RECURSIVE ASPECTS OF DESCRIPTIVE SET THEORY

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GALEN WEITKAMP

with a chapter by Stephen Simpson

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Chapter 9

Bqo Theory and Fraïssé's Conjecture
by Stephen G. Simpson

Let L and M be linearly ordered sets. We write $L \leq M$ to mean that L is embeddable into M, *i.e.* there exists a subset of M isomorphic to L. We write $L \equiv M$ to mean that $L \leq M$ and $M \leq L$. We write $L < M$ to mean that $L \leq M$ and $M \not\leq L$. We write $L \mid M$ to mean that L and M are incomparable under embeddability, *i.e.* $L \not\leq M$ and $M \not\leq L$.

Fraïssé's conjecture [2] is the statement that, among *countable* linearly ordered sets, there are no infinite descending sequences

$$L_0 > L_1 > \ldots > L_n > \ldots (n \in \omega)$$

and no infinite antichains

$$L_i \mid L_j \qquad (i, j \in \omega, i \neq j)\,.$$

The purpose of this chapter is to explain Laver's proof [7] of Fraisse's conjecture. The proof depends heavily on Nash-Williams' theory [12] of better quasiorderings. The latter theory will be presented here as an application of a theorem of Galvin and Prikry [3] on Borel partitions.

In order to motivate Nash-Williams' concept of better quasiordering (bqo), we first discuss the closely related but simpler concept of well quasiordering (wqo).

A *quasiordered* (*i.e.* qo) *set* is a set Q endowed with a binary relation $\leq$ which is transitive ($x \leq y$, $y \leq z$ imply $x \leq z$) and reflexive ($x \leq x$ for

all $x \in Q$). For $x, y \in Q$ we write $x \equiv y$ to mean that $x \leq y$ and $y \leq x$; $x < y$ to mean that $x \leq y$ and $y \not\leq x$; and $x \mid y$ to mean that $x \not\leq y$ and $y \not\leq x$. A quasiordered set Q is said to be *well quasiordered* (wqo) if it contains no infinite descending sequence $x_0 > x_1 > \ldots > x_n > \ldots (n \in \omega)$ and no infinite antichain $x_i \mid x_j (i, j \in \omega, i \neq j)$.

Thus Fraïssé's conjecture may be rephrased as follows: the set of all countable linearly ordered sets is wqo under embeddability. An obvious strategy would be to construct an elaborate theory stating that certain large classes of qo sets are wqo. One could then hope for Fraïssé's conjecture to fall out as a corollary.

Such an elaborate wqo theory does in fact exist. One of the major theorems of of wqo theory reads as follows. Let Q be a wqo. Let $Q^{<\omega}$ be the set of finite sequences of elements of Q quasiordered by $(a_1 \ldots a_m) \leq (b_1 \ldots b_n)$ if and only if there exist $k_1 < \ldots < k_m \leq n$ such that $a_i \leq b_{k_i}$. Then $Q^{<\omega}$ is wqo. (This theorem is due to Graham Higman. For the proof, plus an excellent survey of wqo theory, see Laver [9, § 1].)

Unfortunately, it turns out that wqo theory alone is not sufficiently far-reaching to provide a proof of Fraïssé's conjecture. The difficulty is that the class of wqo sets fails to be closed under certain infinitary closure operations. For instance, let Q^{ω} be the set of ω-sequences of elements of Q quasiordered by $(a_n)_{n \in \omega} \leq (b_n)_{n \in \omega}$ if and only if there exist $n_0 < n_1 < \ldots < n_i < \ldots (i \in \omega)$ such that $a_i \leq b_{n_i}$. It is not hard to devise a wqo Q_1 such that $(Q_1)^{\omega}$ is not wqo. Namely, let $Q_1 = \{(i, j) : i < j < \omega\}$ quasiordered by $(i, j) \leq (k, l)$ if and only if either $i = k$ and $j \leq l$, or $j < k$. This counterexample is due to Richard Rado (see Laver [9, § 1]).

