An Incompleteness Theorem for β_n-Models

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Abstract

Let n be a positive integer. By a β_n-model we mean an ω-model which is elementary with respect to Σ^1_n formulas. We prove the following β_n-model version of Gödel's Second Incompleteness Theorem. For any recursively axiomatized theory S in the language of second order arithmetic, if there exists a β_n-model of S, then there exists a β_n-model of S + "there is no countable β_n-model of S." We also prove a β_n-model version of Löb's Theorem. As a corollary, we obtain a β_n-model which is not a β_{n+1}-model.

1 Introduction

Let ω denote the set of natural numbers $\{0, 1, 2, \ldots\}$. Let $P(\omega)$ denote the set of all subsets of ω. An *ω-model* is a nonempty set $M \subseteq P(\omega)$, viewed as a model for the language of second order arithmetic. Here the number variables range over ω, the set variables range over M, and the arithmetical operations are standard. For n a positive integer, a *β_n-model* is an ω-model which is an elementary submodel of $P(\omega)$ with respect to Σ^1_n formulas of the language of second order arithmetic.

Recently Engström [3] posed the following question: Does there exist a β_n-model which is not a β_{n+1}-model? To our amazement, there seems to be no answer to this question in the literature.

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Previous research has focused on minimum β_n-models. A *minimum β_n-model* is a β_n-model which is included in all β_n-models. If a minimum β_n-model exists, then obviously it is unique, and it is not a β_{n+1}-model. However, the existence of minimum β_n-models is problematic, to say the least. Simpson [10, Corollary VIII.6.9] proves that there is no minimum β_1-model. Shilleto [8] proves the existence of a minimum β_2-model. Enderton and Friedman [2] prove the existence of minimum β_n-models, $n \geq 3$, assuming a basis property which follows from $\mathrm{V} = \mathrm{L}$ but which is not provable in ZFC. We conjecture that the existence of a minimum β_n-model is not provable in ZFC, for $n \geq 3$. We have verified this conjecture for $n \geq 4$. Simpson's book [10, Sections VII.1–VII.7 and VIII.6] contains further results concerning minimum β_1- and β_2-models of specific subsystems of second order arithmetic, as well as β_n-models for $n \geq 3$. See also Remark 3.6 below.

In this paper we answer Engström's question affirmatively. We prove that, for each $n \geq 1$, there exists a β_n-model which is not a β_{n+1}-model (Corollary 3.7). Our proof is based on a β_n-model version of Gödel's Second Incompleteness Theorem (Theorem 2.1). We draw corollaries concerning β_n-models of specific true theories (Corollary 3.3, Remark 3.5). We also obtain a β_n-model version of Löb's Theorem (Theorem 2.3).

Preliminaries

Our results are formulated in terms of L_2, the *language of second order arithmetic*. L_2 has variables of two sorts: first order (number) variables, denoted $i, j, k, m, n, \ldots$ and intended to range over ω, and second order (set) variables, denoted $X, Y, Z, \ldots$ and intended to range over $P(\omega)$. The variables of both sorts are quantified. We also have addition, multiplication, equality, and order for numbers, denoted $+, \cdot, =, <$, as well as set membership, denoted $\in$. Recall that an *ω-model* is a nonempty subset of $P(\omega)$. For M an ω-model and Φ an L_2-sentence with parameters from M, we define $M \models \Phi$ to mean that M *satisfies* Φ, i.e., Φ is true in the L_2-model $(\omega, M, +, \cdot, =, <, \in)$. If S is a set of L_2-sentences, we define $M \models S$ to mean that $M \models \Phi$ for all $\Phi \in S$.

An L_2-formula is said to be *arithmetical* if it contains no set quantifiers. An L_2-formula is said to be Σ^1_n if it is equivalent to a formula of the form

$$\exists X_1 \forall X_2 \exists X_3 \cdots X_n \, \Theta$$

with n alternating set quantifiers, where Θ is arithmetical. An L_2-formula is said to be Π^1_n if its negation is Σ^1_n. A *β_n-model* is an ω-model M such that, for all Σ^1_n formulas $\Phi(X_1, \ldots, X_k)$ with exactly the free variables displayed, and for all $A_1, \ldots, A_k \in M$,

$$P(\omega) \models \Phi(A_1, \ldots, A_k) \quad \Leftrightarrow \quad M \models \Phi(A_1, \ldots, A_k).$$

If X is a subset of ω, then X can be viewed as coding a countable ω-model $\{(X)_i : i \in \omega\}$, where $(X)_i = \{j : 2^i 3^j \in X\}$. Moreover, every countable ω-model can be coded in this way. Therefore we define a *countable coded ω-model* to be simply a subset of ω. A *countable coded β_n-model* is then a countable coded ω-model X such that $\{(X)_i : i \in \omega\}$ is a β_n-model.

