mathbf{N}}$ such that $\forall n \exists k \varphi(n, y[k])$ and let $x = \lim_{i \to \infty} \pi(\tau_0^{*\frown}y[i])$. Then:

(1) $x \in \hat{A}$: For $i < j$ we have

$$\begin{aligned} &d(\pi(\tau_0^{*\frown}y[i]), \pi(\tau_0^{*\frown}y[j])) \\ &\quad \leq d(\pi(\tau_0^{*\frown}y[i]), \pi(\tau_0^{*\frown}y[j])i_0 + i) + d(\pi(\tau_0^{*\frown}y[i])i_0 + i, \pi(\tau_0^{*\frown}y[j])i_0 + j) \\ &\qquad\qquad + d(\pi(\tau_0^{*\frown}y[j])i_0 + j, \pi(\tau_0^{*\frown}y[j])) \\ &\quad < 2^{-(i_0+i)} + 2^{-(i_0+i)} + 2^{-(i_0+j)} \\ &\quad < 3 \cdot 2^{-(i_0+i)} < 2^{-i} \quad (\text{since } i_0 \geq 2). \end{aligned}$$

That $x \in \hat{A}$ now follows from the completeness of $\hat{A}$ [2].

(2) $x \in O_n$ for all $n \in \mathbf{N}$. Fix $n \in \mathbf{N}$ and let k_n be such that $\varphi(n, y[k_n])$. Then $\exists i_n \forall j \geq k_n \forall k d(\pi(\tau_0^{*\frown}y[j], x_{n,k}) \geq 2^{-i_n}$. Choose $j > \max(k_n, i_n)$. Then for all k we have

$$\begin{aligned} 2^{-i_n} &\leq d(\pi(\tau_0^{*\frown}y[j], x_{n,k}) \\ &\leq d(\pi(\tau_0^{*\frown}y[j], x) + d(x, x_{n,k}) \\ &\leq 2 - j + d(x, x_{n,k}). \end{aligned}$$

Thus $d(x, x_{n,k}) \geq 2^{-i_n} - 2^{-j} > 0$, so $x \in O_n$, as claimed.

(3) $x \in (a_0, r_0)$: We have

$$d(x, a_0) \le d(x, \pi(\tau_0^{*\frown} y[i_0])) + d(\pi(\tau_0^{*\frown} y[i_0]), a_0)$$
$$< 2^{-i_0} + 2^{-i_0}$$

(since $\pi(\tau_0^{*\frown} y[i_0])$ begins with a sequence of a_0's of length i_0)

$$< r_0.$$

This completes the proof of the theorem.

Thus we see that B.C.T.II holds in RCA_0^+. In fact RCA_0^+ proves the following stronger result: if $\langle\langle(a_{n,k}, r_{n,k})\colon k \in \mathbf{N}\rangle\colon n \in \mathbf{N}\rangle$ is an arithmetically defined sequence of sequences of basic open sets in a complete separable metric space $\hat{A}$ such that for each n the open set $U_n = \langle(a_{n,k}, r_{n,k})\colon k \in \mathbf{N}\rangle$ is dense, then $\bigcap_{\mathbf{N}} U_n$ is dense (the results of this section only need a Σ_3^0 sequence $\langle\langle(a_{n,k}, r_{n,k})\colon k \in \mathbf{N}\rangle\colon n \in \mathbf{N}\rangle$). Apparently this is the strongest version of the Baire Category theorem provable in RCA_0^+.

§5. The Open Mapping and Closed Graph theorems. We now apply Theorem 4.2. In what follows let $\hat{A}$ and $\hat{B}$ be separable Banach spaces. For $n \in \mathbf{N}$, let $\langle a_{n,k}\colon k \in \mathbf{N}\rangle$ be an enumeration of the Σ_1^0 set

$$S_n = \{a \mid \|a\| < 2^{-n}\}.$$

Note that if $T\colon \hat{A} \to \hat{B}$ then $T(S_n) = \langle T(a_{n,k})\colon k \in \mathbf{N}\rangle$ is a sequence of points in $\hat{B}$.