In order to prove Fraïssé's conjecture we need a concept stronger than well quasiordering, namely better quasiordering (bqo). It will turn out that every bqo is wqo and that every non-pathological wqo is bqo. Furthermore the class of bqos enjoys strong infinitary closure properties; for instance, if Q is bqo then Q^{ω} is bqo (indeed Q^{α} is bqo for all ordinals α). Fraïssé's conjecture will be proved by showing that the class of countable linearly ordered sets (indeed the wider class of scattered linearly ordered sets) is bqo under embeddability.

The following exercises are provided for the convenience of the reader and are not essential for the rest of the chapter.

9.1. Exercise. Verify that Q_1 is wqo but $(Q_1)^\omega$ is not wqo.

9.2. Exercise. An ω-sequence $(a_n) \in Q^\omega$ is called *bad* if $a_m \not\leq a_n$ for all n and $m < n$. Show that the following assertions are pairwise equivalent.

(*i*) Q is wqo.

(*ii*) There is no bad ω-sequence $(a_n) \in Q^\omega$.

(*iii*) For all $(a_n) \in Q^\omega$ there exist $n_0 < n_1 \ldots < n_i < \ldots (i \in \omega)$ such that $a_{n_0} \leq a_{n_1} \leq \ldots \leq a_{n_i} \leq \ldots (i \in \omega)$.

(Hint: use Ramsey's Theorem.)

9.3. Exercise.

(*i*) Show that any well-ordered set is wqo.

(*ii*) Show that if Q is the union of two subsets each wqo in the induced quasiordering, then Q is wqo.

(*iii*) Show that if Q_1 and Q_2 are wqo then $Q_1 \times Q_2$ with the product quasiordering is wqo.

(*iv*) Show that a wqo sum of wqos is wqo.

9.4. Exercise. A bad ω-sequence $(a_n) \in Q^\omega$ is called *minimal bad* if there is no bad ω-sequence $(b_n) \in Q^\omega$ such that $\forall m\, \exists n\, b_m \leq a_n$ and $\exists m\, \exists n\, b_m < a_n$. Show that if Q is well founded but not wqo then there is a minimal bad $(a_n) \in Q^\omega$.

9.5. Exercise. Use the result of the previous exercise to show that if Q

is wqo then $Q^{<\omega}$ is wqo.

Historical Note. The wqo concept occurs in the Ph.D. thesis of Irving Kaplansky [6]. One of the earliest applications of wqo theory is the following result due independently to A.I. Malcev and B.H. Neumann. Let K be a field and let G be a linearly ordered group. Then the group algebra $K(G)$ is embeddable in a skew field. For proof and references see Higman [5].

Theorem of Galvin and Prikry.

Given an infinite set $A \subseteq \omega$, we denote by $[A]^\omega$ the set of infinite subsets of A and $[A]^{<\omega}$ the set of finite subsets of A. For $s \in [\omega]^{<\omega}$ and $U \in [\omega]^\omega$ we write

$$U/s = \{n \in U \,:\, n > i \text{ for all } i \in s\}$$

and

$$[s, U] = \{X \in [\omega]^\omega \,:\, s \subseteq X \subseteq s \cup U\}.$$

We endow $[\omega]^\omega$ with the usual topology whose basic open sets are of the form $[s, \omega/s]$.

The following is a special case of the theorem of Galvin and Prikry.

9.6. Theorem. *Let $\mathcal{O}$ be an open subset of $[\omega]^\omega$. Then there exists $X \in [\omega]^\omega$ such that either $[X]^\omega \subseteq \mathcal{O}$ or $[X]^\omega \cap \mathcal{O} = \emptyset$.*

To prove this we need some special terminology. We call $[s, U]$ ***good*** if there is no $V \in [U]^\omega$ such that $[s, V] \subseteq \mathcal{O}$. We call $[s, U]$ ***strongly good*** if $[s, U]$ is good and, for all $n \in U$, $[s \cup \{n\}, U/\{n\}]$ is good.