2 A β_n-model version of Gödel's Theorem

We now present the main theorem of this paper. Our theorem is a β_n-model version of Gödel's Second Incompleteness Theorem [6]. It was inspired by the ω-model version, due to Friedman [4, Chapter II], as expounded in Simpson [10, Theorem VIII.5.6]. See also Steel [11] and Friedman [5].

Theorem 2.1. *Let S be a recursively axiomatized theory in the language of second order arithmetic. For each $n \geq 1$, if there exists a β_n-model of S, then there exists a β_n-model of $S +$ "there is no countable coded β_n-model of S."*

Proof. We use the subsystem ACA_0^+ of second order arithmetic. The axioms of ACA_0^+ consist of the basic axioms, induction, arithmetical comprehension, and "for every set X, the ωth Turing jump of X exists." See Blass/Hirst/Simpson [1] and Simpson [10, Definition X.3.2]. Note that every β_n-model automatically satisfies ACA_0^+. Furthermore, ACA_0^+ proves that for every countable coded ω-model there exists a full satisfaction predicate. This allows us to write L_2-formulas which assert certain properties of countable coded ω-models. Let $\mathrm{B}_n(X)$ be the L_2-formula asserting that X is a countable coded β_n-model. Let $\mathrm{Sat}(X, S)$ be the L_2-formula asserting that $X \models S$, i.e., the countable ω-model $\{(X)_i : i \in \omega\}$ satisfies Φ for all $\Phi \in S$. For brevity we introduce the L_2-formula

$$\mathrm{B}_n(X, S) \quad \equiv \quad \mathrm{B}_n(X) \wedge \mathrm{Sat}(X, S)$$

asserting that X is a countable coded β_n-model of S.

Consider the L_2-theory T consisting of $\mathsf{ACA}_0^+ + \Phi_1 + \Phi_2$, where

$$\Phi_1 \quad \equiv \quad \exists X\, \mathrm{B}_n(X, S)\,,$$

$$\Phi_2 \quad \equiv \quad \forall Y\, (\mathrm{B}_n(Y, S) \Rightarrow Y \models \exists Z\, \mathrm{B}_n(Z, S))\,.$$

We claim that T proves $\mathrm{Con}(T)$, the standard L_2-sentence asserting consistency of T. To see this, we reason within T. By Φ_1 there exists X such that $\mathrm{B}_n(X, S)$ holds. We claim that X satisfies T. Being a β_n-model, X satisfies ACA_0^+. Furthermore, in light of Φ_2, X satisfies Φ_1. It remains to show that X satisfies Φ_2. For this, let $Y = (X)_i$ be such that X satisfies $\mathrm{B}_n(Y, S)$. Then $\mathrm{B}_n(Y, S)$ is true, because a β_n-submodel of a β_n-model is a β_n-model. Hence by Φ_2 we have $Y \models \exists Z\, \mathrm{B}_n(Z, S)$. We conclude that $X \models \Phi_2$. We have now shown that X is a model of T. Thus T is consistent. Our claim is proved.

We have shown that T proves $\mathrm{Con}(T)$. From this plus Gödel's Second Incompleteness Theorem [6], it follows that T is inconsistent. In other words, $\Phi_1 \Rightarrow \neg\, \Phi_2$ is provable in ACA_0^+. Since ACA_0^+ is true, $\Phi_1 \Rightarrow \neg\, \Phi_2$ is true.

To prove Theorem 2.1, assume the existence of a β_n-model of S. By the Löwenheim/Skolem Theorem, there exists a countable coded β_n-model of S. In other words, Φ_1 holds. Therefore, $\neg\,\Phi_2$ holds, i.e., there exists a β_n-model of S which does not contain a countable coded β_n-model of S. This completes the proof of Theorem 2.1. □

Remark 2.2. In proving Theorem 2.1, we have actually proved more. Namely, we have proved that Theorem 2.1 is provable in ACA_0^+. Actually we could replace ACA_0^+ throughout this paper by the weaker theory $\mathsf{ACA}_0^* = \mathsf{ACA}_0 + \forall n\,\forall X$ (the nth Turing jump of X exists).

Our β_n-model version of Löb's Theorem [7] is as follows.

Theorem 2.3. *Let S be a recursively axiomatized L_2-theory. Let Φ be an L_2-sentence. For each $n \geq 1$, if every β_n-model of S satisfies*

$$\text{“every countable coded } \beta_n\text{-model of } S \text{ satisfies } \Phi\text{”} \Rightarrow \Phi,$$

then every β_n-model of S satisfies Φ.

