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LOGICAL ANALYSIS OF SOME THEOREMS OF COMBINATORICS AND TOPOLOGICAL DYNAMICS

by

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§1. Introduction.

Let $\mathbb{N}$ be the set of nonnegative integers. Given $X \subseteq \mathbb{N}$ let $FS(X)$ be the set of all sums of finite nonempty subsets of X. Hindman's Theorem, HT, is the following statement.

$$(\mathrm{HT})\begin{cases}\underline{\text{If}}\ \ \mathbb{N} = C_0 \cup \ldots \cup C_\ell\ \ \underline{\text{then}} \\ \underline{\text{there exists an infinite set}}\ \ X \subseteq \mathbb{N} \\ \underline{\text{such that}}\ \ FS(X) \subseteq C_i\ \ \underline{\text{for some}}\ \ i \leq \ell .\end{cases}$$

It is well known that all existing proofs of HT are nonconstructive. One of the goals of this paper is to delimit the degree of nonconstructivity which is inherent in Hindman's Theorem. We also discuss some related theorems from combinatorics (Carlson-Simpson) and topological dynamics (Auslander-Ellis).

Our results concerning Hindman's Theorem are of two kinds: axiomatic and recursion-theoretic. The axiomatic results provide partial answers to the following question: Which set existence axioms are sufficient and/or necessary to prove HT? The recursion-theoretic results respond to a somewhat different question. Namely, what can one say about the recursion-theoretic complexity of the homogeneous set X relative to that of the given coloring $C_0,\ldots,C_\ell$?

Our recursion-theoretic work has its precedent in Jockusch's recursion-theoretic analysis of Ramsey's Theorem [17]. Regrettably, our results on Hindman's Theorem are not so complete as those of Jockusch on Ramsey's Theorem. By adapting a device of Jockusch, we prove in §2 the

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following negative result. For all $W \subseteq \mathbb{N}$ there exists a coloring $\mathbb{N} = C_0 \cup C_1$ which is recursive in W, such that for all infinite sets $X \subseteq \mathbb{N}$ and $i \in \{0,1\}$, $FS(X) \subseteq C_i$ implies that X is not recursive in $W^{(1)}$. (Here $W^{(1)}$ denotes the Turing jump of W.) We also prove a similar result with the conclusion "$W^{(1)}$ is recursive in X" in place of "X is not recursive in $W^{(1)}$". In §4 we obtain the following positive result. For all $W \subseteq \mathbb{N}$, if a given coloring $\mathbb{N} = C_0 \cup \ldots \cup C_\ell$ is recursive in W, then there exists an infinite set $X \subseteq \mathbb{N}$ such that $FS(X) \subseteq C_i$ for some $i \leq \ell$, and X is recursive in $W^{(\omega+1)}$. (Here $W^{(\alpha)}$ denotes the αth Turing jump of W.) Thus we have lower and upper bounds $W^{(1)}$ and $W^{(\omega+1)}$ for the recursion-theoretic complexity of X. It would be desirable to narrow or close the gap between these two bounds.

There is a rather extensive literature on Hindman's Theorem. See for instance the papers by Blass [2] and Hindman [16] in this volume. There are four known proofs of Hindman's Theorem: (1) the original combinatorial proof due to Hindman [15]; (2) the simplified combinatorial proof due to Baumgartner [1]; (3) the dynamical proof due to Furstenberg and Weiss [10], [9]; and (4) the ultrafilter proof due to Glazer [12]. A convenient reference for proofs (2), (3) and (4) is the book by Graham, Rothschild and Spencer [13].

Our results in §4 are based on a somewhat delicate analysis of Hindman's original proof. This analysis yields the above-mentioned, recursion-theoretic upper bound. In axiomatic terms, the same analysis shows that Hindman's Theorem is provable in a certain formal system ACA_0^+. Namely ACA_0^+ is the subsystem of second order arithmetic whose principal axiom asserts that arithmetical comprehension can be iterated along the natural numbers. (For information on subsystems of second order arithmetic, see [25], [5], [24], [8], [4].)

In §3 we present a somewhat similar analysis of Baumgartner's proof. This analysis yields no recursion-theoretic information beyond what is provided automatically by the Kleene Basis Theorem. However, the analysis does lead to an interesting axiomatic conclusion. Namely, Baumgartner's proof or something like it can be pushed through in the formal system $\Pi_2^1\text{-}TI_0$ (described in §3). This conclusion is interesting because it applies not only to Hindman's Theorem but also to other results which are proved by methods similar to that of Baumgartner. For instance, Theorem 6.3 of Carlson-Simpson [6] is provable in $\Pi_2^1\text{-}TI_0$. We do not know whether Theorem 6.3 of Carlson-Simpson [6] is provable in any weaker system, e.g. RCA_0 or ACA_0^+ or $\Delta_2^1\text{-}TI_0$.

Furstenberg and Weiss [10], [9] (see also [13]) have made the following very interesting observation: Hindman's Theorem can be deduced rather easily from a theorem of topological dynamics due to Auslander and Ellis. The Auslander-Ellis Theorem, AET, reads as follows.

(AET) Let X be a compact metric space and let $T: X \to X$ be continuous. Regard $(X, \langle T^n \rangle_{n \in \mathbb{N}})$ as a dynamical system. Given $x \in X$, there exists $y \in X$ such that y is uniformly recurrent and proximal to x.

For an explanation of the notions of uniform recurrence and proximality, see e.g. [9] or [13] or §5 below.

The purpose of §5 is to present an axiomatic analysis of AET. The classical proof of AET is extremely nonconstructive, relying as it does on Zorn's Lemma applied to the partial ordering by inclusion of the closed subsets of the nonmetrizable Tychonoff product space X^X. (See the discussion of the "enveloping semigroup" on page 159 of [9] or page 143 of [13].) It is not at all obvious that this classical proof or anything like it can be carried out within full second order arithmetic. In §5 we present an apparently new proof of AET in which Hindman's Theorem is used as a lemma. We show that all parts of the new proof, except possibly the applications of Hindman's Theorem, can be pushed through in ACA_0. Combining this with a result from §4, we conclude: AET is provable in ACA_0^+. Thus our proof of AET is much closer to being constructive than is the classical proof.

§2. Strong recursive counterexample to Hindman's Theorem.

Given $X, W \subseteq \mathbb{N}$ we say that X is recursive in W if the characteristic function of X is computable by a Turing machine using an oracle for the characteristic function of W. We use $W^{(1)}$ to denote the Turing jump of W. In particular $\emptyset^{(1)}$ is the Turing jump of the empty set, i.e. the complete recursively enumerable subset of $\mathbb{N}$. Thus

$\emptyset^{(1)}$ has the same degree of unsolvability as the Halting Problem. The recursion-theoretic notions which we use are explained in Rogers [21].

The purpose of this section is to prove the following theorems.

2.1. Theorem. *There exists a recursive coloring* $\mathbb{N} = C_0 \cup C_1$ *such that for all infinite* $X \subseteq \mathbb{N}$, *if* $FS(X) \subseteq C_i$ *for some* $i \in \{0,1\}$, *then* X *is not recursive in* $\emptyset^{(1)}$.

2.2. Theorem. *There exists a recursive coloring* $\mathbb{N} = C_0 \cup C_1$ *such that, for all infinite* $X \subseteq \mathbb{N}$, *if* $FS(X) \subseteq C_i$ *for some* $i \in \{0,1\}$, *then* $\emptyset^{(1)}$ *is recursive in* X.

Proof of Theorem 2.1. We imitate the proof of Theorem 3.1 of Jockusch [17].

For $A \subseteq \mathbb{N}$ let $c_A: \mathbb{N} \to \{0,1\}$ be the characteristic function of A. By Theorem 2 of Shoenfield [22] there exists a recursive function $f: \mathbb{N}^3 \to \mathbb{N}$ with the following property. For all $A \subseteq \mathbb{N}$, A is recursive in $\emptyset^{(1)}$ if and only if, for some j, $c_A(u) = \lim_s f(j,u,s)$ for all $u \in \mathbb{N}$. Let us write $A = A_j$ in this case.