9.7. Lemma. *If $[s, U]$ is good then there exists $V \in [U]^\omega$ such that $[s, V]$ is strongly good.*

Proof. Suppose the conclusion fails. Put $W_0 = U/s$. Assume inductively that we have chosen $n_0 < \ldots < n_{i-1} < \min(W_i)$ where $W_i \subseteq U$. Then

$[s, W_i]$ is good but not strongly good so choose $n_i \in W_i$ such that $[s \cup \{n_i\}, W_i/\{n_i\}]$ is not good. Choose $W_{i+1} \subseteq W_i/\{n_i\}$ so that $[s \cup \{n_i\}, W_{i+1}] \subseteq \mathcal{O}$. Finally put $V = \{n_i : i \in \omega\}$. Then clearly $[s, V] \subseteq \mathcal{O}$ so $[s, U]$ is not good. This proves the lemma.

We now prove the theorem. If $[U]^\omega \subseteq \mathcal{O}$ for some $U \in [\omega]^\omega$ we are done, so assume no such U exists. Hence $[\emptyset, \omega]$ is good. Put $U_0 = \omega$ and assume inductively that we have chosen $n_0 < \ldots < n_{i-1} < \min(U_i)$ such that $[s, U_i]$ is good for all $s \subseteq \{n_0, \ldots, n_{i-1}\}$. Apply the lemma 2^i times to get $V_i \subseteq U_i$ such that $[s, V_i]$ is strongly good for all $s \subseteq \{n_0, \ldots, n_{i-1}\}$. Put $n_i = \min(V_i)$ and $U_{i+1} = V_i/\{n_i\}$. Finally put $X = \{n_i : i \in \omega\}$.

We claim that $[X]^\omega \cap \mathcal{O} = \emptyset$. Suppose not. Let Y be an element of $[X]^\omega \cap \mathcal{O}$. Since $\mathcal{O}$ is open, we can find $[s, W]$ such that $Y \in [s, W] \subseteq \mathcal{O}$. Let i be such that $s \subseteq \{n_0, \ldots, n_{i-1}\}$ and $Y/s \subseteq U_i$. Then $U_i \cap W$ is infinite and $[s, U_i \cap W] \subseteq \mathcal{O}$ contradicting the goodness of $[s, U_i]$. This completes the proof. □

9.8. Remark. The same proof shows that Theorem 9.6 and Theorem 9.9 remain true if we replace the usual topology on $[\omega]^\omega$ by the Ellentuck topology with basic open sets of the form $[s, U]$ (see Ellentuck [1]).

9.9. Theorem. (Galvin, Prikry [3]). *Given $A \in [\omega]^\omega$ and a Borel set $\mathcal{B}$ in $[A]^\omega$. There exists $X \in [A]^\omega$ such that either $[X]^\omega \subseteq \mathcal{B}$ or $[X]^\omega \cap \mathcal{B} = \emptyset$.*

Proof. If $\mathcal{B}$ is open in $[A]^\omega$ then the desired conclusion follows from Theorem 9.6 since $[A]^\omega$ is homeomorphic to $[\omega]^\omega$ by a homeomorphism $h : [\omega]^\omega \to [A]^\omega$ such that $X \subseteq Y$ if and only if $h(X) \subseteq h(Y)$. It is also clear that the theorem holds for $\mathcal{B}$ if and only if it holds for $[A]^\omega - \mathcal{B}$.

It remains to show that if the theorem holds for Borel sets of rank $< \rho$ then it holds for Borel sets of rank ρ. So suppose $\mathcal{B} = \bigcup \{\mathcal{B}_i : i \in \omega\}$ where each $\mathcal{B}_i$ has smaller rank than $\mathcal{B}$. Put $A_0 = A$. Having defined A_i let $n_i = \min(A_i)$ and apply the induction hypothesis 2^{i+1} times to get

$A_{i+1} \subseteq A_i/\{n_i\}$ such that for all $s \subseteq \{n_0, \dots, n_i\}$ either $[s, A_{i+1}] \subseteq \mathcal{B}_i$ or $[s, A_{i+1}] \cap \mathcal{B}_i = \emptyset$. Finally put $Z = \{n_i : i \in \omega\}$. For each $Y \in [Z]^\omega$ we have by construction $Y \in \mathcal{B}_i$ if and only if $[s, A_{i+1}] \subseteq \mathcal{B}_i$ where $s = Y \cap \{n_0, \dots, n_i\}$. Hence, for each i, $\mathcal{B}_i \cap [Z]^\omega$ is open (in fact clopen) in $[Z]^\omega$. Hence $\mathcal{B} \cap [Z]^\omega$ is open in $[Z]^\omega$. Hence by Theorem 9.6 we can find $X \subseteq Z$ such that either $[X]^\omega \subseteq \mathcal{B}$ or $[X]^\omega \cap \mathcal{B} = \emptyset$. □

The following consequence of the Galvin-Prikry theorem will be used at a crucial point in the proof of Theorem 9.17.