LEMMA 5.1 (RCA_0^+). *Let T be a bounded linear operator from $\hat{A}$ onto $\hat{B}$. Then there is an $r \in \mathbf{Q}^+$ such that, for all $y \in \hat{B}$, if $\|y\| < r$ then $y = T(x)$ for some $x \in \hat{A}$ with $\|x\| < 1$.*

PROOF. Since $\hat{A} = \bigcup_{\mathbf{N}} n\overline{S_1}$ and T is onto, we have $\hat{B} = \bigcup_{\mathbf{N}} n\overline{T(S_1)}$. By Theorem 4.2 there must exist an n_0 such that $\langle n_0 T(a_{1,k})\colon k \in \mathbf{N}\rangle = n_0 T(S_1)$ is not nowhere dense. Thus there is some basic open set (b'_0, r'_0) contained in $n_0\overline{T(S_1)}$. Then $\overline{T(S_1)}$ contains $(b'_0/n_0, r'_0/n_0)$. Let $b_0 = b'_0/n_0$ and $r_0 = r'_0/n_0$. Then $b_0 + (0, r_0) \subset \overline{T(S_1)}$ and so, noting that $-b_0 \in \overline{T(S_1)}$,

$$(0, r_0) \subset \overline{T(S_1)} - b_0 \subset 2\overline{T(S_1)} \subset \overline{T(S_0)}.$$

More generally, by linearity, $(0, r_0/2^n) \subset \overline{T(S_n)}$ for any $n \in \mathbf{N}$. We claim that if $\|y\| < r_0/2$ then there is an $x \in \hat{A}$ such that $\|x\| < 1$ and $T(x) = y$. Indeed, suppose $y \in (0, r_0/2)$. Then $y \in \overline{T(S_1)}$, so there is an $a_1 \in S_1$ with $\|y - T(a_1)\| < r_0/4$. Suppose that we continue in this fashion to choose $a_1, \ldots, a_n$ such that $a_i \in S_i$, $1 \le i \le n$, and $\|y - \sum_{i=1}^n T(a_i)\| < r_0/2^{n+1}$. Then $y - \sum_{i=1}^n T(a_i) \in \overline{T(S_{n+1})}$ and hence there is an $a_{n+1} \in S_{n+1}$ with

$$\left\| y - \sum_{i=1}^{n} T(a_i) - T(a_{n+1}) \right\| < r_0/2^{n+2}.$$

Applying Σ_1^0 induction we then obtain a sequence $\langle a_n\colon 1 \le n \in \mathbf{N}\rangle$ such that $a_n \in S_n$ and $\|y - \sum_{i=1}^n T(a_i)\| < r_0/2^{n+1}$. Let $x = \langle \sum_{i=1}^n a_i\colon 1 \le n \in \mathbf{N}\rangle$. Then for any $j \ge 1$

and $n \in \mathbf{N}$,

$$\left\|\sum_{i=1}^{n} a_i - \sum_{i=1}^{j+n} a_i\right\| = \left\|\sum_{i=j+1}^{j+n} a_i\right\| \leq \|a_{j+1}\| + \cdots + \|a_{j+n}\|$$
$$< 2^{-(j+1)} + \cdots + 2^{-(j+n)} < 2^{-j}.$$

Thus $x \in \hat{A}$. We also have

$$\|x\| = \lim_{n\to\infty}\left\|\sum_{i=1}^{n} a_i\right\| < \lim_{n\to\infty}(2^{-1} + \cdots + 2^{-n}) = 1.$$

Thus $\|x\| < 1$. Since $\|y - \sum_{i=1}^{n} T(a_i)\| < r_0/2^{n+1}$ for all $n \in \mathbf{N}$, it follows that $T(x) = y$, as desired.

Thus, within RCA_0^+, the image of an open unit ball in $\hat{A}$ contains a ball about zero in $\hat{B}$. Note that RCA_0^+ was needed only to show that $\overline{T(S_1)}$ contains a basic open set; the remainder of the proof requires only RCA_0. We are now able to prove the Open Mapping theorem for separable Banach spaces.

THEOREM 5.2 (RCA_0^+). *If T is a bounded linear operator from $\hat{A}$ onto $\hat{B}$ and U is an open subset of $\hat{A}$, then there exists an open set V in $\hat{B}$ such that $y \in V$ iff $y = T(x)$ for some $x \in U$.*

PROOF. With Lemma 5.1 in hand we can carry out the usual proof using only the axioms of RCA_0. See [2].

Thus the Open Mapping theorem for open sets is provable in RCA_0^+. Two questions remain unanswered (as of this writing):

(1) Can the Open Mapping theorem for open sets be proved in a system weaker than RCA_0?

(2) What axioms are needed to prove the Open Mapping theorem for separably open sets?

Since ACA_0 proves that every separably open set is open, it is immediate from Theorem 5.2 that ACA_0 proves the Open Mapping theorem for separably open sets. Thus we have a partial answer to (2).

From Theorem 5.2 we may now obtain the Bounded Inverse theorem for separable Banach spaces.

THEOREM 5.3 (RCA_0^+). *Suppose T is a bounded linear operator mapping $\hat{A}$ one-one onto $\hat{B}$. Then T^{-1} is bounded.*

PROOF. Here again the usual proof can be carried out once we have Theorem 5.2 [2].