If A_j is defined and has at least $2j + 2$ elements, let D_j consist of the smallest $2j + 2$ elements of A_j. Otherwise let D_j be undefined. We shall now define a finite set D_j^s to approximate D_j at stage s. If there are at least $2j + 2$ numbers u such that $u \leq s$ and $f(j,u,s) = 1$, let D_j^s consist of the smallest $2j + 2$ such numbers. Otherwise let D_j^s be undefined.

Given $n \geq 1$ let us write $\lambda(n) = n_1$ and $\mu(n) = n_k$ where $n = 2^{n_1} + \dots + 2^{n_k}$, $n_1 < \dots < n_k$. Note that $\lambda(m+n) = \lambda(m)$ and $\mu(m+n) = \mu(n)$ provided $\mu(m) < \lambda(n)$.

The recursive coloring $\mathbb{N} = C_0 \cup C_1$ will be constructed in stages. At stage s of the construction we shall place each of the finitely many numbers n with $\mu(n) = s$ into exactly one of the color classes C_0 and C_1.

Stage s. By induction on $j \le s$, let u_j^s and v_j^s be two effectively chosen numbers which are different from each other and from all u_i^s and v_i^s, $i < j$, and which belong to D_j^s if D_j^s is defined. This can be done since $|D_j^s| = 2j + 2$ if D_j^s is defined. Now for all n such that $\mu(n) = s$, put $n \in C_0$ if $\lambda(n) = u_j^s$ for some $j \le s$, otherwise $n \in C_1$.

This completes the construction. Clearly C_0 and C_1 are recursive.

Let X be an infinite set such that $FS(X) \subseteq C_0$ or $FS(X) \subseteq C_1$. We claim that X is not recursive in $\emptyset^{(1)}$. To see this we first let Y be an infinite set such that Y is recursive in X, $FS(Y) \subseteq FS(X)$, and $\mu(m) < \lambda(n)$ for all $m \in Y$, $n \in Y$, $m < n$. (See Lemma 4.1 below.) Put $Z = \{\lambda(n): n \in Y\}$. Suppose that X is recursive in $\emptyset^{(1)}$. Then Z is recursive in $\emptyset^{(1)}$ so let j be such that $Z = A_j$. Since Z is infinite, D_j is defined and $D_j \subseteq Z$. Choose $n \in Y$ so large that $\max(D_j) < \lambda(n)$ and $D_j^s = D_j$ where $s = \mu(n)$. Then u_j^s and v_j^s are distinct elements of $D_j^s = D_j \subseteq Z$. Let $m_0, m_1 \in Y$ be such that $\lambda(m_0) = u_j^s$ and $\lambda(m_1) = v_j^s$. Then $\max(\mu(m_0), \mu(m_1)) < \lambda(n)$, hence $m_0 + n, m_1 + n \in FS(Y) \subseteq FS(X)$ and $m_0 + n \in C_0$, $m_1 + n \in C_1$. This contradiction completes the proof.

Proof of Theorem 2.2. We view each $n \in \mathbb{N}$ as a code for the finite set $\{n_1, \dots, n_k\}$, where $n = 2^{n_1} + \dots + 2^{n_k}$ and $n_1 < \dots < n_k$.

Accordingly, we refer to the pairs (n_i, n_{i+1}), $i = 1, \ldots, k-1$, as the <u>gaps</u> of n. Notice that if $\mu(m) < \lambda(n)$, where μ and λ are as in the proof of Theorem 2.1, then the gaps of $m + n$ are those of m, those of n, and the pair $(\mu(m), \lambda(n))$.

Fix a recursive algorithm enumerating the r.e. set $\emptyset^{(1)}$; let $\emptyset^{(1)}(k)$ be the finite subset of $\emptyset^{(1)}$ enumerated by this algorithm in its first k computation steps. For any $n \in \mathbb{N}$ and any gap (a,b) of n, we say that (a,b) is a <u>short</u> gap of n if there exists $x \le a$ such that $x \in \emptyset^{(1)}$ but $x \notin \emptyset^{(1)}(b)$. We say that (a,b) is a <u>very short</u> gap of n if there is $x \le a$ such that $x \in \emptyset^{(1)}(\mu(n))$ but $x \notin \emptyset^{(1)}(b)$. Let $SG(n)$ (resp. $VSG(n)$) be the number of short (resp. very short) gaps of n. Observe that, given n, one can effectively compute $VSG(n)$ (but not $SG(n)$). Thus, the following coloring $\mathbb{N} = C_0 \cup C_1$ is a recursive one.

$$C_i = \{n \in \mathbb{N} \mid VSG(n) \equiv i \pmod 2\}.$$

Suppose X is an infinite set with $FS(X) \subseteq C_i$ for some $i \in \{0,1\}$. We shall show that $\emptyset^{(1)}$ is recursive in X.

As in the proof of Theorem 2.1, we first use Lemma 4.1 to find an infinite Y such that Y is recursive in X, $FS(Y) \subseteq FS(X) \subseteq C_i$, and $\mu(m) < \lambda(n)$ for all $m < n$ in Y. It suffices to show that $\emptyset^{(1)}$ is recursive in Y.

<u>Claim 1</u>. For every $m \in FS(Y)$, $SG(m)$ is even.

<u>Proof</u>. Let $m \in FS(Y)$ be given, and choose an $n \in Y$ so large that, for all $x \le \mu(m)$, if $x \in \emptyset^{(1)}$ then $x \in \emptyset^{(1)}(\lambda(n))$. This can be done because every such x is in $\emptyset^{(1)}(k)$ for some k and λ is strictly increasing on Y. We compute the number $VSG(m+n)$ of very short gaps of $m+n$ by considering separately the gaps of m, the gaps of n,

and the gap $(\mu(m),\lambda(n))$. The last of these is not very short, by our choice of n. A gap of n is very short in $m+n$ if and only if it is very short in n, because $\mu(m+n) = \mu(n)$. A gap (a,b) of m is very short in $m+n$ if and only if it is short (not necessarily very short) as a gap of m, because, for $x \leq a < \mu(m)$, our choice of n ensures that $x \in \emptyset^{(1)}$ if and only if $x \in \emptyset^{(1)}(\lambda(n))$ if and only if $x \in \emptyset^{(1)}(\mu(m+n))$. Therefore,

$$VSG(m+n) = SG(m) + VSG(n).$$

By our choice of Y, the two VSG terms have the same parity. So the other term, $SG(m)$, must be even, and the claim is proved.

<u>Claim 2</u>. Assume that $m < n$ are in Y and that $x \leq \mu(m)$. Then $x \in \emptyset^{(1)}$ if and only if $x \in \emptyset^{(1)}(\lambda(n))$.

<u>Proof</u>. The "if" part is trivial, and the "only if" asserts that the gap $(\mu(m),\lambda(n))$ of $m+n$ is not short. Suppose, toward a contradiction, that it were short. Then the short gaps of $m+n$ would be those of m, those of n, and $(\mu(m),\lambda(n))$. Thus, we would have

$$SG(m+n) = SG(m) + SG(n) + 1,$$

which contradicts Claim 1. Thus, Claim 2 is proved.

We can now complete the proof of the theorem by giving an algorithm, with an oracle for Y, that computes membership in $\emptyset^{(1)}$. Given an input x, use the oracle to find an $m \in Y$ with $x \leq \mu(m)$ and to find an $n \in Y$ with $m < n$. Then run the algorithm enumerating $\emptyset^{(1)}$ for $\lambda(n)$ steps to decide whether $x \in \emptyset^{(1)}(\lambda(n))$. By Claim 2, this also decides whether $x \in \emptyset^{(1)}$.

2.3. <u>Remark</u>. Brackin [3] has proved a weaker version of Theorem 2.1 in which "X is not recursive in $\emptyset^{(1)}$" is replaced by "X is not recursive". As a consequence, he also obtained a weaker version of Theorem 2.6 asserting that HT is not provable in RCA_0.

It is straightforward to generalize Theorems 2.1 and 2.2 as follows.