9.10. Theorem. (cf. Mathias [11, § 6]). *Given $A \in [\omega]^\omega$ and a Borel measurable function $f: [A]^\omega \to X$ where X is a metric space. There exists $B \in [A]^\omega$ such that the restriction of f to $[B]^\omega$ is continuous.*

Proof. In order to prove the theorem we use the following lemma (but see Remark 9.12 below).

9.11. Lemma. *The image of f is separable.*

Proof. Suppose not. Since $\mathrm{im}(f)$ is a nonseparable metric space, it contains a closed discrete set S of power $\aleph_1$. Let $T \subseteq \mathbf{R}$ be a set of reals of power $\aleph_1$ with no perfect subset. Let $h: S \to T$ be a $1-1$ mapping of S onto T. Define $g: X \to \mathbf{R}$ by

$$g(x) = \begin{cases} h(x) & \text{if } x \in S \\ 0 & \text{otherwise.} \end{cases}$$

Since S is closed discrete, g is Borel measurable. Hence the composition $gf: [A]^\omega \to \mathbf{R}$ is Borel measurable. Hence the range of gf is analytic. But $T \subseteq \mathrm{im}(gf) \subseteq T \cup \{0\}$ so $\mathrm{im}(gf)$ is uncountable with no perfect subset. This contradicts the well known theorem that every uncountable analytic set in $\mathbf{R}$ has a perfect subset. □

9.12. Remark. Lemma 9.11 says essentially that a Borel measurable function from a complete separable metric space into a metric space has

separable image (cf. Stone [15]). We do not really need this lemma since in the application of Theorem 9.10 to be made later, $\mathsf{im}(f)$ can be assumed separable. We have included the lemma because it is interesting in its own right and not widely known. See also Louveau-Simpson [16].

We shall now prove Theorem 9.10. By Lemma 9.11 $\mathsf{im}(f)$ is separable so let $\{U_i : i \in \omega\}$ be a countable open base for the topology of $\mathsf{im}(f)$. Define $A_0 = A$. Supposing A_i has been defined, let $n_i = \min(A_i)$ and apply Theorem 9.9 2^{i+1} times to get $A_{i+1} \subseteq A_i/\{n_i\}$ such that for all $s \subseteq \{n_0, \ldots, n_i\}$ either $[s, A_{i+1}] \subseteq f^{-1}(U_i)$ or $[s, A_{i+1}] \cap f^{-1}(U_i) = \emptyset$. Finally put $B = \{n_i : i \in \omega\}$. Then for all $X \in [B]^\omega$ we have $X \in f^{-1}(U_i)$ if and only if $[s, A_{i+1}] \subseteq f^{-1}(U_i)$ where $s = X \cap \{n_0, \ldots, n_i\}$. Hence $f^{-1}(U_i) \cap [B]^\omega$ is open (in fact clopen) in $[B]^\omega$. Hence f is continuous on $[B]^\omega$. □

Better quasiordering.

Let Q be a qo set. We endow Q with the discrete topology. A *Q-array* is a Borel measurable function $f : [A]^\omega \to Q$ where $A \in [\omega]^\omega$. A Q-array $f : [A]^\omega \to Q$ is called *bad* if there is no $X \in [A]^\omega$ such that $f(X) \leq f(X/\{\min(X)\})$. A qo Q is said to be *better quasiordered (bqo)* if there is no bad Q-array. This concept is due to Nash-Williams [12].