We conclude this section by considering the Closed Graph theorem. Suppose A and B are codes for separable Banach spaces with norms $\|\cdot\|_A$ and $\|\cdot\|_B$ respectively. Consider the set $A \times B$ with operations:

(i) $(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$,

(ii) $q(a_1, b_1) = (qa_1, qb_1)$,

where $a_1, a_2 \in A$, $b_1, b_2 \in B$, and $q \in \mathbf{Q}$. For $a \in A$ and $b \in B$ define

$$\|(a, b)\| = \|a\|_A + \|b\|_B.$$

It is then easy to see that the completion of $A \times B$ under this norm yields the separable Banach space $\hat{A} \times \hat{B}$, i.e., the set of pairs (x, y) where $x \in \hat{A}$ and $y \in \hat{B}$. If $T\colon \hat{A} \to \hat{B}$ is a linear operator (not necessarily bounded), we define the *graph of* T

to be the "set" of pairs (x, y) in $\hat{A} \times \hat{B}$ such that $T(x) = y$. Note that the graph of T does not formally exist within RCA_0.

THEOREM 5.4 (RCA_0^+). *Suppose $T: \hat{A} \to \hat{B}$ is a linear operator such that the graph of T is separably closed in $\hat{A} - \hat{B}$. Then T is continuous.*

PROOF. Let $G_T = \langle (x_n, y_n): n \in \mathbf{N} \rangle$ code the graph of T. Then $\overline{G_T}$ is a separable Banach space by Theorem 1.5. Define $P: \overline{G_T} \xrightarrow[\text{onto}]{1\text{-}1} \hat{A}$ by $P((x_n, y_n)) = x_n$. Then P is linear and

$$\begin{aligned} \|P((x_n, y_n))\|_{\hat{A}} &= \|x_n\|_{\hat{A}} \\ &\le \|x_n\|_{\hat{A}} + \|y_n\|_{\hat{B}} = \|(x_n, y_n)\|_{\hat{A} \times \hat{B}}, \end{aligned}$$

so P is bounded. Thus by Theorem 5.3, $P^{-1}: \hat{A} \to \overline{G_T}$ is bounded. Therefore there is an M such that for all $x \in \hat{A}$

$$\begin{aligned} \|P^{-1}(x)\|_{\overline{G_T}} \le M\|x\|_{\hat{A}} &\Rightarrow \|(x, T(x)\|_{\overline{G_T}} \le M\|x\|_{\hat{A}} \\ &\Rightarrow \|x\|_{\hat{A}} + \|T(x)\|_{\hat{B}} \le M\|x\|_{\hat{A}} \\ &\Rightarrow \|T(x)\|_{\hat{B}} \le (M - 1)\|x\|_{\hat{A}}. \end{aligned}$$

Thus T is bounded and therefore continuous.

In Theorem 5.4 we have the Closed Graph theorem for linear operators with separably closed graphs. It is an open question as to what axioms are necessary to prove the Closed Graph theorem for linear operators with graphs which are closed sets. We obtain a partial answer by noting that, since Π_1^1-CA_0 proves that every closed set is separably closed, it is immediate from Theorem 5.4 that no system stronger than Π_1^1-CA_0 is needed to obtain this result.

§6. Model theory of WKL_0^+. In this section we demonstrate the existence of a model of WKL_0^+ and, in the course of this, establish that WKL_0^+ is logically a weak subsystem of axioms in that it is conservative over Primitive Recursive Arithmetic (PRA) with respect to Π_2^0 sentences.

Let L_2 be the language of second-order arithmetic. A *model for* L_2 is an ordered 7-tuple

$$M = \langle |M|, S^M, +^M, \cdot^M, <^M, 0^M, 1^M \rangle,$$

where $|M|$ is a set, S^M a collection of subsets of $|M|$, $+^M$ and $\cdot^M$ are functions from $|M| \times |M|$ into $|M|$, and 0^M and 1^M are distinguished elements of $|M|$. For any theory T in the language L_2 we say that *M is a model of T* or *M satisfies T*, written $M \models T$, if the axioms of T are universally true in M when the first-order variables range over $|M|$, the second-order variables range over S^M, and $+, \cdot, 0, 1$ are interpreted in the obvious manner.

LEMMA 6.1. *Let M be a model of the basic axioms of* RCA_0 *plus Σ_1^0 induction. Then there exists a model M' of* RCA_0 *such that M is a submodel of M' and $|M| = |M'|$.*

PROOF. Let

$$M' = \langle |M|, \Delta_1^0\text{-Def } M, +^M, \cdot^M, <^M, 0^M, 1^M \rangle,$$

where Δ_1^0-Def M is the set of all $X \subset |M|$ such that X is Δ_1^0 definable over M allowing parameters from $|M| \cup S^M$. It follows immediately that $|M| = |M'|$ and $S^M \subset S^{M'}$. Thus M is a submodel of M' and M' satisfies the basic axioms of RCA_0.