2.4. Theorem. Given $W \subseteq \mathbb{N}$, there exists a coloring $\mathbb{N} = C_0 \cup C_1$ with the following properties. C_0 and C_1 are recursive in W and, for all infinite $X \subseteq \mathbb{N}$, if $FS(X) \subseteq C_i$ for some $i \in \{0,1\}$, then X is not recursive in $W^{(1)}$.

2.5 Theorem. For any $W \subseteq \mathbb{N}$, there exists a coloring $\mathbb{N} = C_0 \cup C_1$ with the following properties. C_0 and C_1 are recursive in W and, for all infinite $X \subseteq \mathbb{N}$, if $FS(X) \subseteq C_i$ for some $i \in \{0,1\}$, then $W^{(1)}$ is recursive in X.

The proof of Theorem 2.5 can be modified to yield a result concerning the set existence axioms which are needed to prove Hindman's Theorem. By RCA_0 (respectively ACA_0) we mean the subsystem of second order arithmetic with restricted induction and recursive (respectively arithmetical) comprehension [25,24,5]. It is well known that ACA_0 can be obtained from RCA_0 by adding an axiom asserting the existence of Turing jumps. An inspection of the proof of Theorem 2.5 shows that this proof goes through in RCA_0. Hence we have the following axiomatic result.

2.6. Theorem (RCA_0). Hindman's Theorem HT implies ACA_0.

In other words, no set existence axioms weaker than those of ACA_0 can suffice to prove Hindman's Theorem.

§3. Analysis of Baumgartner's proof

The purpose of this section is to present an axiomatic analysis of Baumgartner's proof [1] of Hindman's Theorem. We show that a version of Baumgartner's proof can be carried out within a certain formal system $\Pi^1_2 - TI_0$ (to be described below).

The reader of this section is assumed to have some familiarity with subsystems of second order arithmetic [25,7,23]. We also assume that the reader has access to Baumgartner's proof as presented on pages 69 through 71 of [13].

The following notions are basic to Baumgartner's proof. Let E and F be nonempty finite subsets of $\mathbb{N}$. We write $E < F$ if $\max(E) < \min(F)$. A <u>disjoint</u> <u>collection</u> is an infinite collection $\mathcal{D} = \{D_n: n \in \mathbb{N}\}$ of nonempty finite subsets of $\mathbb{N}$ such that $D_n < D_{n+1}$ for all $n \in \mathbb{N}$. From now on $\mathcal{D}$ denotes a disjoint collection. We use $FU(\mathcal{D})$ to denote the set of all unions of nonempty finite subcollections of $\mathcal{D}$. We write $\mathcal{D}' \leq \mathcal{D}$ to mean that $\mathcal{D}'$ is a disjoint collection and $\mathcal{D}' \subseteq FU(\mathcal{D})$. We say that C is <u>large</u> <u>for</u> $\mathcal{D}$ if $C \cap FU(\mathcal{D}') \neq \emptyset$ for all $\mathcal{D}' \leq \mathcal{D}$. We write

$$C/E = \{F \in C: E < F\}.$$

Consider the following statement HTU which is a version of Hindman's Theorem (see Corollary 3.3 of [15]).

$$(\text{HTU}) \begin{cases} \underline{\text{If}}\ \ FU(\mathcal{D}) = C_0 \cup \dots \cup C_\ell\ \ \underline{\text{then}} \\ \underline{\text{there}}\ \underline{\text{exists}}\ \ \mathcal{D}' \leq \mathcal{D}\ \ \underline{\text{such}}\ \underline{\text{that}} \\ FU(\mathcal{D}') \subseteq C_i\ \ \underline{\text{for}}\ \underline{\text{some}}\ \ i \leq \ell. \end{cases}$$

Hindman's Theorem HT follows immediately from HTU by taking $\mathcal{D} = \{\{n\}: n \in \mathbb{N}\}$ and

$$C_i = \{D \in FU(\mathcal{D}): \Sigma\{2^m: m \in D\} \in C_i\}$$

for all $i \leq \ell$. Baumgartner's method for proving Hindman's Theorem is to prove HTU by means of a sequence of lemmas involving the notion of largeness. We now present our axiomatic analysis of Baumgartner's proof.

We use Z_2 to denote the formal system of second order arithmetic. Recall [25,24,5] that RCA_0 (respectively ACA_0) are the subsystems of Z_2 with restricted induction and recursive (respectively arithmetical) comprehension.

An ω-<u>model</u> is a set $M \subseteq P(\mathbb{N}) = \{X: X \subseteq \mathbb{N}\}$ regarded as a model for L_2, the language of Z_2. If φ is a sentence of L_2 with parameters from M, we say that M <u>satisfies</u> φ if φ is true when the set variables range over M, the number variables range over $\mathbb{N}$, and the remaining symbols have their standard interpretation.

A β-<u>model</u> is an ω-model M such that, for all Σ^1_1 sentences φ with parameters from M, φ is true if and only if M satisfies φ. A <u>countable</u> <u>coded</u> ω-<u>model</u> is a set $Z \subseteq \mathbb{N}$ viewed as (a code for) the ω-model $M = \{(Z)_n: n \in \mathbb{N}\}$ where $(Z)_n = \{m: (m,n) \in Z\}$, $(m,n) = \frac{1}{2}(m+n)(m+n+1)+m$. A <u>countable</u> <u>coded</u> β-<u>model</u> is a countable coded ω-model which is also a β-model. We assume that the notion of countable coded β-model has been defined formally within RCA_0 as in §VII.2 of [25]. (In particular a countable coded β-model comes equipped with a satisfaction predicate for a "universal lightface Σ^1_1 formula.")

We shall now state four lemmas which are modeled on Lemmas 17, 20 and 21 and Theorem 18 on pages 70 and 71 of [13]. The proofs of our lemmas are obtained by straightforward adaptation of the original proofs.

3.1. Lemma. The following is provable in RCA_0. Let M be a countable coded β-model. Suppose that $C_0, \ldots, C_\ell$, $\mathcal{D} \in M$ and that $C_0 \cup \ldots \cup C_\ell$ is large for $\mathcal{D}$. Then some C_i, $i \le \ell$ is large for some $\mathcal{D}' \le \mathcal{D}$, $\mathcal{D}' \in M$.

3.2. Lemma. The following is provable in RCA_0. Let M be a countable coded β-model. Suppose that $C, \mathcal{D} \in M$ and that C is large for $\mathcal{D}$. Then there exists $E \in FU(\mathcal{D})$ such that

$$C' = \{F \in C/E: E \cup F \in C\}$$

is large for some $\mathcal{D}' \le \mathcal{D}/E$, $\mathcal{D}' \in M$.

3.3. Lemma. The following is provable in RCA_0. Let M be a countable coded β-model. Suppose that $C, \mathcal{D} \in M$ and that C is large for $\mathcal{D}$. Then there exists $E \in C \cap FU(\mathcal{D})$ such that

$$C' = \{F \in C/E: E \cup F \in C\}$$

is large for some $\mathcal{D}' \le \mathcal{D}/E$, $\mathcal{D}' \in M$.

3.4. Lemma. The following is provable in RCA_0. Let M be a countable coded β-model. Suppose that $C, \mathcal{D} \in M$ and that C is large for $\mathcal{D}$. Then there exists $\mathcal{D}' \le \mathcal{D}$, $\mathcal{D}' \in M$ such that $FU(\mathcal{D}') \subseteq C$.

As an immediate consequence of Lemmas 3.1 and 3.4, we have:

3.5. Lemma. The following is provable in RCA_0. Let M be a countable coded β-model. Then M satisfies HTU. Hence M satisfies Hindman's Theorem HT.

Recall that Π^1_1-CA_0 is the subsystem of Z_2 with restricted induction and Π^1_1 comprehension. The following proposition, essentially due to Friedman [7], is also proved in §VII.2 of [25].

3.6. Proposition. The following is provable in Π^1_1-CA_0. For all $X \subseteq \mathbb{N}$ there exists a countable coded β-model M such that $X \in M$.

