

LOGICAL ANALYSIS OF SOME THEOREMS OF COMBINATORICS  
AND TOPOLOGICAL DYNAMICS

by

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§1. Introduction.

Let  $\mathbb{N}$  be the set of nonnegative integers. Given  $X \subseteq \mathbb{N}$  let  $FS(X)$  be the set of all sums of finite nonempty subsets of  $X$ . Hindman's Theorem, HT, is the following statement.

(HT)  $\left\{ \begin{array}{l} \text{If } \mathbb{N} = C_0 \cup \dots \cup C_\ell \text{ then} \\ \text{there exists an infinite set } X \subseteq \mathbb{N} \\ \text{such that } FS(X) \subseteq C_i \text{ for some } i \leq \ell. \end{array} \right.$

It is well known that all existing proofs of HT are nonconstructive. One of the goals of this paper is to delimit the degree of nonconstructivity which is inherent in Hindman's Theorem. We also discuss some related theorems from combinatorics (Carlson-Simpson) and topological dynamics (Auslander-Ellis).

Our results concerning Hindman's Theorem are of two kinds: axiomatic and recursion-theoretic. The axiomatic results provide partial answers to the following question: Which set existence axioms are sufficient and/or necessary to prove HT? The recursion-theoretic results respond to a somewhat different question. Namely, what can one say about the recursion-theoretic complexity of the homogeneous set  $X$  relative to that of the given coloring  $C_0, \dots, C_\ell$ ?

Our recursion-theoretic work has its precedent in Jockusch's recursion-theoretic analysis of Ramsey's Theorem [17]. Regrettably, our results on Hindman's Theorem are not so complete as those of Jockusch on Ramsey's Theorem. By adapting a device of Jockusch, we prove in §2 the

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following negative result. For all  $W \subseteq \mathbb{N}$  there exists a coloring  $\mathbb{N} = C_0 \cup C_1$  which is recursive in  $W$ , such that for all infinite sets  $X \subseteq \mathbb{N}$  and  $i \in \{0,1\}$ ,  $FS(X) \subseteq C_i$  implies that  $X$  is not recursive in  $W^{(1)}$ . (Here  $W^{(1)}$  denotes the Turing jump of  $W$ .) We also prove a similar result with the conclusion " $W^{(1)}$  is recursive in  $X$ " in place of " $X$  is not recursive in  $W^{(1)}$ ". In §4 we obtain the following positive result. For all  $W \subseteq \mathbb{N}$ , if a given coloring  $\mathbb{N} = C_0 \cup \dots \cup C_\ell$  is recursive in  $W$ , then there exists an infinite set  $X \subseteq \mathbb{N}$  such that  $FS(X) \subseteq C_i$  for some  $i \leq \ell$ , and  $X$  is recursive in  $W^{(\omega+1)}$ . (Here  $W^{(\alpha)}$  denotes the  $\alpha$ th Turing jump of  $W$ .) Thus we have lower and upper bounds  $W^{(1)}$  and  $W^{(\omega+1)}$  for the recursion-theoretic complexity of  $X$ . It would be desirable to narrow or close the gap between these two bounds.

There is a rather extensive literature on Hindman's Theorem. See for instance the papers by Blass [2] and Hindman [16] in this volume. There are four known proofs of Hindman's Theorem: (1) the original combinatorial proof due to Hindman [15]; (2) the simplified combinatorial proof due to Baumgartner [1]; (3) the dynamical proof due to Furstenberg and Weiss [10], [9]; and (4) the ultrafilter proof due to Glazer [12]. A convenient reference for proofs (2), (3) and (4) is the book by Graham, Rothschild and Spencer [13].

Our results in §4 are based on a somewhat delicate analysis of Hindman's original proof. This analysis yields the above-mentioned, recursion-theoretic upper bound. In axiomatic terms, the same analysis shows that Hindman's Theorem is provable in a certain formal system  $ACA_0^+$ . Namely  $ACA_0^+$  is the subsystem of second order arithmetic whose principal axiom asserts that arithmetical comprehension can be iterated along the natural numbers. (For information on subsystems of second order arithmetic, see [25], [5], [24], [8], [4].)