Since Σ_1^0 induction in M implies Σ_1^0 collection in M' [11] it follows that we have Δ_1^0 comprehension in M' (see [23]). Thus we need only show that M' satisfies Σ_1^0 induction. Suppose $\varphi(n)$ is a Σ_1^0 formula in M', say

$$\varphi(n) \equiv \exists k \theta(n, k, \vec{X}),$$

where θ is Σ_0^0 and $\vec{X}$ is a complete list of the parameters from $S^{M'}$ occurring in θ. Since $\vec{X} \subset \Delta_1^0$-Def M, there is a Σ_1^0 formula θ_Σ containing only parameters from $|M|$ and S^M such that $\vec{a} \in \vec{X}$ iff $\theta_\Sigma(\vec{a})$ and, over M', $\theta(n,k,\vec{X}) \equiv \theta(n,k,\theta_\Sigma(\vec{a}))$ (see [23]). Thus, over M',

$$\varphi(n) \equiv \exists k \theta(n, k, \theta_\Sigma(\vec{a})),$$

i.e., φ is equivalent over M' to a Σ_1^0 formula containing only parameters from $|M| \cup S^M$. Σ_1^0 induction in M' then follows easily from Σ_1^0 induction in M, for:

$$\begin{aligned} M' &\models [\varphi(0) \wedge \forall n(\varphi(n) \to \varphi(n+1))] \\ &\Rightarrow M \models [\exists k\theta(0,k,\theta_\Sigma(\vec{a})) \wedge \forall n(\exists k\theta(n,k,\theta_\Sigma(\vec{a})) \to \exists k\theta(n+1,k,\theta_\Sigma(\vec{a}))] \\ &\Rightarrow M \models \forall n \exists k \theta(n,k,\theta_\Sigma(\vec{a})) \\ &\Rightarrow M' \models \forall n \exists k \theta(n,k,\vec{X}) \\ &\Rightarrow M' \models \forall n \varphi(n). \end{aligned}$$

Thus M' satisfies Σ_1^0 induction and so M' is a model of RCA_0.

Let M be any countable model (i.e., $|M|$ is countable) of RCA_0. Let $2_M^{<\mathbf{N}}$ be the set of M-finite sequences of zeros and ones. A set $D \subset 2_M^{<\mathbf{N}}$ is *dense* if for all $\sigma \in 2_M^{<\mathbf{N}}$ there is a $\tau \in D$ such that $\tau \supset \sigma$. A set $D \subset 2_M^{<\mathbf{N}}$ is *definable* if D is definable over M with parameters from $|M| \cup S^M$. A set $X \subset |M|$ is *M-generic* if for all definable dense sets $D \subset 2_M^{<\mathbf{N}}$ there is a $\sigma \in D$ such that $\sigma \in X$. As usual, if X is M-generic we can consider X to be an infinite sequence of zeros and ones, so $X \in 2_M^{\mathbf{N}}$ and we will write $\sigma \subset X$ for $\sigma \in X$.

LEMMA 6.2. *Let M be a countable model of* RCA_0. *There is a model M' of* RCA_0 *such that*:

(i) *M is a submodel of M'*;
(ii) $|M| = |M'|$;
(iii) *there is an $X \in S^{M'}$ such that X intersects all M-definable dense sets D.*

PROOF. Let X be M-generic, and let

$$M[X] = \langle |M|, S^M \cup \{X\}, +^M, \cdot^M, <^M, 0^M, 1^M \rangle.$$

Clearly M is a submodel of $M[X]$ and $|M| = |M[X]|$. Since the integers of $M[X]$ are those of M, it follows that $M[X]$ satisfies the basic axioms of RCA_0. Claim $M[X]$ satisfies Σ_1^0 induction. Indeed suppose

$$M[X] \models \varphi(0) \wedge \forall n(\varphi(n) \to \varphi(n+1)),$$

where $\varphi(n)$ is Σ_1^0 in $M[X]$. Write $\varphi(n)$ as $\exists k \theta(n, X[k])$ where $\theta(n,k)$, $\sigma \in 2_M^{<\mathbf{N}}$, is Σ_0^0 with parameters from $|M| \cup S^M$ only (see [23]). For each $n \in |M|$, define $E \subset 2_M^{<\mathbf{N}}$ by

$$\sigma \in E \text{ iff } \forall \tau \supset \sigma \forall k < lh(\tau) \sim \theta(0, \tau[k]) \vee \exists m < n[\exists k < lh(\sigma)\theta(m, \sigma[k]) \wedge \forall t \supset \sigma \forall k < lh(\tau) \sim \theta(m+1, \tau[k])].$$