Combining this with Lemma 3.5 we obtain:

3.7. Theorem. Hindman's Theorem HT is provable in Π^1_1-CA_0.

We now describe the formal systems Π^1_k-TI_0, $k \in \mathbb{N}$. Let $WO(X)$ stand for the Π^1_1 formula which asserts that X is a code for a countable well ordering. If $\psi(n)$ is any L_2-formula, let $TI(X,\psi)$ stand for the formula

$$\forall n(\forall m(m <_X n \to \psi(m)) \to \psi(n)) \to \forall n\psi(n).$$

This expresses transfinite induction along X with respect to $\psi(n)$. We define Π^1_k-TI_0 to be the subsystem of Z_2 whose axioms are those of ACA_0 plus all formulas of the form

$$WO(X) \to TI(X,\psi)$$

where $\psi(n)$ is Π^1_k. Also

$$\Pi^1_\infty\text{-}TI_0 = \bigcup_{k=0}^{\infty}\Pi^1_k\text{-}TI_0.$$

The following proposition, due to Simpson [25] §VII.2, is related to some unpublished work of Harrington [14].

3.8. __Proposition__. __Let__ φ __be any__ Π^1_2 __sentence__. __The following assertions are equivalent__: (i) Π^1_∞-TI_0 __proves__ φ; (ii) ACA_0 __proves that any countable coded__ β-__model satisfies__ φ.

Since Hindman's Theorem HT is a Π^1_2 sentence, we can combine Proposition 3.8 with Lemma 3.5 to obtain:

3.9. __Theorem__. __Hindman's Theorem__ HT __is provable in__ Π^1_∞-TI_0.

In order to sharpen the previous result, we apply the following proposition due to Simpson [25]. For the proof see the exercises at the end of §VII.2 of [25].

3.10. __Proposition__. __Let__ φ __be any sentence of the form__ $\forall X \exists Y \theta$ __where__ θ __is__ Π^0_2. __The following assertions are equivalent__: (i) Π^1_2-TI_0 __proves__ φ; (ii) RCA_0 __proves that any countable coded__ β-__model satisfies__ φ.

Since Hindman's Theorem is of the required form, we obtain:

3.11. __Theorem__. __Hindman's Theorem__ HT __is provable in__ Π^1_2-TI_0.

We now consider a result of Carlson and Simpson [6]. Let A be a fixed finite alphabet. Let A* be the set of finite words over A, i.e. finite sequences of elements of A. An __infinite variable word__ is an infinite sequence W of elements of the disjoint union $A \cup \{x_i : i \in \mathbb{N}\}$ in which each x_i occurs at least once, each x_i occurs only finitely many times, and for each i the last occurence of x_i comes before the first occurrence of x_{i+1}. The x_i's are to be regarded as variables ranging over A. Given $s = a_0a_1\ldots a_{j-1} \in A^*$, let $W(s)$ be the element of A^* which results from W by first substituting a_i for each occurrence of x_i, $i < j$, then truncating just before the first occurrence of x_j. Put

$$W(A^*) = \{W(s): s \in A^*\}.$$

Theorem 6.3 of Carlson-Simpson [6] asserts the following:

(CST) $\begin{cases} \text{\underline{If} } A^* = C_0 \cup \ldots \cup C_{\ell-1} \text{ \underline{then there}} \\ \text{\underline{exists an infinite variable word} } W \\ \text{\underline{such that} } W(A^*) \subseteq C_i \text{ \underline{for some} } i < \ell. \end{cases}$

The proof of CST in [6] is broadly similar to Baumgartner's proof of Hindman's Theorem. Our analysis in terms of countable coded β-models can be adapted so as to apply to the proof of CST. In this way we obtain:

3.12. Theorem. *Theorem 6.3 of Carlson-Simpson [6], i.e. CST, is provable in* Π^1_2-TI_0.

We do not know whether CST is provable in any weaker system such as RCA_0, ACA_0^+, or Δ^1_2-TI_0.

§4. Analysis of Hindman's Proof.

The purpose of this section is to present an analysis of Hindman's original proof of Hindman's Theorem. This yields both recursion-theoretic and axiomatic information. Our main recursion-theoretic result is as follows: Given a coloring $\mathbb{N} = C_0 \cup \ldots \cup C_\ell$, there exists an infinite set $X \subseteq \mathbb{N}$ such that $FS(X) \subseteq C_i$ for some $i \leq \ell$, and X is recursive in the $(\omega+1)$st Turing jump of the given coloring. Axiomatically, our result is that Hindman's Theorem HT is provable in a certain formal system ACA_0^+ (to be described below).

The reader of this section is assumed to have access to Hindman's original paper [15]. To the extent possible, we use Hindman's notation and terminology. The heart of our analysis is a sequence of recursion-theoretic lemmas which correspond to Lemmas 2.2, 2.5, 2.6, 2.8, 2.9, 2.10, 2.11 and 2.12 of Hindman [15]. The proofs of our lemmas can be obtained by straightforwardly modifying Hindman's proofs of his lemmas. The recursion-theoretic notions which we use are taken from Rogers [21].

4.1. Lemma (corresponding to Hindman's 2.2). *If* $\langle x_i \rangle_{i=1}^{\infty}$ *is a sequence in* $\mathbb{N}$, *then there is a sequence* $\langle y_i \rangle_{i=1}^{\infty}$ *recursive in* $\langle x_i \rangle_{i=1}^{\infty}$ *such that* $FS(\langle y_n \rangle_{n=1}^{\infty}) \subseteq FS(\langle x_n \rangle_{n=1}^{\infty})$ *and* $2^s | y_{n+1}$ *whenever* $2^{s-1} \leq y_n$.

4.2. Lemma (corresponding to Hindman's 2.5). *Let* $\langle x_n \rangle_{n=1}^{\infty}$ *be a sequence in* $\mathbb{N}$ *such that* $2^s | x_{n+1}$ *whenever* $2^{s-1} \leq x_n$. *Let* τ *be the natural map for* $FS(\langle x_n \rangle_{n=1}^{\infty})$ *and let* $\langle y_n \rangle_{n=1}^{\infty}$ *be any sequence in* $\mathbb{N}$ *such that* $FS(\langle y_n \rangle_{n=1}^{\infty}) \subseteq FS(\langle x_n \rangle_{n=1}^{\infty})$. *Then there is a sequence* $\langle z_n \rangle_{n=1}^{\infty}$ *recursive in* $\langle y_n \rangle_{n=1}^{\infty}$ *and* $\langle x_n \rangle_{n=1}^{\infty}$ *such that* $FS(\langle z \rangle_{n=1}^{\infty}) \subseteq FS(\langle y_n \rangle_{n=1}^{\infty})$ *and* $\tau(\sum_{n \in F} z_n) = \sum_{n \in F} \tau(z_n)$ *whenever* $F \subseteq_f \mathbb{N}$.

4.3. Lemma (corresponding to Hindman's 2.6). *Let* $k \in \mathbb{N}$ *and let* $C = \langle A(i,n)\colon 1 \leq i \leq k,\ n \in \mathbb{N} \rangle$ *be a collection of sets such that if* $n \in \mathbb{N}$ *and* $1 \leq i \leq k$, *then* $A(i,n+1) \subseteq A(i,n)$. *Then there exist a subset* S *of* $\{1,2,\ldots,k\}$, *a sequence* $\langle x_m \rangle_{m=1}^{\infty}$ *recursive in* C, *and an integer* M, *such that if* $n \geq M$ *and* $\langle y_m \rangle_{m=1}^{\infty}$ *is a sequence recursive in* C *with* $FS(\langle y_m \rangle_{m=1}^{\infty}) \subseteq FS(\langle x_m \rangle_{m=1}^{\infty})$ *then* $FS(\langle y_m \rangle_{m=1}^{\infty}) \cap A(i,n) \neq \emptyset$ *if and only if* $i \in S$.