In §3 we present a somewhat similar analysis of Baumgartner's proof. This analysis yields no recursion-theoretic information beyond what is provided automatically by the Kleene Basis Theorem. However, the analysis does lead to an interesting axiomatic conclusion. Namely, Baumgartner's proof or something like it can be pushed through in the formal system  $\Pi_2^1-TI_0$  (described in §3). This conclusion is interesting because it applies not only to Hindman's Theorem but also to other results which are proved by methods similar to that of Baumgartner. For instance, Theorem 6.3 of Carlson-Simpson [6] is provable in  $\Pi_2^1-TI_0$ . We do not know whether Theorem 6.3 of Carlson-Simpson [6] is provable in any weaker system, e.g.  $RCA_0$  or  $ACA_0^+$  or  $\Delta_2^1-TI_0$ .

Furstenberg and Weiss [10], [9] (see also [13]) have made the following very interesting observation: Hindman's Theorem can be deduced

rather easily from a theorem of topological dynamics due to Auslander and Ellis. The Auslander-Ellis Theorem, AET, reads as follows.

(AET)  $\left\{ \begin{array}{l} \text{Let } X \text{ be a compact metric space} \\ \text{and let } T: X \rightarrow X \text{ be continuous.} \\ \text{Regard } (X, \langle T^n \rangle_{n \in \mathbb{N}}) \text{ as a dynamical} \\ \text{system. Given } x \in X, \text{ there exists} \\ y \in X \text{ such that } y \text{ is uniformly} \\ \text{recurrent and proximal to } x. \end{array} \right.$

For an explanation of the notions of uniform recurrence and proximality, see e.g. [9] or [13] or §5 below.

The purpose of §5 is to present an axiomatic analysis of AET. The classical proof of AET is extremely nonconstructive, relying as it does on Zorn's Lemma applied to the partial ordering by inclusion of the closed subsets of the nonmetrizable Tychonoff product space  $X^X$ . (See the discussion of the "enveloping semigroup" on page 159 of [9] or page 143 of [13].) It is not at all obvious that this classical proof or anything like it can be carried out within full second order arithmetic. In §5 we present an apparently new proof of AET in which Hindman's Theorem is used as a lemma. We show that all parts of the new proof, except possibly the applications of Hindman's Theorem, can be pushed through in  $ACA_0$ . Combining this with a result from §4, we conclude: AET is provable in  $ACA_0^+$ . Thus our proof of AET is much closer to being constructive than is the classical proof.

## §2. Strong recursive counterexample to Hindman's Theorem.

Given  $X, W \subseteq \mathbb{N}$  we say that  $X$  is recursive in  $W$  if the characteristic function of  $X$  is computable by a Turing machine using an oracle for the characteristic function of  $W$ . We use  $W^{(1)}$  to denote the Turing jump of  $W$ . In particular  $\emptyset^{(1)}$  is the Turing jump of the empty set, i.e. the complete recursively enumerable subset of  $\mathbb{N}$ . Thus

$\mathcal{P}^{(1)}$  has the same degree of unsolvability as the Halting Problem. The recursion-theoretic notions which we use are explained in Rogers [21].

The purpose of this section is to prove the following theorems.

2.1. Theorem. There exists a recursive coloring  $N = C_0 \cup C_1$  such that for all infinite  $X \subseteq N$ , if  $FS(X) \subseteq C_i$  for some  $i \in \{0,1\}$ , then  $X$  is not recursive in  $\mathcal{P}^{(1)}$ .

2.2. Theorem. There exists a recursive coloring  $N = C_0 \cup C_1$  such that, for all infinite  $X \subseteq N$ , if  $FS(X) \subseteq C_i$  for some  $i \in \{0,1\}$ , then  $\mathcal{P}^{(1)}$  is recursive in  $X$ .