9.13. Theorem. *If Q is* bqo *then Q is* wqo.

Proof. Suppose that Q is qo but not wqo. Then there exists an ω-sequence $(a_n)_{n\in\omega}$ of elements of Q such that $a_m \not\leq a_n$ for all n and $m < n$. Define $f : [\omega]^\omega \to Q$ by $f(X) = a_m$ where $m = \min(X)$. It is easy to check that f is a bad Q-array. □

The reader probably finds the bqo concept somewhat mysterious. For the reader's edification we provide the following exercises.

9.14. Exercise.

(*i*) Show that any well ordered set is a bqo.

(*ii*) Show that if Q is the union of two bqo subsets then Q is bqo.

(*iii*) Show that if Q_1 and Q_2 are bqo then $Q_1 \times Q_2$ is bqo.

(*iv*) Show that a bqo sum of bqos is bqo.

9.15. Exercise. Show that the following assertions are pairwise equivalent.

(*i*) Q is bqo.

(*ii*) There is no bad continuous Q-array.

(*iii*) For every Q-array $f\colon [A]^\omega \to Q$ there exists $B \in [A]^\omega$ such that $f(X) \leq f(X/\{\min(X)\})$ for all $X \in [B]^\omega$.

We shall now develop a powerful technique due to Nash-Williams [12] for proving that a given qo, Q, suspected to be bqo, is in fact bqo. Frequently such a Q comes equipped with some sort of ordinal ranking of its elements. We formalize this idea as follows.

9.16. Definition. (Laver [10]). Let Q be a qo whose quasiorder relation is denoted $\leq$. A *partial ranking* of Q is a well founded partial ordering of $\leq'$ of the elements of Q such that $x \leq' y$ implies $x \leq y$.

Now let Q be a qo which is not bqo and suppose we have in mind a particular partial ranking $\leq'$ of Q . We write $x <' y$ to mean that $x \leq' y$ and $x \neq y$. Let $f\colon [A]^\omega \to Q$ and $g\colon [B]^\omega \to Q$ be bad Q-arrays. We write $g \leq^* f$ to mean that $B \subseteq A$ and $g(X) \leq' f(X)$ for all $X \in [B]^\omega$. We write $g <^* f$ to mean that $B \subseteq A$ and $g(X) <' f(X)$ for all $X \in [B]^\omega$. (Caution: $g <^* f$ is not equivalent to the conjunction of $g \leq^* f$ and $g \neq f$.) We say that a Q-array f is *minimal bad* (with respect to the given partial ranking $\leq'$) if f is bad and there is no bad Q-array $g <^* f$.

The following theorem is essentially due to Nash-Williams [12] although it was first enunciated explicitly by Laver [10].

9.17. Theorem. *Let Q be a* qo *equipped with a partial ranking. Let $f_0 : [A_0]^\omega \to Q$ be a bad Q-array. Then there exists a minimal bad Q-array $f \leq^* f_0$.*

Proof. Assume not. Using this assumption we shall define an uncountable transfinite sequence of bad Q-arrays $f_\xi : [A_\xi]^\omega \to Q$ such that $f_\eta \leq^* f_\xi$ and $A_\eta \neq A_\xi$ for all countable ordinals $\xi < \eta < \aleph_1$. This is clearly impossible.

We begin by letting $f_0 : [A_0]^\omega \to Q$ be a bad Q-array such that there is no minimal bad $f \leq^* f_0$.

Let ξ be a countable ordinal. Assume inductively tht we have defined a bad Q-array $f_\xi \leq^* f_\gamma$ for all $\gamma < \xi$. In particular $f_\xi \leq^* f_0$ so f_ξ is not minimal bad. Let $g_\xi : [B_\xi]^\omega \to Q$ be a bad Q-array such that $g_\xi <^* f_\xi$. Use Theorem 9.10 to shrink B_ξ if necessary so that g_ξ is continuous. By further shrinking we may also assume that $A_\xi - B_\xi$ is infinite. By continuity of g_ξ there exists a nonempty initial segment s_ξ of B_ξ so that $g_\xi(X) = g_\xi(B_\xi)$ for all $X \in [s_\xi, B_\xi]$. Define

$$A_{\xi+1} = B_\xi \cup \{n \in A_\xi \,:\, n \leq \max(s_\xi)\}$$

and

$$f_{\xi+1} = \begin{cases} g_\xi(X) & \text{if } X \in [B_\xi]^\omega \\ f_\xi(X) & \text{if } X \in [A_{\xi+1}]^\omega - [B_\xi]^\omega. \end{cases}$$

Clearly $f_{\xi+1} : [A_{\xi+1}]^\omega \to Q$ is a Q-array, *i.e.* Borel measurable. Using the fact that $g_\xi(X) \leq f_\xi(X)$ for all $X \in [B_\xi]^\omega$, it is easy to check that $f_{\xi+1}$ is bad. It is also clear that $f_{\xi+1} \leq^* f_\xi$ and $A_{\xi+1} \subset A_\xi$.