4.4. Lemma (corresponding to Hindman's 2.8). *Let*

$\alpha = \langle A_i : 1 \le i \le a\rangle$ be a partition of $\mathbb{N}$. Suppose that for every $n \in \mathbb{N}$ and every $\langle y_m\rangle_{m=1}^{\infty}$ recursive in α, one has $FS(\langle y_m\rangle_{m=1}^{\infty}) \setminus \bigcup_{k=1}^{n-1} F_\alpha(k,n) \neq \emptyset$. Then we can define, arithmetically in α, an $i \in \{1,2,\ldots,a\}$, sequences $\langle x_n\rangle_{n=1}^{\infty}$ and $\langle M_n\rangle_{n=1}^{\infty}$, sequences $\langle x_{n,m}\rangle_{m=1}^{\infty}$ recursive in α for each n, and for each n a sequence of sets $\langle U(n,p): p \in \mathbb{N}\rangle$, recursive in α, such that the following six properties are satisfied:

(1) For all $m \in \mathbb{N}$, $2^{s-1} \le x_{n,m}$ implies $2^s \mid x_{n,m+1}$;

(2) If $p \ge M_n$ and $\langle y_m\rangle_{m=1}^{\infty}$ is a sequence recursive in α such that $FS(\langle y_m\rangle_{m=1}^{\infty}) \subseteq FS(\langle x_{n,m}\rangle_{m=1}^{\infty})$ then $FS(\langle y_m\rangle_{m=1}^{\infty}) \cap U(n,p) \neq \emptyset$;

(3) If $p \ge M_n$ then $U(n,p+1) \subseteq U(n,p) \subseteq A_i$;

(4) If $n \ge 1$ then $M_n \ge M_{n-1}$ and $M_n > \sum_{j=1}^{n} x_j$;

(5) If $n \ge 1$ and $p \ge M_n$ then $U(n,p) \subseteq U(n-1,p)$;

(6) If $n \ge 1$ and $p \ge M_n$ and $x \in U(n,p)$, then $x + x_n \in U(n-1,M_{n-1})$.

The proof of Lemma 4.4 is similar to Hindman's proof of his Lemma 2.8. The only difference is that the sequences $\langle z_m\rangle_{m=1}^{\infty}$ are restricted to be recursive in α.

4.5. Lemma (corresponding to Hindman's 2.9). Let $\alpha = \{A_j\}_{j=1}^{n}$ be a partition of $\mathbb{N}$. If for every n in $\mathbb{N}$ and every sequence $\langle y_m\rangle_{m=1}^{\infty}$ in $\mathbb{N}$ which is recursive in α one has $FS(\langle y_m\rangle_{m=1}^{\infty}) \setminus \bigcup_{k=1}^{n-1} F_\alpha(k,n) \neq \emptyset$, then there exist an $i \in \{1,\ldots,a\}$ and a sequence $\langle x_n\rangle_{n=1}^{\infty}$ in $\mathbb{N}$ which is arithmetical in α such that $FS(\langle x_n\rangle_{n=1}^{\infty}) \cap A_i = \emptyset$.

4.6. Lemma (corresponding to Hindman's 2.10). Let $\alpha = \{A_i\}_{i=1}^{a}$ be a partition of $\mathbb{N}$. Then there exist $n \in \mathbb{N}$ and a sequence $\langle x_m\rangle_{m=1}^{\infty}$ arithmetical in α such that $FS(\langle x_m\rangle_{m=1}^{\infty}) \subseteq \bigcup_{k=1}^{n-1} F_\alpha(k,n)$.

4.7. Lemma (corresponding to Hindman's 2.11). For every partition α of $\mathbb{N}$ with $\alpha = \{A_i\}_{i=1}^{a}$, there exists a function $f_\alpha\colon \mathbb{N} \to \mathbb{N}$ recursive in $\alpha^{(\omega)}$ such that for each r in $\mathbb{N}$ there exist $i \in \{1,\ldots,a\}$ and $\langle y_j\rangle_{j=1}^{r}$ such that:

(1) $FS(\langle y_j\rangle_{j=1}^{r}) \subseteq A_i$;

(2) if $j \in \{1,2,\ldots,r-1\}$ and $2^{s-1} \le y_j$ then $2^s | y_{j+1}$.

(3) if $j \in \{1,2,\ldots,r\}$ then $y_j \le f_\alpha(j)$.

Proof. Let $\alpha_1 = \alpha$ and suppose α_n is defined. By Lemma 4.6 we may choose p_n and a sequence $\langle x_{n,m}\rangle_{m=1}^{\infty}$ arithmetical in α_n such that $FS(\langle x_{n,m}\rangle_{m=1}^{\infty}) \subseteq \bigcup_{k=1}^{p_n-1} F_{\alpha_n}(k,p_n)$ and $2^s | x_{n,m+1}$ whenever $2^{s-1} \le x_{n,m}$. We may further assume that, for each m, $2^s | x_{n,m}$ whenever $2^{s-1} \le p_n$. Let τ_n be the natural map for $FS(\langle x_{n,m}\rangle_{m=1}^{\infty})$. Then τ_n and τ_n^{-1} are arithmetical in α_n. Let $\alpha_{n+1} = \{A_{n+1,k}\colon 1 \le k \le p_n - 1\} =$

$\{\tau_n(F_{\alpha_n}(k,p_n)) : 1 \le k \le p_n - 1\}$. Then α_{n+1} is a partition of $\mathbb{N}$ which is arithmetical in α_n. Let $T(n,m) = \tau_n^{-1}(m)$. Then $T(n,m)$ is recursive in $\alpha^{(\omega)}$. Define f by $f(n,1) = T(n,1)$ and $f(n,m+1) = T(n,f(n+1,m))$. Since f is recursive in T, f is recursive in $\alpha^{(\omega)}$.

We now show, by induction on r, that for each $n \in \mathbb{N}$, $f(n,m)$ satisfies the lemma for α_n. The case for $r = 1$ is trivial for each n. Let $r > 1$ and assume that properties (1), (2) and (3) hold for each n at $r - 1$. Fix n. Let $\langle w_j\rangle_{j=1}^{r-1}$ and $k \in \{1,2,\dots,p_{n+1} - 1\}$ be as guaranteed by the lemma for α_{n+1} at $r - 1$. Let $i < p_n - 1$ be the integer such that $k \in A_{n,i}$. Let $y_1 = k$ and for $j \in \{2,3,\dots,r\}$, let $y_j = \tau_n^{-1}(w_{j-1})$.

To verify property (1), fix $F \subseteq \{2,\dots,r\}$. By Hindman's Lemma 2.4 and property (2) for α_{n+1}, $\tau_n(\sum_{j\in F} y_j) = \sum_{j\in F} w_{j-1}$. By definitions of α_{n+1} and $\langle w_j\rangle_{j=1}^{r-1}$, $\sum_{j\in F} w_{j-1} \in \tau_n(F_{\alpha_n}(k,p_n))$, so $\sum_{j\in F} y_j \in F_{\alpha_n}(k,p_n)$. Thus $\sum_{j\in F} y_j \in A_{n,i}$, and $y_1 + \sum_{j\in F} y_j \in A_{n,i}$. Since $y_1 \in A_{n,i}$, property (1) is satisfied.

To see that property (2) holds, note that if $2^{s-1} \le y_1$, then $2^{s-1} < p_n$, so $2^s | x_{n,m}$ for every m. Consequently, $2^s | y_2$. By Hindman's Lemma 2.4, $2^s | y_{j+1}$ for $j \in \{2,\dots,r - 1\}$.

Now consider property (3). First, $y_1 = k \le p_n - 1 \le f(n,1)$. Let $j \in \{2,\dots,r\}$. Then $w_{j-1} \le f(n+1,j-1)$, so

$y_j = \tau_n^{-1}(w_{j-1}) \le \tau_n^{-1}(f(n+1,j-1)) = T(n,f(n+1,j-1)) = f(n,j)$, as desired.

Since $f(n,m)$ satisfies the lemma for α_n, $f(1,m) = f_\alpha$ satisfies the lemma for $\alpha_1 = \alpha$. This completes the proof.