Proof of Theorem 2.1. We imitate the proof of Theorem 3.1 of Jockusch [17].

For  $A \subseteq N$  let  $c_A: N \rightarrow \{0,1\}$  be the characteristic function of  $A$ . By Theorem 2 of Shoenfield [22] there exists a recursive function  $f: N^3 \rightarrow N$  with the following property. For all  $A \subseteq N$ ,  $A$  is recursive in  $\mathcal{P}^{(1)}$  if and only if, for some  $j$ ,  $c_A(u) = \lim_s f(j, u, s)$  for all  $u \in N$ . Let us write  $A = A_j$  in this case.

If  $A_j$  is defined and has at least  $2j + 2$  elements, let  $D_j$  consist of the smallest  $2j + 2$  elements of  $A_j$ . Otherwise let  $D_j$  be undefined. We shall now define a finite set  $D_j^s$  to approximate  $D_j$  at stage  $s$ . If there are at least  $2j + 2$  numbers  $u$  such that  $u \leq s$  and  $f(j, u, s) = 1$ , let  $D_j^s$  consist of the smallest  $2j + 2$  such numbers. Otherwise let  $D_j^s$  be undefined.

Given  $n \geq 1$  let us write  $\lambda(n) = n_1$  and  $\mu(n) = n_k$  where  $n = 2^{n_1} + \dots + 2^{n_k}$ ,  $n_1 < \dots < n_k$ . Note that  $\lambda(m+n) = \lambda(m)$  and  $\mu(m+n) = \mu(n)$  provided  $\mu(m) < \lambda(n)$ .

The recursive coloring  $N = C_0 \cup C_1$  will be constructed in stages. At stage  $s$  of the construction we shall place each of the finitely many numbers  $n$  with  $\mu(n) = s$  into exactly one of the color classes  $C_0$  and  $C_1$ .

Stage  $s$ . By induction on  $j \leq s$ , let  $u_j^s$  and  $v_j^s$  be two effectively chosen numbers which are different from each other and from all  $u_i^s$  and  $v_i^s$ ,  $i < j$ , and which belong to  $D_j^s$  if  $D_j^s$  is defined. This can be done since  $|D_j^s| = 2j + 2$  if  $D_j^s$  is defined. Now for all  $n$  such that  $\mu(n) = s$ , put  $n \in C_0$  if  $\lambda(n) = u_j^s$  for some  $j \leq s$ , otherwise  $n \in C_1$ .

This completes the construction. Clearly  $C_0$  and  $C_1$  are recursive.

Let  $X$  be an infinite set such that  $FS(X) \subseteq C_0$  or  $FS(X) \subseteq C_1$ . We claim that  $X$  is not recursive in  $\emptyset^{(1)}$ . To see this we first let  $Y$  be an infinite set such that  $Y$  is recursive in  $X$ ,  $FS(Y) \subseteq FS(X)$ , and  $\mu(m) < \lambda(n)$  for all  $m \in Y$ ,  $n \in Y$ ,  $m < n$ . (See Lemma 4.1 below.) Put  $Z = \{\lambda(n) : n \in Y\}$ . Suppose that  $X$  is recursive in  $\emptyset^{(1)}$ . Then  $Z$  is recursive in  $\emptyset^{(1)}$  so let  $j$  be such that  $Z = A_j$ . Since  $Z$  is infinite,  $D_j$  is defined and  $D_j \subseteq Z$ . Choose  $n \in Y$  so large that  $\max(D_j) < \lambda(n)$  and  $D_j^s = D_j$  where  $s = \mu(n)$ . Then  $u_j^s$  and  $v_j^s$  are distinct elements of  $D_j^s = D_j \subseteq Z$ . Let  $m_0, m_1 \in Y$  be such that  $\lambda(m_0) = u_j^s$  and  $\lambda(m_1) = v_j^s$ . Then  $\max(\mu(m_0), \mu(m_1)) < \lambda(n)$ , hence  $m_0 + n, m_1 + n \in FS(Y) \subseteq FS(X)$  and  $m_0 + n \in C_0$ ,  $m_1 + n \in C_1$ . This contradiction completes the proof.