Let $D \subset 2_M^{<\mathbf{N}}$ be defined by

$$\sigma \in D \quad \text{iff} \quad \sigma \in E \vee \sim\exists\tau \in E \quad (\tau \supset \sigma).$$

Then D is M-definable. Fix $\sigma \in 2_M^{<\mathbf{N}}$. Either there is a $\tau \in E \subset D$ such that $\tau \supset \sigma$ or no such τ exists, in which case $\sigma \in D$. In either event it follows that D is dense. Let $\sigma_0 \in D \cap X$, and suppose $\sigma_0 \in E$. Then either

(i) $\forall t \supset \sigma_0 \forall k < lh(\tau) \sim \theta(0, \tau[k]) \Rightarrow \forall k \sim \theta(0, X[k])$, contradicting our assumption that $M[x] \models \varphi(0)$, or

(ii) $\exists m < n[\exists k < lh(\sigma_0)\, \theta(m, \sigma_0[k]) \wedge \forall t \supset \sigma_0 \forall k < lh(\tau) \sim \theta(m+1, \tau[k])]$.

Fix m as in case (ii). Then

$$\begin{aligned} \exists k < lh(\sigma_0)\, \theta(m, \sigma_0[k]) &\Rightarrow \exists k \theta(m, X[k]) \\ &\Rightarrow \exists k \theta(m+1, X[k]) \quad \text{(induction hypothesis)} \\ &\Rightarrow \exists \tau \supset \sigma_0 \exists k < lh(\tau)\, \theta(m+1, \tau[k]), \end{aligned}$$

which contradicts $\sigma_0 \in E$. Thus $\sigma_0 \notin E$ so, by the definition of D, if $\tau \supset \sigma_0$ then $\tau \notin E$ and hence

(1) $\forall m[\exists k < lh(\tau)\, \theta(m, \tau[k]) \rightarrow \exists \gamma \supset \tau \exists k < lh(\gamma)\, \theta(m+1, \gamma[k])]$.

Since $\sigma_0 \notin E$, we have

$$\exists \tau_0 \supset \sigma_0 \exists k < lh(\tau_0)\, \theta(0, \tau_0[k]).$$

Now for $m \in |M|$ suppose we have $\tau_m \supset \sigma_0$ such that

$$\exists k < lh(\tau_m)\, \theta(m, \tau_m[k]).$$

Then by (1) there is a $\tau_{m+1} \supset \tau_m$ such that $\exists k < lh(\tau_{m+1})\, \theta(i, \tau_{m+1}[k])$. Thus we have

$$\begin{aligned} &\exists \tau \supset \sigma_0 \exists k < lh(\tau)\, \theta(0, \tau[k]) \wedge \forall n(\exists \tau \supset \sigma_0 \exists k < lh(\tau)\, \theta(n, \tau[k]) \\ &\quad \rightarrow \exists \tau \supset \sigma_0 \exists k < lh(\tau)\, \theta(n+1, \tau[k]). \end{aligned}$$

It follows from Σ_1^0 induction in M that

(2) $\forall n \exists \tau \supset \sigma_0 \exists k < lh(\tau)\, \theta(n, \tau[k])$,

and this in fact holds with σ_0 replaced by any $\sigma \supset \sigma_0$. Now, for each $n \in |M|$, define $D_n \subset 2_M^{<\mathbf{N}}$ by

$$\sigma \in D_n \quad \text{iff} \quad (\sigma \text{ is incompatible with } \sigma_0) \vee \exists k < lh(\sigma)\, \theta(n, \sigma[k]).$$

Then D_n is M-definable for each $n \in |M|$. Fix $\sigma \in 2_M^{<\mathbf{N}}$. If σ is incompatible with σ_0 then $\sigma \in D_n$. Otherwise either:

(i) $\sigma \subset \sigma_0$, in which case by (2) $\exists \tau \supset \sigma_0 \supset \sigma$ such that $\exists k < lh(\tau)\, \theta(n, \tau[k])$ so that $\tau \in D_n$, or

(ii) $\sigma_0 \subset \sigma$, so by the generalized version of (2) we get $\tau \supset \sigma$ with $\tau \in D_n$.