4.8. <u>Lemma</u> (corresponding to Hindman's 2.12). <u>For every partition</u> α <u>of</u> $\mathbb{N}$, <u>with</u> $\alpha = \{A_i\}_{i=1}^{a}$, <u>there exists a function</u> $f_\alpha: \mathbb{N} \to \mathbb{N}$ <u>recursive in</u> $\alpha^{(\omega)}$ <u>and an</u> $i \in \{1,2,\ldots,a\}$ <u>such that, for every</u> r <u>in</u> $\mathbb{N}$, <u>there exists</u> $\langle y_j \rangle_{j=1}^{r}$ <u>such that</u> $FS(\langle y_j \rangle)_{j=1}^{r} \subseteq A_i$ <u>and</u> $y_j \le f_\alpha(j)$ <u>whenever</u> $j \in \{1,2,\ldots,r\}$.

We can now present the main recursion-theoretic result of this section:

4.9. <u>Theorem</u> (corresponding to Hindman's 3.1). <u>Let</u> α <u>be a finite partition of</u> $\mathbb{N}$ <u>with</u> $\alpha = \{A_i\}_{i=1}^{a}$. <u>There exists an</u> $i \in \{1,2,\ldots,a\}$, <u>and a sequence</u> $\langle x_m \rangle_{m=1}^{\infty}$ <u>recursive in</u> $\alpha^{(\omega+1)}$ <u>such that</u> $FS(\langle x_m \rangle_{m=1}^{\infty}) \subseteq A_i$.

<u>Proof</u>. Let i and f_α be as in Lemma 4.8. Let T be the tree given by

$$T = \{\langle x_i \rangle_{i=1}^{n} : n \in \mathbb{N} \wedge (1 \le i < j \le n \to x_i < x_j) \wedge$$

$$FS(\langle x_i \rangle_{i=1}^{n}) \subseteq A_i \wedge \forall j \le n (x_j \le f_\alpha(j))\}.$$

Intuitively, T is the tree of finite sequences which are homogeneous in the sense of Hindman's theorem and bounded by f_α. T is finitely branching since it is bounded by f_α. Lemma 4.8 guarantees that T is infinite. By König's Lemma, T has an infinite path, say $\langle x_m \rangle_{m=1}^{\infty}$. Clearly this path is the desired sequence. Since T is recursive in $\alpha^{(\omega)}$, by the Kreisel Basis Theorem [20] (see also Kleene [19, p.398]), we may assume that $\langle x_m \rangle_{m=1}^{\infty}$ is recursive in $\alpha^{(\omega+1)}$. This completes the proof.

Note that by using the Low Basis Theorem (Theorem 2.1 of Jockusch and Soare [18]) relativized to $\alpha^{(\omega)}$, we can improve the conclusion. Not only the sequence $\langle x_m\rangle_{m=1}^{\infty}$ but also its Turing jump can be made recursive in $\alpha^{(\omega+1)}$.

The above theorem, Theorem 4.9, is the main recursion-theoretic result of this section. For use in §5 we now present a generalization of Theorem 4.9 involving a countable sequence of partitions. First we prove the following generalization of Lemma 4.7.

4.10. Lemma. For every sequence of partitions $\langle\beta_i\rangle_{i=1}^{\infty} = \langle\{B_{i,n}\}_{n=1}^{a_i}\rangle_{i=1}^{\infty}$, there is a function $f: \mathbb{N} \to \mathbb{N}$ recursive in $(\langle\beta_i\rangle_{i=1}^{\infty})^{(\omega)}$ such that for each r in $\mathbb{N}$ there is a sequence $\langle y_i\rangle_{i=1}^{r}$ and a sequence $\langle j_i\rangle_{i=1}^{r}$ such that

(1) For all $k \le r$, $FS(\langle y_i\rangle_{i=k}^{r}) \subseteq B_{k,j_k}$.

(2) If $1 \le k < r$ and $2^{s-1} \le y_k$ then $2^s | y_{k+1}$.

(3) If $1 \le k \le r$ then $y_k \le f(k)$.

Proof. Let $\alpha_1 = \beta_1$ and $S(1,m) = m$. Suppose α_n and $S(n,m)$ are defined. By Lemma 4.6 we may choose p_n and a sequence $\langle x_{n,m}\rangle_{m=1}^{\infty}$ arithmetical in α_n such that $FS(\langle x_{n,m}\rangle_{m=1}^{\infty}) \subseteq \bigcup_{k=1}^{p_n-1} F_{\alpha_n}(k,p_n)$ and $2^s | x_{n,m+1}$ whenever $2^{s-1} \le x_{n,m}$. We may further assume that, for each m, $2^s | x_{n,m}$ whenever $2^{s-1} \le p_n$. Let τ_n be the natural map for $FS(\langle x_{n,m}\rangle_{m=1}^{\infty})$. Then τ_n and τ_n^{-1} are arithmetical in α_n. Let $S(n+1,m) = \tau_n(S(n,m))$. Let $\langle j,k\rangle$ denote the pairing function applied to the pair (j,k). Define the partition α_{n+1} by:

$$\alpha_{n+1} = \{A_{n+1,<k,\ell>}: 1 \le k \le p_n - 1 \text{ and } 1 \le \ell \le a_{n+1}\}$$
$$= \{\tau_n(F_{\alpha_n}(k,p_n)) \cap S(n+1,B_{n+1,\ell}): 1 \le k \le p_n - 1 \text{ and } 1 \le \ell \le a_{n+1}\}.$$

Then α_{n+1} is a partition of $\mathbb{N}$ which is arithmetical in α_n and β_{n+1}. Let $T(n,m) = \tau_n^{-1}(m)$. Then $T(n,m)$ is recursive in $(<\beta_n>_{n=1}^{\infty})^{(\omega)}$. Define f by $f(n,1) = T(n,1)$ and $f(n,m+1) = T(n,f(n+1,m))$. Since f is recursive in T, f is recursive in $(<\beta_n>_{n=1}^{\infty})^{(\omega)}$.

The proof is completed by showing that for all $r,n \in \mathbb{N}$, there exist an $i \in \mathbb{N}$, a sequence $<y_j>_{j=1}^{r}$, and a sequence $<j_{n+k-1}>_{k=1}^{r}$ such that

(0') For all $k \le r$, $FS(<y_i>_{k=1}^{r}) \subseteq S(n,B_{n+k-1,j_{n+k-1}})$.

(1') $FS(<y_k>_{k=1}^{r}) \subseteq A_{n,i}$,

(2') if $1 \le k \le r - 1$ and $2^{s-1} \le y_k$ then $2^s | y_{k+1}$,

(3') if $1 \le k \le r$ then $y_k \le f(n,k)$.

Letting $f(m) = f(1,m)$, this suffices to complete the proof. The verification of properties (0') through (3') is carried out by induction on r, exactly as in the proof of Lemma 4.7.

Using the previous lemma we can imitate the proofs of Lemma 4.8 and Theroem 4.9 to arrive at the following theorem.

4.11. Theorem. Let $<\beta_n>_{n=1}^{\infty}$ be a sequence of partitions of $\mathbb{N}$ with $\beta_n = \{B_{n,i}\}_{i=1}^{a_n}$. Then there exist sequences $<x_m>_{m=1}^{\infty}$ and $<j_m>_{m=1}^{\infty}$, both recursive in $(<\beta_n>_{n=1}^{\infty})^{(\omega+1)}$ such that for all n,

$$FS(\langle x_m\rangle_{m=n}^{\infty}) \subseteq B_{n,j_n}.$$

We now present our axiomatic results. The reader is assumed to have some familiarity with subsystems of Z_2.

Recall [25,24] that ACA_0 is the subsystem of Z_2 whose principal axiom is arithmetical comprehension, i.e. all formulas of the form

$$\exists X \forall n (n \in X \leftrightarrow \varphi(n))$$

where $\varphi(n)$ is any arithmetical formula in which the set variable X does not occur.

An inspection of the proofs of Lemmas 4.1 through 4.6 reveals that these proofs can be carried out formally within ACA_0.