Now let δ be a countable limit ordinal and assume that we have defined $f_\xi : [A_\xi]^\omega \to Q$ as above for all $\xi < \delta$. Define $A_\delta = \bigcap \{A_\xi \,:\, \xi < \delta\}$.

We claim that A_δ is infinite. Suppose for a contradiction that A_δ is finite, say $A_\delta \subseteq m < \omega$. For each $\xi < \delta$ let n_ξ be the least $n \geq m$ such

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that $n \in A_\xi$. Clearly there are infinitely many ξ such that $n_\xi \notin A_{\xi+1}$. For any such $\xi, n_\xi > \max(s_\xi)$ by the definition of $A_{\xi+1}$. Hence, for any such $\xi, m > \max(s_\xi)$ by the definition of n_ξ . Hence there are infinitely many ξ for which the s_ξ are all the same. But if $\xi < \eta$ and $s_\xi = s_\eta$, then $B_\eta \in [s_\xi, B_\xi]$, hence

$$f_\eta(B_\eta) \leq' f_{\xi+1}(B_\eta) = g_\xi(B_\eta) = g_\xi(B_\xi) <' f_\xi(B_\xi) .$$

This contradicts the well foundedness of $<'$ and so proves the claim.

We now define $f_\delta : [A_\delta]^\omega \to Q$ by $f_\delta(X) = \lim_{\xi<\delta} f_\xi(X)$. This limit exists since $\leq'$ is well founded and $f_\eta(X) \leq' f_\xi(X)$ for all $\xi < \eta < \delta$. Also f_δ is Borel measurable since it is the pointwise limit of a countable sequence of Borel measurable functions. It is also easy to check that f_δ is bad and $f_\delta \leq^* f_\xi$ for all $\xi < \delta$. This completes the proof. □

Transfinite sequences.

Let Q be a qo set. A *transfinite Q-sequence* is a function $s : \alpha \to Q$ where $\alpha = lh(s)$ is an ordinal called the *length* of s . If $\theta \leq lh(s)$ we denote by $s\restriction\theta$ the *restriction* of s to θ, *i.e.* the unique s' of length θ such that $s'(\xi) = s(\xi)$ for all $\xi < \theta$. The set of all transfinite Q-sequences is denoted $\tilde{Q}$. We quasiorder $\tilde{Q}$ by $s \leq t$ if and only if there exists a strictly order preserving $h : lh(s) \to lh(t)$ such that $s(\xi) \leq t(h(\xi))$ for all $\xi < lh(s)$. We shall prove that if Q is bqo then $\tilde{Q}$ is bqo.

9.18. Lemma. *If* $s, t, \in \tilde{Q}$ *and* $s \not\leq t$ *then there exists* $\theta < lh(s)$ *such that* $s\restriction\theta \leq t$ *and* $s\restriction\theta + 1 \not\leq t$.

Proof. Given $s \not\leq t$ define h by induction as follows. Let $h(\xi)$ be the least $\eta < lh(t)$ such that $s(\xi) \leq t(\eta)$ and $\eta > h(\xi')$ for all $\xi' < \xi$. Let θ be the least ξ such that $h(\xi)$ is undefined. Clearly $s\restriction\theta \leq t$ but $s\restriction\theta + 1 \not\leq t$. □

9.19. Theorem. (Nash-Williams [13]). *Given a bad* $\tilde{Q}$*-array* $(s_X : X \in [A]^\omega)$. *There exists* $B \in [A]^\omega$ *and a bad Q-array* $(f(X) : X \in [B]^\omega)$ *such*

that, for all $X \in [B]^\omega$, $f(X)$ is a term of the transfinite Q-sequence s_X .