Thus we see that each D_n is dense. Let $\gamma_n \in X \cap D_n$ for $n \in |M|$. Since $\sigma_0 \subset X$, it follows that each γ_n is compatible with σ_0, so we must have

$$\exists k < lh(\gamma_n)\, \theta(n, \gamma_n[k]).$$

Therefore

$$\begin{aligned} M \models \forall n \exists k < lh(\gamma_n)\, \theta(n, \gamma_n[k]) \\ \Rightarrow M[X] \models \forall n \exists k \theta(n, X[k]) \\ \Rightarrow M[X] \models \forall n \varphi(n). \end{aligned}$$

Therefore $M[X]$ satisfies Σ_1^0 induction, as desired. Finally, apply Lemma 6.1 to obtain a model M' of RCA_0 such that $M[X]$ is a submodel of M' and $|M[X]| = |M'|$. Clearly M' has the desired properties.

LEMMA 6.3. *Any countable model M of* RCA_0 *can be expanded to a countable model M' of* WKL_0 *with* $|M| = |M'|$.

PROOF. Harrington (see [21]).

THEOREM 6.4. *Let M be a countable model of* RCA_0. *There exists a model M' of* WKL_0^+ *such that M is a submodel of M' and* $|M| = |M'|$.

PROOF. Apply Lemmas 6.2 and 6.3 repeatedly to obtain a sequence $\langle M_i : i \in \omega \rangle$ of models such that:

(i) $M_0 = M$;

(ii) each M_i is a submodel of M_{i+1} with $|M_i| = |M_{i+1}|$;

(iii) $M_{2i+1} \models \mathrm{RCA}_0 + (\exists X[D \cap X \neq \varnothing])$, for all M_{2i}-definable dense sets D;

(iv) $M_{2i} \models \mathrm{WKL}_0$.

Let $M' = \langle |M|, \bigcup_\omega S^{M_i}, +^M, \cdot^M, <^M, 0^M, 1^M \rangle$. Clearly M is a submodel of M' and $|M| = |M'|$, and hence M' satisfies the basic axioms of RCA_0. Suppose $\varphi(n)$ is a Σ_1^0 formula such that

$$M' \models \varphi(0) \wedge \forall n(\varphi(n) \rightarrow \varphi(n+1)).$$

Let i be such that all parameters in φ appear by stage $2i+1$. Then

$$M_{2i+1} \models \varphi(0) \wedge \forall n(\varphi(n) \rightarrow \varphi(n+1))$$

and, since $M_{2i+1} \models \mathrm{RCA}_0$, $M_{2i+1} \models \forall n \varphi(n)$ so $M' \models \forall n \varphi(n)$. Therefore M' satisfies Σ_1^0 induction. Similarly, if T is a tree in $2^{\mathbf{N}}_{M'}$ satisfied by M' to be infinite, then, for some i, M_{2i} satisfies that T is an infinite tree in $2^{\mathbf{N}}_{M_{2i}}$ and hence, since $M_{2i} \models \mathrm{WKL}_0$, that there is a path through T. Thus M' satisfies that there exists a path through T and so $M' \models \mathrm{WKL}_0$. Finally, let φ be an arithmetic formula such that M' satisfies

$$\forall n \forall \sigma \in 2^{<\mathbf{N}}_{M'} \exists \tau \in 2^{<\mathbf{N}}_{M'} (\tau \supset \sigma \wedge \varphi(n, \tau)).$$

Let i be such that all parameters in φ appear by stage $2i+1$. Then each set $D_n = \{\sigma \mid \varphi(n, \sigma)\}$ is an M_{2i+1}-definable dense set and so $M_{2i+1} \models \exists X \in 2^{\mathbf{N}}_{M_{2i+1}}$ $\forall n (D_n \cap X \neq \varnothing)$. Therefore $M' \models \exists X \in 2^{\mathbf{N}}_{M'} \forall n (D_n \cap X \neq \varnothing)$, i.e., M' satisfies

$$\exists X \in 2^{\mathbf{N}}_{M'} \forall n \exists k \varphi(n, X[k]).$$

Thus $M' \models \mathrm{WKL}_0^+$, as desired.

COROLLARY 6.5. WKL_0^+ *is a conservative extension of* RCA_0 *with respect to* Π_1^1 *sentences; i.e., any* Π_1^1 *sentence provable in* WKL_0^+ *is provable in* RCA_0.

PROOF. Suppose $\forall X \varphi$, φ arithmetic, is not provable in RCA_0. Then $\exists X \sim \varphi$ is consistent with RCA_0 so, by Gödel's completeness theorem, there is a countable model M of $\mathrm{RCA}_0 + \exists X \sim \varphi$. By Theorem 6.4 there is a model M' of WKL_0^+ with M a submodel of M' and $|M| = |M'|$. Then $M' \models \exists X \sim \varphi$ so, by soundness, WKL_0^+ cannot prove $\forall X \varphi$.

LEMMA 6.6. RCA_0 *is a conservative extension of* PRA *with respect to* Π_2^0 *sentences.*

PROOF. Parsons (see [21]).