We now define ACA_0^+. (See also [5].) Given $X \subseteq \mathbb{N}$ and $j \in \mathbb{N}$, we put $(X)_j = \{n: (n,j) \in X\}$ and $(X)^j = \{(m,i): (m,i) \in X \wedge i < j\}$. ACA_0^+ is defined to be the subsystem of Z_2 whose axioms are those of ACA_0 plus all formulas of the form

$$\exists X \forall j \forall n (n \in (X)_j \leftrightarrow \varphi(n,(X)^j))$$

where $\varphi(n,Y)$ is any arithmetical formula in which the set variable X does not occur. Thus ACA_0^+ contains axioms which assert that arithmetical comprehension can be iterated along $\mathbb{N}$. In the presence of ACA_0, these additional axioms are equivalent to the single axiom $\forall W \exists X (X = W^{(\omega)})$ where $W^{(\omega)}$ denotes the ωth Turing jump of W. Since ACA_0 is finitely axiomatizable, it follows that ACA_0^+ is finitely axiomatizable.

An inspection of the proofs of Lemmas 4.7, 4.8 and 4.10 and Theorems 4.9 and 4.11 reveals that these proofs can be carried out formally within

ACA_0^+. This yields the following results.

4.12. <u>Theorem</u>. <u>Hindman's Theorem</u> HT <u>is provable in</u> ACA_0^+.

4.13. <u>Theorem</u>. <u>The following generalization of Hindman's Theorem is provable in</u> ACA_0^+. <u>Given a countable sequence of colorings</u> $\mathbb{N} = C_{n0} \cup \ldots \cup C_{n\ell_n}$, $n \in \mathbb{N}$, <u>there exists an infinite set</u> $X = \{x_i\}_{i=0}^{\infty} \subseteq \mathbb{N}$ <u>such that for each</u> n, $FS(\{x_i\}_{i=n}^{\infty}) \subseteq C_{nj}$ <u>for some</u> $j \leq \ell_n$.

Theorem 4.13 will be applied in the next section.

§5. <u>Analysis of the Auslander-Ellis Theorem</u>.

In this section we turn our attention to topological dynamics. We present an apparently new proof of the Auslander-Ellis Theorem (AET of §1). Our proof is much more constructive than all previously known proofs. We show that our proof can be carried out within the formal system ACA_0^+ (defined in §4). This is of interest because it is not at all obvious that any of the previously known proofs of AET can be carried out within ACA_0^+ or even within full second order arithmetic.

We begin by defining some notions from topological dynamics. Let X be a compact metric space. We use d to denote the distance function on X. Let $T: X \to X$ be a fixed continuous function. We use $i, j, k, \ell, m, n, \ldots$ to denote nonnegative integers, i.e. elements of $\mathbb{N}$. A point $x \in X$ is called <u>recurrent</u> if for all $\varepsilon > 0$ there exist infinitely many n such that $d(T^n x, x) < \varepsilon$. We say that x is <u>uniformly recurrent</u> if for all $\varepsilon > 0$ there exists m such that for all n there exists $k < m$ such that $d(T^{n+k}x, x) < \varepsilon$. Two points $x, y \in X$ are said to be <u>proximal</u> if for all $\varepsilon > 0$ there exist infinitely many n such that

$d(T^n x, T^n y) < \varepsilon$. The Auslander-Ellis Theorem reads as follows: For all $x \in X$ there exists $y \in X$ such that y is proximal to x and y is uniformly recurrent.

We now give our proof of the Auslander-Ellis Theorem. The idea of the proof is to use Hindman's Theorem to obtain proximal points. This idea comes from Chapter 8 of Furstenberg [9].

5.1. Definition. Let $x, y \in X$ and let $A \subseteq \mathbb{N}$ be an infinite set. We write $y = \lim_A x$ to mean that for all $\varepsilon > 0$ there exists a finite set $F \subseteq A$ such that $d(T^n x, y) < \varepsilon$ for all $n \in FS(A \setminus F)$.

5.2. Lemma. *If* $y = \lim_A x$ *then* y *is proximal to* x.

Proof. Suppose $y = \lim_A x$. Given $\varepsilon > 0$ let F be a finite subset of A such that $d(T^k x, y) < \varepsilon$ for all $k \in FS(A \setminus F)$. Fix $m \in FS(A \setminus F)$ and choose $\delta > 0$ so that $d(z, y) < \delta$ implies $d(T^m z, T^m y) < \varepsilon$ for all $z \in X$. Let G be a finite subset of $A \setminus F$ such that $m \in FS(G)$. Fix $n \in FS(A \setminus (F \cup G))$ such that $d(T^n x, y) < \delta$. Then $d(T^{m+n} x, T^m y) < \varepsilon$. Also m and $m + n$ belong to $FS(A \setminus F)$ so $d(T^m x, y) < \varepsilon$ and $d(T^{m+n} x, y) < \varepsilon$. Combining these inequalities we obtain $d(T^m x, T^m y) < 3\varepsilon$. (Compare Lemma 8.15 of [9].)

5.3. Lemma. *Given* $x \in X$ *and an infinite set* $A \subseteq \mathbb{N}$, *there exists an infinite set* B *such that* $FS(B) \subseteq FS(A)$ *and* $\lim_B x$ *exists*.

Proof. Let $\langle a_n : n \in \mathbb{N} \rangle$ be the elements of A in increasing order. Given $m = 2^{m_1} + \ldots + 2^{m_k}$, $m_1 < \ldots < m_k$ we put $\lambda(m) = m_1$, $\mu(m) = m_k$, and

$$m^* = a_{m_1} + \ldots + a_{m_k} \in FS(A).$$

For each n let $\langle y_{ni}: i \leq \ell_n \rangle$ be a finite sequence of points such that for all $y \in X$ there exists $i \leq \ell_n$ such that $d(y, y_{ni}) < 2^{-n}$. Define

$$C_{ni} = \{m: d(T^{m^*}x, y_{ni}) < 2^{-n}\}.$$

Thus $\langle\langle C_{ni}: i \leq \ell_n \rangle: n \in \mathbb{N}\rangle$ is a countable sequence of colorings of $\mathbb{N}$. Apply Hindman's Theorem countably many times to obtain an infinite set Z such that for each n there exists a finite set $F \subseteq Z$ such that $FS(Z \backslash F) \subseteq C_{ni}$ for some $i \leq \ell_n$. Define sequences $\langle z_n: n \in \mathbb{N}\rangle$ and $\langle F_n: n \in \mathbb{N}\rangle$ by induction on n as follows. Choose $z_0 \in FS(Z)$, $z_0 > 1$. Let F_0 be a finite subset of Z such that $z_0 \in FS(F_0)$. Choose $z_{n+1} \in FS(Z \backslash (F_0 \cup \ldots \cup F_n))$ so that $\lambda(z_{n+1}) > \mu(z_n)$. Let F_{n+1} be a finite subset of $Z \backslash (F_0 \cup \ldots \cup F_n)$ such that $z_{n+1} \in FS(F_{n+1})$. Finally put $B = \{z_n^*: n \in \mathbb{N}\}$. Then B is infinite, $FS(B) \subseteq FS(A)$, and it is straightforward to check that $\lim_B x$ exists. (Compare Theorem 8.14 of [9].)

Given $x \in X$ we denote by $\overline{x}$ the <u>orbit closure</u> of x, i.e. the set of points y such that for all $\varepsilon > 0$ there exists n such that $d(T^n x, y) < \varepsilon$.

5.4. <u>Lemma</u>. <u>For all</u> $y \in \overline{x}$ <u>there exists</u> $z \in \overline{y}$ <u>such that</u> $z = \lim_B x$ <u>for some infinite set</u> $B \subseteq \mathbb{N}$.

<u>Proof</u>. For each $k \in \mathbb{N}$ let U_k be the set of points which are at distance $< 2^{-k}$ from $\overline{y}$. Note that for all n and k, $\overline{y} \subseteq T^{-n}U_k$ and $T^{-n}U_k$ is open. By induction on k define a sequence $\langle n_k: k \in \mathbb{N}\rangle$ as follows. Let n_0 be arbitrary. Given n_k let $n_{k+1} > n_k$ be such that

$$T^{n_{k+1}}x \in \bigcap\{T^{-n}U_k: n \leq n_0 + \ldots + n_k\}.$$

Put $A = \{n_k : k \in \mathbb{N}\}$. Thus for each k, $T^n x \in U_k$ for all sufficiently large $n \in FS(A)$. By Lemma 5.3 let B be an infinite set such that $FS(B) \subseteq FS(A)$ and $z = \lim_B x$ exists. Then $z \in \bigcap_{k=0}^{\infty}(\text{closure of } U_k) = \overline{y}$.