Proof. For $s, t, \in \tilde{Q}$ define $s \leq' t$ to mean that s is in initial segment of t, *i.e.* $s = t\restriction\gamma$ for some $\gamma \leq lh(t)$. Clearly $\leq'$ is a partial ranking of $\tilde{Q}$. Note that if $x \in Q$ is a term of $s \leq' t$ then x is a term of t . Hence, by the minimal bad array Theorem 9.17, we may safely assume that the $\tilde{Q}$-array $(s_X : X \in [A]^\omega)$ is minimal bad.

Given $X \in [A]^\omega$ and $Y = X/\{\min(X)\}$ we have $s_X \not\leq s_Y$. By Lemma 9.18 let θ_X be such that $s_X\restriction\theta_X \leq s_Y$ and $s_X\restriction\theta_X + 1 \not\leq s_Y$. Note that

$$(s_X\restriction\theta_X : X \in [A]^\omega) <^* (s_X : X \in [A]^\omega) .$$

Hence, by minimality, there is no bad Q-array $\leq^* (s_X\restriction\theta_X : X \in [A]^\omega)$. Hence, by the Galvin-Prikry Theorem 9.9, there exists $B \in [A]^\omega$ such that $s_X\restriction\theta_X \leq s_Y\restriction\theta_Y$ for all $X \in [B]^\omega$, $Y = X/\{\min(X)\}$. Thus we have $s_X\restriction\theta_X \leq s_Y\restriction\theta_Y$ but $s_X\restriction\theta_X + 1 \not\leq s_Y\restriction\theta_Y + 1$. It follows that $s_X(\theta_X) \not\leq s_Y(\theta_Y)$. Thus $(s_X(\theta_X) : X \in [B]^\omega)$ is a bad Q-array. □

9.20. Corollary. (Nash-Williams [13]). *If Q is* bqo *then $\tilde{Q}$ is* bqo.

Proof. Immediate from the theorem. □

We mention without proof the following characterization of better quasiordering due to Pouzet [14]: Q is bqo if and only if $\tilde{Q}$ is wqo.

Proof of Fraïssé's Conjecture.

A linearly ordered set L is called *scattered* if it has no subset isomorphic to the rational numbers. We shall prove that the class of scattered linearly ordered sets is bqo under embeddability. In order to apply the method of minimal bad arrays (Theorem 9.17), we need an appropriate partial ranking. This will be provided by the following characterization of scattered sets due to Hausdorff.

Let S_0 be the class of one-point linearly ordered sets. For any ordinal $\rho > 0$ let S_ρ be the class of linearly ordered sets L such that L is isomorphic to either a well ordered sum

$$L_0 + L_1 + \cdots + L_\xi + \cdots \qquad (\xi < \alpha)$$

or a converse well ordered sum

$$\cdots + L_\xi + \cdots + L_1 + L_0 \qquad (\xi < \alpha)$$

where each L_ξ belongs to $\bigcup\{S_\pi : \pi < \rho\}$. Let $S = \bigcup\{S_\rho : \rho \text{ an ordinal}\}$.

9.21. Theorem. (Hausdorff [4]). *S is the class of scattered linearly ordered sets.*

Proof. It is easy to prove by induction on ρ that if $L \in S_\rho$ then L is scattered. Conversely, let L be scattered. For $x, y \in L$ define $x \approx y$ if and only if the nonempty interval $[x, y]$ or $[y, x]$ belongs to S. Clearly $\approx$ is a congruence relation on L. If the linearly ordered set $L/\approx$ contains more than one point, then it is densely ordered, hence L is not scattered, a contradiction. So $L/\approx$ consists of a single point, *i.e.* $x \approx y$ for all $x, y, \in L$. By considering a well ordered cofinal set and a converse well ordered coinitial set, it is now easy to see that $L \in S$. □

9.22. Theorem. (Laver [7]). *The class S of scattered linearly ordered sets is* bqo *under embeddability.*

Proof. We define the *rank* of a scattered linearly ordered set L to be the least ordinal ρ such that $L \in S_\rho$. We write $L <' M$ if and only if $L \leq M$ and $\mathsf{rank}(L) < \mathsf{rank}(M)$. We employ the partial ranking $\leq'$ of S defined by $L \leq' M$ if and only if $L <' M$ or $L = M$.