Proof of Theorem 2.2. We view each  $n \in N$  as a code for the finite set  $\{n_1, \dots, n_k\}$ , where  $n = 2^{n_1} + \dots + 2^{n_k}$  and  $n_1 < \dots < n_k$ .

Accordingly, we refer to the pairs  $(n_i, n_{i+1})$ ,  $i = 1, \dots, k-1$ , as the gaps of  $n$ . Notice that if  $\mu(m) < \lambda(n)$ , where  $\mu$  and  $\lambda$  are as in the proof of Theorem 2.1, then the gaps of  $m+n$  are those of  $m$ , those of  $n$ , and the pair  $(\mu(m), \lambda(n))$ .

Fix a recursive algorithm enumerating the r.e. set  $\mathcal{P}^{(1)}$ ; let  $\mathcal{P}^{(1)}(k)$  be the finite subset of  $\mathcal{P}^{(1)}$  enumerated by this algorithm in its first  $k$  computation steps. For any  $n \in \mathbb{N}$  and any gap  $(a, b)$  of  $n$ , we say that  $(a, b)$  is a short gap of  $n$  if there exists  $x \leq a$  such that  $x \in \mathcal{P}^{(1)}$  but  $x \notin \mathcal{P}^{(1)}(b)$ . We say that  $(a, b)$  is a very short gap of  $n$  if there is  $x \leq a$  such that  $x \in \mathcal{P}^{(1)}(\mu(n))$  but  $x \notin \mathcal{P}^{(1)}(b)$ . Let  $SG(n)$  (resp.  $VSG(n)$ ) be the number of short (resp. very short) gaps of  $n$ . Observe that, given  $n$ , one can effectively compute  $VSG(n)$  (but not  $SG(n)$ ). Thus, the following coloring  $\mathbb{N} = C_0 \cup C_1$  is a recursive one.

$$C_i = \{n \in \mathbb{N} \mid VSG(n) \equiv i \pmod{2}\}.$$

Suppose  $X$  is an infinite set with  $FS(X) \subseteq C_i$  for some  $i \in \{0, 1\}$ . We shall show that  $\mathcal{P}^{(1)}$  is recursive in  $X$ .

As in the proof of Theorem 2.1, we first use Lemma 4.1 to find an infinite  $Y$  such that  $Y$  is recursive in  $X$ ,  $FS(Y) \subseteq FS(X) \subseteq C_i$ , and  $\mu(m) < \lambda(n)$  for all  $m < n$  in  $Y$ . It suffices to show that  $\mathcal{P}^{(1)}$  is recursive in  $Y$ .

Claim 1. For every  $m \in FS(Y)$ ,  $SG(m)$  is even.

Proof. Let  $m \in FS(Y)$  be given, and choose an  $n \in Y$  so large that, for all  $x \leq \mu(m)$ , if  $x \in \mathcal{P}^{(1)}$  then  $x \in \mathcal{P}^{(1)}(\lambda(n))$ . This can be done because every such  $x$  is in  $\mathcal{P}^{(1)}(k)$  for some  $k$  and  $\lambda$  is strictly increasing on  $Y$ . We compute the number  $VSG(m+n)$  of very short gaps of  $m+n$  by considering separately the gaps of  $m$ , the gaps of  $n$ ,

and the gap  $(\mu(m), \lambda(n))$ . The last of these is not very short, by our choice of  $n$ . A gap of  $n$  is very short in  $m+n$  if and only if it is very short in  $n$ , because  $\mu(m+n) = \mu(n)$ . A gap  $(a, b)$  of  $m$  is very short in  $m+n$  if and only if it is short (not necessarily very short) as a gap of  $m$ , because, for  $x \leq a < \mu(m)$ , our choice of  $n$  ensures that  $x \in \wp^{(1)}$  if and only if  $x \in \wp^{(1)}(\lambda(n))$  if and only if  $x \in \wp^{(1)}(\mu(m+n))$ . Therefore,

$$\text{VSG}(m+n) = \text{SG}(m) + \text{VSG}(n).$$

By our choice of  $Y$ , the two VSG terms have the same parity. So the other term,  $\text{SG}(m)$ , must be even, and the claim is proved.