5.5. Lemma. Given $x \in X$ there exists $y \in \overline{x}$ such that every $z \in \overline{y}$ is uniformly recurrent.

Proof. Consider the class of nonempty closed sets $Y \subseteq \overline{x}$ such that $Y \subseteq T^{-1}(Y)$. By Zorn's Lemma, let Y_0 be a minimal element of this class. It is easy to see that every $y \in Y_0$ is uniformly recurrent. (See §1.4 of [9] or §6.1 of [13] or the proof of Lemma 5.10 below.)

We can now finish the proof of the Auslander-Ellis Theorem.

5.6. Theorem (AET). Given $x \in X$ there exists $z \in \overline{x}$ such that x and z are proximal and z is uniformly recurrent.

Proof. By Lemma 5.5 let $y \in \overline{x}$ be such that every $z \in \overline{y}$ is uniformly recurrent. By Lemma 5.4 let $z \in \overline{y}$ be such that $z = \lim_B x$ for some infinite $B \subseteq \mathbb{N}$. By Lemma 5.2 z is proximal to x. This completes the proof.

We now present an axiomatic analysis of the above proof of AET.

We assume familiarity with the formal systems ACA_0 and ACA_0^+ which were defined in §4. We also assume familiarity with the formal development of the theory of complete separable metric spaces within ACA_0. (See Brown [4] or §4 of Brown-Simpson [5].) Within ACA_0 (actually in the weaker system RCA_0), we define a compact metric space to be a complete separable metric space X such that there exists a countable sequence of finite sequences of points $<<x_{ni}: i \le k_n>: n \in \mathbb{N}>$, $x_{ni} \in X$, such that for all n and all $x \in X$ there exists $i \le k_n$ such that $d(x,x_{ni}) < 2^{-n}$. (For details about the formal development of the theory of compact metric spaces within ACA_0 and weaker systems, see Brown [4].)

5.7. Lemma. The following is provable in ACA_0 (actually in the weaker system WKL_0). Let X be a compact metric space. Let Y be a closed set in X and let $<U_k: k \in \mathbb{N}>$ be a sequence of open sets in X. If $Y \subseteq \bigcup_{k=0}^{\infty} U_k$ then $Y \subseteq \bigcup_{k=0}^{n} U_k$ for some n.

Proof. See Simpson [24] and Brown [4].

5.8. Lemma. The following is provable in ACA_0 (actually in WKL_0). Let X be a compact metric space. Let Y and U denote (codes for) a closed and open subset of X respectively. Then the formula $Y \subseteq U$ is equivalent to a Σ_1^0 formula.

Proof. We reason within WKL_0. If r is a positive rational number and $y \in X$, let $B(y,r)$ be the open ball of radius r about y. Let $<y_m: m \in \mathbb{N}>$ and $<r_m: m \in \mathbb{N}>$ be such that

$$U \cup (X\setminus Y) = \bigcup_{m=0}^{\infty} B(y_m, r_m)$$

$$= \bigcup_{m=0}^{\infty} \bigcup_{q<r_m} B(y_m, q).$$

By Lemma 5.7, $Y \subseteq U$ if and only if there exist n and $<m_i: i \le k_n>$ such that

$$d(x_{ni}, y_{m_i}) + 2^{-n} < r_{m_i}.$$

for all $i \le k_n$. This is Σ^0_1.

The next lemma is related to Annex 7.E (pages 481 et seq.) of Girard [11].

5.9. Lemma. The following is provable in ACA_0. Let X be a compact metric space and let $T: X \to X$ be a continuous function. Consider the class of nonempty closed sets $Y \subseteq X$ such that $Y \subseteq T^{-1}(Y)$. Then this class contains a minimal element.

Proof. We reason within ACA_0. Let $\langle V_m : m \in \mathbb{N}\rangle$ be an enumeration of the basic open sets of X. Let $\varphi(F)$ be a formula which says that F is a (code for a) finite subset of $\mathbb{N}$ and

$$X \setminus \bigcup_{n=0}^{\infty} T^{-n}(\bigcup_{m\in F} V_m) \neq \emptyset.$$

By Lemmas 5.7 and 5.8, $\varphi(F)$ is equivalent to a Π^0_1 formula. Hence $S = \{F: \varphi(F)\}$ exists by arithmetical comprehension. Define $f: \mathbb{N} \to \{0,1\}$ inductively by putting $f(m) = 1$ if

$$\{k < m: f(k) = 1\} \cup \{m\} \in S,$$

otherwise $f(m) = 0$. Put $Y = X \setminus U$ where

$$U = \bigcup\{V_m : f(m) = 1\}.$$

Thus Y is a closed subset of X.

Put $Z = \bigcap_{n=0}^{\infty} T^{-n}(Y)$. We claim that Z is nonempty. To see this, put

$$U_m = \bigcup\{V_k : k < m \wedge f(k) = 1\},$$

$$Y_m = X \setminus U_m, \text{ and}$$

$$Z_m = \bigcap_{n=0}^{\infty} T^{-n}(Y_m).$$

By Lemma 5.8 and arithmetical induction on m, $Z_m \neq \emptyset$ for all m. Since Z_m is closed, it follows by Lemma 5.7 that $Z = \bigcap_{m=0}^{\infty} Z_m \neq \emptyset$.

We claim that $Y = \overline{z}$ for any $z \in Z$. Clearly $\overline{z} \subseteq Y$. For the converse let m be such that $\overline{z} \cap V_m = \emptyset$. Then $f(m) = 1$. Hence $Y \cap V_m = \emptyset$. Since $\overline{z}$ is closed it follows that $Y \subseteq \overline{z}$.

From the above two claims, it follows easily that $Y \subseteq T^{-1}(Y)$ and that Y has the other desired properties. This completes the proof of Lemma 5.9.

5.10. <u>Lemma</u>. <u>The following is provable in</u> ACA_0. <u>Let</u> X <u>be a compact metric space and let</u> $T: X \to X$ <u>be a continuous function</u>. <u>For all</u> $x \in X$ <u>there exists</u> $y \in \overline{x}$ <u>such that every</u> $z \in \overline{y}$ <u>is uniformly recurrent</u>.

<u>Proof</u>. We reason within ACA_0. Clearly $\overline{x} \subseteq X$ is closed and $\overline{x} \subseteq T^{-1}(\overline{x})$. Applying Lemma 5.9 to the compact metric space $\overline{x}$, we obtain a nonempty closed set $Y \subseteq \overline{x}$ with $Y \subseteq T^{-1}(Y)$ and minimal with these properties. Thus $\overline{y} = Y$ for all $y \in Y$. It follows that for all $y, z \in Y$ and $\varepsilon > 0$ there exists k such that $d(T^k y, z) < \varepsilon$. In other words, for any $z \in Y$ and $\varepsilon > 0$, $Y \subseteq \bigcup_{k=0}^{\infty} U_k$ where

$$U_k = \{y : d(T^k y, z) < \varepsilon\}.$$

By Lemma 5.7 it follows that $Y \subseteq \bigcup_{k=0}^{m} U_k$ for some m. This implies that z is uniformly recurrent.

5.11. Theorem. The Auslander-Ellis Theorem AET is provable in ACA_0^+.

Proof. We follow the proof of Theorem 5.6 via Lemmas 5.2 through 5.5. The proof of Lemma 5.2 goes through in ACA_0 without difficulty. The proof of Lemma 5.3 goes through without difficulty in ACA_0 except for the countably many applications of Hindman's Theorem. At this point we invoke Theorem 4.13 which tells us that the desired construction can be performed within ACA_0^+. The proof of Lemma 5.4 goes through without difficulty in ACA_0 except for the application of Lemma 5.3. Thus Lemmas 5.3 and 5.4 are provable in ACA_0^+. Finally Lemma 5.10 tells us that Lemma 5.5 is provable in ACA_0. Combining these results as in the proof of Theorem 5.6, we see that AET is provable in ACA_0^+.

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