Suppose that S is not bqo. By Theorem 9.17 let $(L_X : X \in [A]^\omega)$ be a minimal bad S-array. Each L_X is either

(1) a well ordered sum of scattered sets of smaller rank, or

(2) a converse well ordered sum of scattered sets of smaller rank, or

(3) a one-point set.

By the Galvin-Prikry Theorem 9.9 we may shrink A if necessary so that all the L_X's in the array are of the same kind: (1), (2), or (3).

Clearly Case (3) does not always hold, since the array is bad. Assume that Case (1) always holds (Case (2) is similar). Thus for each $X \in [A]^\omega$ we have

$$L_X = L_X^0 + L_X^1 + \ldots + L_X^\xi + \ldots (\xi < \alpha_X)$$

where $\mathsf{rank}(L_X^\xi) < \mathsf{rank}(L_X)$, hence $L_X^\xi <' L_X$, for each $\xi < \alpha_X$. Consider the transfinite S-sequence $s_X = (L_X^\xi : \xi < \alpha_X) \in \tilde{S}$. Clearly $(s_X : X \in [A]^\omega)$ is an $\tilde{S}$-array.

We claim that $(s_X : X \in [A]^\omega)$ is bad. If not, then for some $X \in [A]^\omega$ and $Y = X/\{\min(X)\}$ we have $s_X \leq s_Y$. Hence there exists a strictly order preserving map $h : \alpha_X \to \alpha_Y$ such that $L_X^\xi \leq L_Y^{h(\xi)}$ for all $\xi < \alpha_X$. Hence $L_X \leq L_Y$. This contradicts the assumed badness of $(L_X : X \in [A]^\omega)$.

Now by the transfinite sequence Theorem 9.19 there exists $B \in [A]^\omega$ and a bad S-array $f : [B]^\omega \to S$ such that $f(X)$ is a term of s_X for all $X \in [B]^\omega$. In other words, $f(X) = L_X^{\theta_X}$ for some $\theta_X < \alpha_X$. Thus $(L_X^{\theta_X} : X \in [B]^\omega)$ is a bad S-array. But we also have

$$(L_X^{\theta_X} : X \in [B]^\omega) <^* (L_X : X \in [A]^\omega).$$

contradicting the assumed minimality of $(L_X : X \in [A]^\omega)$. This completes the proof. □

The following corollary is the solution to Fraïssé's conjecture.

9.23. Corollary. (Laver [7]). *The class of countable linearly ordered sets is* wqo *under embeddability.*

Proof. Every countable linearly ordered set is embeddable in the rationals. Hence, every countable linearly ordered set is either scattered or equivalent

sets of smaller rank,

nk A if necessary so
), (2), or (3).

array is bad. Assume
or each $X \in [A]^\omega$ we

$< \alpha_X$)
or each $\xi < \alpha_X$.
$< \alpha_X) \in \bar{S}$. Clearly

then for some $X \in$
Hence there exists a
$L_X^\xi \leq L_Y^{h(\xi)}$ for all
assumed badness of

here exists $B \in [A]^\omega$
a term of s_X for all
$\theta_X < \alpha_X$. Thus

$A]^\omega$) .
$1]^\omega$) . This completes

's conjecture.

linearly ordered sets

able in the rationals.
attered or equivalent

to the rationals under mutual embeddability. It follows from Theorem 9.22 that the countable linearly ordered sets are bqo under embeddability. Hence by Theorem 9.13 they are wqo under embeddability. □

We mention without proof one further theorem of Laver [7]. Call a linearly ordered set *σ-scattered* if it is the union of countably many scattered subsets. Then the class of σ-scattered linearly ordered sets is bqo under embeddability. The proof of this theorem uses a Hausdorff-style characterization of the σ-scattered sets due to Fred Galvin (see Laver [7]).

For other major applications of bqo theory, the reader may consult Laver [8], [10] and Nash-Williams [12].

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