Claim 2. Assume that  $m < n$  are in  $Y$  and that  $x \leq \mu(m)$ . Then  $x \in \wp^{(1)}$  if and only if  $x \in \wp^{(1)}(\lambda(n))$ .

Proof. The "if" part is trivial, and the "only if" asserts that the gap  $(\mu(m), \lambda(n))$  of  $m+n$  is not short. Suppose, toward a contradiction, that it were short. Then the short gaps of  $m+n$  would be those of  $m$ , those of  $n$ , and  $(\mu(m), \lambda(n))$ . Thus, we would have

$$\text{SG}(m+n) = \text{SG}(m) + \text{SG}(n) + 1,$$

which contradicts Claim 1. Thus, Claim 2 is proved.

We can now complete the proof of the theorem by giving an algorithm, with an oracle for  $Y$ , that computes membership in  $\wp^{(1)}$ . Given an input  $x$ , use the oracle to find an  $m \in Y$  with  $x \leq \mu(m)$  and to find an  $n \in Y$  with  $m < n$ . Then run the algorithm enumerating  $\wp^{(1)}$  for  $\lambda(n)$  steps to decide whether  $x \in \wp^{(1)}(\lambda(n))$ . By Claim 2, this also decides whether  $x \in \wp^{(1)}$ .

2.3. Remark. Brackin [3] has proved a weaker version of Theorem 2.1 in which "X is not recursive in  $\wp^{(1)}$ " is replaced by "X is not

recursive". As a consequence, he also obtained a weaker version of Theorem 2.6 asserting that HT is not provable in  $RCA_0$ .

It is straightforward to generalize Theorems 2.1 and 2.2 as follows.

2.4. Theorem. Given  $W \subseteq \mathbb{N}$ , there exists a coloring  $N = C_0 \cup C_1$  with the following properties.  $C_0$  and  $C_1$  are recursive in  $W$  and, for all infinite  $X \subseteq \mathbb{N}$ , if  $FS(X) \subseteq C_i$  for some  $i \in \{0,1\}$ , then  $X$  is not recursive in  $W^{(1)}$ .

2.5 Theorem. For any  $W \subseteq \mathbb{N}$ , there exists a coloring  $N = C_0 \cup C_1$  with the following properties.  $C_0$  and  $C_1$  are recursive in  $W$  and, for all infinite  $X \subseteq \mathbb{N}$ , if  $FS(X) \subseteq C_i$  for some  $i \in \{0,1\}$ , then  $W^{(1)}$  is recursive in  $X$ .

The proof of Theorem 2.5 can be modified to yield a result concerning the set existence axioms which are needed to prove Hindman's Theorem. By  $RCA_0$  (respectively  $ACA_0$ ) we mean the subsystem of second order arithmetic with restricted induction and recursive (respectively arithmetical) comprehension [25,24,5]. It is well known that  $ACA_0$  can be obtained from  $RCA_0$  by adding an axiom asserting the existence of Turing jumps. An inspection of the proof of Theorem 2.5 shows that this proof goes through in  $RCA_0$ . Hence we have the following axiomatic result.

2.6. Theorem ( $RCA_0$ ). Hindman's Theorem HT implies  $ACA_0$ .

In other words, no set existence axioms weaker than those of  $ACA_0$  can suffice to prove Hindman's Theorem.

### §3. Analysis of Baumgartner's proof

The purpose of this section is to present an axiomatic analysis of Baumgartner's proof [1] of Hindman's Theorem. We show that a version of Baumgartner's proof can be carried out within a certain formal system  $\Pi_2^1 - TI_0$  (to be described below).

