

LOGICAL ANALYSIS OF SOME THEOREMS OF COMBINATORICS
AND TOPOLOGICAL DYNAMICS

by

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§1. Introduction.

Let \mathbb{N} be the set of nonnegative integers. Given $X \subseteq \mathbb{N}$ let $FS(X)$ be the set of all sums of finite nonempty subsets of X . Hindman's Theorem, HT, is the following statement.

(HT) $\left\{ \begin{array}{l} \text{If } \mathbb{N} = C_0 \cup \dots \cup C_\ell \text{ then} \\ \text{there exists an infinite set } X \subseteq \mathbb{N} \\ \text{such that } FS(X) \subseteq C_i \text{ for some } i \leq \ell. \end{array} \right.$

It is well known that all existing proofs of HT are nonconstructive. One of the goals of this paper is to delimit the degree of nonconstructivity which is inherent in Hindman's Theorem. We also discuss some related theorems from combinatorics (Carlson-Simpson) and topological dynamics (Auslander-Ellis).

Our results concerning Hindman's Theorem are of two kinds: axiomatic and recursion-theoretic. The axiomatic results provide partial answers to the following question: Which set existence axioms are sufficient and/or necessary to prove HT? The recursion-theoretic results respond to a somewhat different question. Namely, what can one say about the recursion-theoretic complexity of the homogeneous set X relative to that of the given coloring C_0, \dots, C_ℓ ?

Our recursion-theoretic work has its precedent in Jockusch's recursion-theoretic analysis of Ramsey's Theorem [17]. Regrettably, our results on Hindman's Theorem are not so complete as those of Jockusch on Ramsey's Theorem. By adapting a device of Jockusch, we prove in §2 the

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following negative result. For all $W \subseteq \mathbb{N}$ there exists a coloring $\mathbb{N} = C_0 \cup C_1$ which is recursive in W , such that for all infinite sets $X \subseteq \mathbb{N}$ and $i \in \{0,1\}$, $FS(X) \subseteq C_i$ implies that X is not recursive in $W^{(1)}$. (Here $W^{(1)}$ denotes the Turing jump of W .) We also prove a similar result with the conclusion " $W^{(1)}$ is recursive in X " in place of " X is not recursive in $W^{(1)}$ ". In §4 we obtain the following positive result. For all $W \subseteq \mathbb{N}$, if a given coloring $\mathbb{N} = C_0 \cup \dots \cup C_\ell$ is recursive in W , then there exists an infinite set $X \subseteq \mathbb{N}$ such that $FS(X) \subseteq C_i$ for some $i \leq \ell$, and X is recursive in $W^{(\omega+1)}$. (Here $W^{(\alpha)}$ denotes the α th Turing jump of W .) Thus we have lower and upper bounds $W^{(1)}$ and $W^{(\omega+1)}$ for the recursion-theoretic complexity of X . It would be desirable to narrow or close the gap between these two bounds.

There is a rather extensive literature on Hindman's Theorem. See for instance the papers by Blass [2] and Hindman [16] in this volume. There are four known proofs of Hindman's Theorem: (1) the original combinatorial proof due to Hindman [15]; (2) the simplified combinatorial proof due to Baumgartner [1]; (3) the dynamical proof due to Furstenberg and Weiss [10], [9]; and (4) the ultrafilter proof due to Glazer [12]. A convenient reference for proofs (2), (3) and (4) is the book by Graham, Rothschild and Spencer [13].

Our results in §4 are based on a somewhat delicate analysis of Hindman's original proof. This analysis yields the above-mentioned, recursion-theoretic upper bound. In axiomatic terms, the same analysis shows that Hindman's Theorem is provable in a certain formal system ACA_0^+ . Namely ACA_0^+ is the subsystem of second order arithmetic whose principal axiom asserts that arithmetical comprehension can be iterated along the natural numbers. (For information on subsystems of second order arithmetic, see [25], [5], [24], [8], [4].)

In §3 we present a somewhat similar analysis of Baumgartner's proof. This analysis yields no recursion-theoretic information beyond what is provided automatically by the Kleene Basis Theorem. However, the analysis does lead to an interesting axiomatic conclusion. Namely, Baumgartner's proof or something like it can be pushed through in the formal system $\Pi_2^1\text{-TI}_0$ (described in §3). This conclusion is interesting because it applies not only to Hindman's Theorem but also to other results which are proved by methods similar to that of Baumgartner. For instance, Theorem 6.3 of Carlson-Simpson [6] is provable in $\Pi_2^1\text{-TI}_0$. We do not know whether Theorem 6.3 of Carlson-Simpson [6] is provable in any weaker system, e.g. RCA_0 or ACA_0^+ or $\Delta_2^1\text{-TI}_0$.

Furstenberg and Weiss [10], [9] (see also [13]) have made the following very interesting observation: Hindman's Theorem can be deduced