COROLLARY 6.7. WKL_0^+ *is a conservative extension of* PRA *with respect to* Π_2^0 *sentences.*

PROOF. The corollary is immediate from 6.5 and 6.6.

From the remarks made in the introduction we see that any Π_2^0 sentence provable in WKL_0^+ is provable in PRA, i.e., is provable finitistically. Thus the results above give a partial realization of Hilbert's program in terms of the Open Mapping and Closed Graph theorems.

COROLLARY 6.8. $\mathrm{WKL}_0 \subsetneq \mathrm{WKL}_0^+ \subsetneq \mathrm{ACA}_0$.

PROOF. By Theorem 4.4 WKL_0^+ proves B.C.T.II while, by Theorem 3.2, WKL_0 does not. Thus $\mathrm{WKL}_0 \subsetneq \mathrm{WKL}_0^+$. For the other half, we note that ACA_0 proves arithmetic induction and hence, in particular, Σ_2^0 induction. Σ_2^0 induction is not provable in RCA_0 [11] so, since Σ_2^0 induction is trivially a Π_1^1 sentence, WKL_0^+ does not prove Σ_2^0 induction by Corollary 6.5. Thus $\mathrm{WKL}_0^+ \subsetneq \mathrm{ACA}_0$.

We can sharpen the statement of Corollary 6.8 by providing ω-models which exhibit the strict inclusions. In fact, the proof of Theorem 3.2 yields an ω-model of WKL_0 which is not a model of WKL_0^+. Thus we need only show that there is an ω-model of WKL_0^+ which is not a model of ACA_0. We begin with the following lemma.

LEMMA 6.9. *Let* $M = \mathrm{Rec}$, *an* ω-*model of* RCA_0. *Then the model* M' *of Lemma* 6.2 *is not a model of* WKL_0.

PROOF. Let $S_1 = \{e \in \omega \mid \{e\}(e) = 0\}$ and $S_2 = \{e \in \omega \mid \{e\}(e) = 1\}$ be disjoint, recursively enumerable, recursively inseparable sets (Kleene [12]), and let f_1, f_2: $\mathbf{N} \to \mathbf{N}$ be 1-1 recursive functions which enumerate S_1 and S_2, respectively. Define $T \subset 2^{<\omega}$ by:

$$\sigma \in T \quad \text{iff} \quad \forall n \leq lh(\sigma)[(f_1(n) < lh(\sigma) \to \sigma(f_1(n)) = 1 \\ \wedge (f_2(n) < lh(\sigma) \to \sigma(f_2(n)) = 0].$$

Then T is an infinite tree with no recursive path (Simpson [21]) and hence has no path in M. Let X be M-generic and let M' be the model in Lemma 6.2. Then $S^{M'} = \Delta_1^0\text{-Def}(M \cup \{X\})$, so every set in $S^{M'}$ is either recursive or recursive in X, and M' is an ω-model of RCA_0. We claim that T has no path in M'. Indeed, suppose to the contrary that $f \in 2^\omega$ is a path through T and that $f \in S^{M'}$. Then f is recursive in X and hence $f = \{e\}$ for some $e \in \omega$. For each $n \in \omega$, let $D_n \subset 2^{<\omega}$ be defined by

$$\sigma \in D_n \quad \text{iff} \quad \exists k(\{n\}^\sigma[k] \notin T).$$

Clearly each D_n is M-definable. Claim each D_n is dense. Indeed, suppose not. Then, for some $n \in \omega$ and $\tau \in 2^{<\omega}$, $\forall \sigma \supset \tau\ (\sigma \notin D_n)$ so that $\forall \sigma \supset \tau\ \forall k(\{n\}^\sigma[k] \in T)$. In particular, we would have $\forall k(\{n\}^\tau[k] \in T)$, i.e., $\{n\}^\tau$ is a path through T. But $\{n\}^\tau \in M$, a contradiction. Therefore each D_n is dense. In particular D_e is dense and, since X is M-generic, there is a $\sigma \in X \cap D_e$. Let $k \in \omega$ be such that $\{e\}^\sigma \notin T$. Then $\{e\}^X \notin T$, contradicting the assumption that $\{e\}^X$ is a path through T. Therefore M' contains no path through T and it follows that M' is not a model of WKL_0.

LEMMA 6.10. *There is an* ω-*model of* RCA_0^+ *which is not a model of* WKL_0.

PROOF. Apply Lemma 6.9 repeatedly to obtain a sequence $\langle M_i \colon i \in \omega\rangle$ of ω-models such that:

(i) $M_0 = \mathrm{Rec}$;

(ii) $M_{i+1} \models \mathrm{RCA}_0 + (\exists X[D \cap X \neq \varnothing])$, for all M_i-definable dense sets D. Then, as in the proof of Theorem 6.4, $M = \bigcup_\omega M_i$ is an ω-model of RCA_0^+.