The reader of this section is assumed to have some familiarity with subsystems of second order arithmetic [25,7,23]. We also assume that the reader has access to Baumgartner's proof as presented on pages 69 through 71 of [13].

The following notions are basic to Baumgartner's proof. Let  $E$  and  $F$  be nonempty finite subsets of  $\mathbb{N}$ . We write  $E < F$  if  $\max(E) < \min(F)$ . A disjoint collection is an infinite collection  $\mathcal{D} = \{D_n : n \in \mathbb{N}\}$  of nonempty finite subsets of  $\mathbb{N}$  such that  $D_n < D_{n+1}$  for all  $n \in \mathbb{N}$ . From now on  $\mathcal{D}$  denotes a disjoint collection. We use  $FU(\mathcal{D})$  to denote the set of all unions of nonempty finite subcollections of  $\mathcal{D}$ . We write  $\mathcal{D}' \leq \mathcal{D}$  to mean that  $\mathcal{D}'$  is a disjoint collection and  $\mathcal{D}' \subset FU(\mathcal{D})$ . We say that  $C$  is large for  $\mathcal{D}$  if  $C \cap FU(\mathcal{D}') \neq \emptyset$  for all  $\mathcal{D}' \leq \mathcal{D}$ . We write

$$C/E = \{F \in C : E < F\}.$$

Consider the following statement HTU which is a version of Hindman's Theorem (see Corollary 3.3 of [15]).

$$(HTU) \left\{ \begin{array}{l} \text{If } FU(\mathcal{D}) = C_0 \cup \dots \cup C_\ell \text{ then} \\ \text{there exists } \mathcal{D}' \leq \mathcal{D} \text{ such that} \\ FU(\mathcal{D}') \subset C_i \text{ for some } i \leq \ell. \end{array} \right.$$

Hindman's Theorem HT follows immediately from HTU by taking  $\mathcal{D} = \{\{n\} : n \in \mathbb{N}\}$  and

$$C_i = \{D \in FU(\mathcal{D}) : \sum\{2^m : m \in D\} \in C_i\}$$

for all  $i \leq \ell$ . Baumgartner's method for proving Hindman's Theorem is to prove HTU by means of a sequence of lemmas involving the notion of largeness. We now present our axiomatic analysis of Baumgartner's proof.

We use  $Z_2$  to denote the formal system of second order arithmetic. Recall [25,24,5] that  $RCA_0$  (respectively  $ACA_0$ ) are the subsystems of  $Z_2$  with restricted induction and recursive (respectively arithmetical) comprehension.

An  $\omega$ -model is a set  $M \subseteq P(\mathbb{N}) = \{X: X \subseteq \mathbb{N}\}$  regarded as a model for  $L_2$ , the language of  $Z_2$ . If  $\varphi$  is a sentence of  $L_2$  with parameters from  $M$ , we say that  $M$  satisfies  $\varphi$  if  $\varphi$  is true when the set variables range over  $M$ , the number variables range over  $\mathbb{N}$ , and the remaining symbols have their standard interpretation.

A  $\beta$ -model is an  $\omega$ -model  $M$  such that, for all  $\Sigma_1^1$  sentences  $\varphi$  with parameters from  $M$ ,  $\varphi$  is true if and only if  $M$  satisfies  $\varphi$ . A countable coded  $\omega$ -model is a set  $Z \subseteq \mathbb{N}$  viewed as (a code for) the  $\omega$ -model  $M = \{(Z)_n: n \in \mathbb{N}\}$  where  $(Z)_n = \{m: (m,n) \in Z\}$ ,  $(m,n) = \frac{1}{2}(m+n)(m+n+1)+m$ . A countable coded  $\beta$ -model is a countable coded  $\omega$ -model which is also a  $\beta$ -model. We assume that the notion of countable coded  $\beta$ -model has been defined formally within  $RCA_0$  as in §VII.2 of [25]. (In particular a countable coded  $\beta$ -model comes equipped with a satisfaction predicate for a "universal lightface  $\Sigma_1^1$  formula.")

We shall now state four lemmas which are modeled on Lemmas 17, 20 and 21 and Theorem 18 on pages 70 and 71 of [13]. The proofs of our lemmas are obtained by straightforward adaptation of the original proofs.

3.1. Lemma. The following is provable in  $RCA_0$ . Let  $M$  be a countable coded  $\beta$ -model. Suppose that  $C_0, \dots, C_\ell, D \in M$  and that  $C_0 \cup \dots \cup C_\ell$  is large for  $D$ . Then some  $C_i, i \leq \ell$  is large for some  $D' \leq D, D' \in M$ .

3.2. Lemma. The following is provable in  $RCA_0$ . Let  $M$  be a countable coded  $\beta$ -model. Suppose that  $C, D \in M$  and that  $C$  is large for  $D$ . Then there exists  $E \in FU(D)$  such that

$$C' = \{F \in C/E: E \cup F \in C\}$$

is large for some  $D' \leq D/E, D' \in M$ .

3.3. Lemma. The following is provable in  $RCA_0$ . Let  $M$  be a countable coded  $\beta$ -model. Suppose that  $C, D \in M$  and that  $C$  is large for  $D$ . Then there exists  $E \in C \cap FU(D)$  such that

$$C' = \{F \in C/E: E \cup F \in C\}$$

is large for some  $D' \leq D/E, D' \in M$ .

3.4. Lemma. The following is provable in  $RCA_0$ . Let  $M$  be a countable coded  $\beta$ -model. Suppose that  $C, D \in M$  and that  $C$  is large for  $D$ . Then there exists  $D' \leq D, D' \in M$  such that  $FU(D') \subset C$ .

As an immediate consequence of Lemmas 3.1 and 3.4, we have:

3.5. Lemma. The following is provable in  $RCA_0$ . Let  $M$  be a countable coded  $\beta$ -model. Then  $M$  satisfies HTU. Hence  $M$  satisfies Hindman's Theorem HT.

Recall that  $\Pi_1^1\text{-CA}_0$  is the subsystem of  $Z_2$  with restricted induction and  $\Pi_1^1$  comprehension. The following proposition, essentially due to Friedman [7], is also proved in §VII.2 of [25].

3.6. Proposition. The following is provable in  $\Pi_1^1\text{-CA}_0$ . For all  $X \subset \mathbb{N}$  there exists a countable coded  $\beta$ -model  $M$  such that  $X \in M$ .

Combining this with Lemma 3.5 we obtain:

3.7. Theorem. Hindman's Theorem HT is provable in  $\Pi_1^1\text{-CA}_0$ .

We now describe the formal systems  $\Pi_k^1\text{-TI}_0$ ,  $k \in \mathbb{N}$ . Let  $\text{WO}(X)$  stand for the  $\Pi_1^1$  formula which asserts that  $X$  is a code for a countable well ordering. If  $\psi(n)$  is any  $L_2$ -formula, let  $\text{TI}(X, \psi)$  stand for the formula

$$\forall n(\forall m(m <_X n \rightarrow \psi(m)) \rightarrow \psi(n)) \rightarrow \forall n\psi(n).$$

This expresses transfinite induction along  $X$  with respect to  $\psi(n)$ . We define  $\Pi_k^1\text{-TI}_0$  to be the subsystem of  $Z_2$  whose axioms are those of  $\text{ACA}_0$  plus all formulas of the form

$$\text{WO}(X) \rightarrow \text{TI}(X, \psi)$$

where  $\psi(n)$  is  $\Pi_k^1$ . Also

$$\Pi_\infty^1\text{-TI}_0 = \bigcup_{k=0}^{\infty} \Pi_k^1\text{-TI}_0.$$

The following proposition, due to Simpson [25] §VII.2, is related to

some unpublished work of Harrington [14].