Since access to the complete recursively enumerable degree $0'$ would allow us to find a path through the tree T of Lemma 6.9 and, conversely, access to a path through T would allow us to compute $0'$, we may state the conclusion of Lemma 6.9 as follows: *adding a generic set to a model of* RCA_0 *which does not already compute* $0'$ *results in a model which still does not compute* $0'$.

A set $X \subset \omega$ is said to be *almost recursive* (or *hyperimmune free*) if, for all functions $f: \omega \to \omega$, whenever f is recursive in X (denoted by $f \leq_T X$) there is a recursive function $g: \omega \to \omega$ such that $\forall n(f(n) \leq g(n))$. More generally, given sets $X, Y \subset \omega$, we say that X is *almost recursive in* Y if, for all functions $f \leq_T X$, there is a function $g \leq_T Y$ such that $\forall n(f(n) \leq g(n))$.

LEMMA 6.11. *Let M be an ω-model of* RCA_0. *There is a model M' of* WKL_0 *such that $|M| = |M'|$ and every set in $S^{M'}$ is almost recursive in some element of S^M.*

PROOF. This is just a relativization of Theorem 2.4 of Jockusch and Soare [10].

LEMMA 6.12. *If X is almost recursive in Y and $g: \omega \to \omega$ is any function such that $g \leq_T$ (degree of Y)$'$, then $g \leq_T Y$.*

PROOF. This follows immediately from the Upward Domination lemma (pg. 53, Lerman [13]).

COROLLARY 6.13. *There is an ω-model of* WKL_0 *which does not contain* $0'$.

PROOF. Consider Rec and let M be a model of WKL_0 in which every set is almost recursive (Lemma 6.11). If $0' \in M$ then $0'$ is almost recursive. Since $0'$ is not recursive, it then follows from Lemma 6.12 that $0' <_T 0'$, a contradiction.

THEOREM 6.14. *There is an ω-model of* WKL_0^+ *which does not contain* $0'$.

PROOF. We construct a sequence $\langle M_i : i \in \omega \rangle$ of ω-models as follows:

$i = 0$. Let M_0 be the ω-model of WKL_0 of Corollary 6.13. We note that this model is obtained by adding a generic set X_0 to Rec and closing the result under Δ_1^0 comprehension (see Theorem 2.1 of [10]). Thus every set in M_0 is recursive in X_0;

$i = 1$. Apply Lemma 6.9 to obtain an ω-model M_1 of $\mathrm{RCA}_0 + (\exists X[D \cap X_1 \neq \varnothing])$, for all M_0-definable dense sets D, by adding a generic set X_1 to M_0 and closing under Δ_1^0 comprehension. Then every set in S^{M_1} is recursive in $X_0 \vee X_1$, the join of X_0 and X_1. Note that $0' \notin M_1$, so $X_0 \vee X_1 <_T 0'$;

$i = 2$. Apply Lemma 6.11 to obtain an ω-model of WKL_0 such that every set is almost recursive in some element of S^{M_1}. Since every element of S^{M_1} is recursive in $X_0 \vee X_1$, it follows that every set in M_2 is almost recursive in $X_0 \vee X_1$. Again, M_2 is obtained by adding a generic set X_2 to M_1. Suppose that $0' \in M_2$, so $0'$ is almost recursive in $X_0 \vee X_1$. Then, since $X_0 \vee X_1 <_T 0'$, it follows from Lemma 6.12 that either $0' <_T 0'$ or $(X_0 \vee X_1)' <_T 0'$; both are contradictions. Thus $0' \notin M_2$. Continuing inductively we have, in general:

(i) M_{2i+1} is an ω-model of $\mathrm{RCA}_0 + (\exists X_{2i+1}[X_{2i+1} \cap D \neq \varnothing])$ for all M_{2i}-definable dense sets D, $\forall X \in S^{M_{2i+1}}(X \leq_T \bigvee_{k=0}^{2i+1} X_k)$, and $0' \notin M_{2i+1}$;

(ii) M_{2i} is an ω-model of WKL_0 with every set in $S^{M_{2i}}$ almost recursive in $\bigvee_{k=0}^{2i+1} X_k$ and, arguing as above, $0' \notin M_{2i}$.

Finally, let $M = \bigcup_\omega M_i$. As in the proof of Theorem 6.4, it follows that M is an ω-model of WKL_0^+ and that $0' \notin M$.

COROLLARY 6.15. *There is an ω-model of* WKL_0^+ *which is not a model of* ACA_0.

PROOF. Any model of ACA_0 contains $0'$.

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