3.8. Proposition. Let  $\varphi$  be any  $\Pi_2^1$  sentence. The following assertions are equivalent: (i)  $\Pi_\infty^1\text{-TI}_0$  proves  $\varphi$ ; (ii)  $\text{ACA}_0$  proves that any countable coded  $\beta$ -model satisfies  $\varphi$ .

Since Hindman's Theorem HT is a  $\Pi_2^1$  sentence, we can combine Proposition 3.8 with Lemma 3.5 to obtain:

3.9. Theorem. Hindman's Theorem HT is provable in  $\Pi_\infty^1\text{-TI}_0$ .

In order to sharpen the previous result, we apply the following proposition due to Simpson [25]. For the proof see the exercises at the end of §VII.2 of [25].

3.10. Proposition. Let  $\varphi$  be any sentence of the form  $\forall X\exists Y\theta$  where  $\theta$  is  $\Pi_2^0$ . The following assertions are equivalent: (i)  $\Pi_2^1\text{-TI}_0$  proves  $\varphi$ ; (ii)  $\text{RCA}_0$  proves that any countable coded  $\beta$ -model satisfies  $\varphi$ .

Since Hindman's Theorem is of the required form, we obtain:

3.11. Theorem. Hindman's Theorem HT is provable in  $\Pi_2^1\text{-TI}_0$ .

We now consider a result of Carlson and Simpson [6]. Let  $A$  be a fixed finite alphabet. Let  $A^*$  be the set of finite words over  $A$ , i.e. finite sequences of elements of  $A$ . An infinite variable word is an infinite sequence  $W$  of elements of the disjoint union  $A \cup \{x_i : i \in \mathbb{N}\}$  in which each  $x_i$  occurs at least once, each  $x_i$  occurs only finitely many times, and for each  $i$  the last occurrence of  $x_i$  comes before the first occurrence of  $x_{i+1}$ . The  $x_i$ 's are to be regarded as variables

ranging over  $A$ . Given  $s = a_0 a_1 \dots a_{j-1} \in A^*$ , let  $W(s)$  be the element of  $A^*$  which results from  $W$  by first substituting  $a_i$  for each occurrence of  $x_i$ ,  $i < j$ , then truncating just before the first occurrence of  $x_j$ . Put

$$W(A^*) = \{W(s) : s \in A^*\}.$$

Theorem 6.3 of Carlson-Simpson [6] asserts the following:

(CST)  $\left\{ \begin{array}{l} \text{If } A^* = C_0 \cup \dots \cup C_{\ell-1} \text{ then there} \\ \text{exists an infinite variable word } W \\ \text{such that } W(A^*) \subset C_i \text{ for some } i < \ell. \end{array} \right.$

The proof of CST in [6] is broadly similar to Baumgartner's proof of Hindman's Theorem. Our analysis in terms of countable coded  $\beta$ -models can be adapted so as to apply to the proof of CST. In this way we obtain:

3.12. Theorem. Theorem 6.3 of Carlson-Simpson [6], i.e. CST, is provable in  $\Pi_2^1\text{-TI}_0$ .

We do not know whether CST is provable in any weaker system such as  $\text{RCA}_0$ ,  $\text{ACA}_0^+$ , or  $\Delta_2^1\text{-TI}_0$ .

#### §4. Analysis of Hindman's Proof.

The purpose of this section is to present an analysis of Hindman's original proof of Hindman's Theorem. This yields both recursion-theoretic and axiomatic information. Our main recursion-theoretic result is as follows: Given a coloring  $N = C_0 \cup \dots \cup C_\ell$ , there exists an infinite set  $X \subset N$  such that  $\text{FS}(X) \subset C_i$  for some  $i \leq \ell$ , and  $X$  is recursive in the  $(\omega+1)$ st Turing jump of the given coloring. Axiomatically, our result is that Hindman's Theorem HT is provable in a certain formal system  $\text{ACA}_0^+$  (to be described