Proof. This is a reformulation of Theorem 2.1 with S replaced by $S + \neg\,\Phi$. □

3 Some corollaries of Theorem 2.1

In this section we draw corollaries concerning β_n-models which are not β_{n+1}-models. In order to do so, we need the following lemmas, which are well known.

Lemma 3.1. *For each $n \geq 1$, the formula $\mathrm{B}_n(X, S)$ is equivalent to a Π^1_n formula. The equivalence is provable in ACA_0^+.*

Proof. Note that an ω-model M is a β_n-model if and only if

$$\forall e\; \forall Y, Z \in M\, (\Psi_n(e, Y, Z) \Rightarrow M \models \Psi_n(e, Y, Z)),$$

where $\Psi_n(e, X, Y)$ is a universal Σ^1_n formula. (The existence of such a formula is provable in ACA_0^+, or actually in ACA_0. See Simpson [10, Lemma V.1.4 and pages 252, 306, 333].) Applying this observation to the countable ω-model $M = \{(X)_i : i \in \omega\}$ coded by X, we have in ACA_0^+ that $\mathrm{B}_n(X)$ holds if and only if

$$\forall e\, \forall i\, \forall j\, (\Psi_n(e, (X)_i, (X)_j) \Rightarrow X \models \Psi_n(e, (X)_i, (X)_j)).$$

Thus $\mathrm{B}_n(X)$ is Π^1_n. Furthermore, ACA_0^+ proves the existence of a full satisfaction predicate for X which is implicitly defined by an arithmetical formula. Thus $\mathrm{Sat}(X, S)$ is both Σ^1_1 and Π^1_1. We now see that $\mathrm{B}_n(X, S)$ is Π^1_n. □

Lemma 3.2. *Let S be a recursively axiomatized L_2-theory. Assume the existence of a β_n-model of S. Let M be an ω-model of $\mathsf{ACA}_0^+ +$ “there is no countable coded β_n-model of S.” Then M is not a β_{n+1}-model.*

Proof. Lemma 3.1 implies that the sentence $\exists X\, \mathrm{B}_n(X, S)$ is Σ^1_{n+1}. Our hypotheses imply that this sentence holds in $P(\omega)$ but not in M. Thus M is not a β_{n+1}-model. □

We now present our corollaries.

Corollary 3.3. *Let S be a recursively axiomatized L_2-theory. For each $n \geq 1$, if there exists a β_n-model of S, then there exists a β_n-model of $S +$ "there is no countable coded β_n-model of S." Such a β_n-model is not a β_{n+1}-model.*

Proof. This is immediate from Theorem 2.1 and Lemma 3.2, noting that any β_n-model satisfies ACA_0^+. □

Corollary 3.4. *Let S be a recursively axiomatized L_2-theory which is true, i.e., which holds in $P(\omega)$. Then for each $n \geq 1$ there exists a β_n-model of $S +$ "there is no countable coded β_n-model of S." Such a β_n-model is not a β_{n+1}-model.*

Proof. This is immediate from Corollary 3.3, since $P(\omega)$ is a β_n-model. □

Remark 3.5. In Corollary 3.4, S can be any true recursively axiomatized L_2-theory. For example, we may take S to be any of the following specific L_2-theories, which have been discussed in Simpson [10]: Π^1_m comprehension, Π^1_m transfinite recursion, Σ^1_m choice, Σ^1_m dependent choice, strong Σ^1_m dependent choice, $m \geq 1$, or any union of these, e.g., Π^1_∞ comprehension, Σ^1_∞ choice, Σ^1_∞ dependent choice. Note that Π^1_∞ comprehension is full second order arithmetic, called Z_2 in [10].

Remark 3.6. Let S be any of the specific L_2-theories mentioned in Remark 3.5, except Σ^1_1 choice and Σ^1_1 dependent choice. By a *minimum β_n-model of S* we mean a β_n-model of S which is included in all β_n-models of S. For $n = 1, 2$ a minimum β_n-model of S can be obtained by methods of Simpson [10, Chapter VII] and Shilleto [8] respectively. For $n \geq 3$ a minimum β_n-model of S can be obtained by methods of Simpson [10, Chapter VII] and Enderton/Friedman [2] assuming $\mathrm{V} = \mathrm{L}$.

We answer Engström's question [3] affirmatively as follows.

Corollary 3.7. *For each $n \geq 1$ there exists a β_n-model which is not a β_{n+1}-model.*

Proof. In Corollary 3.4 let S be the trivial L_2-theory with no axioms. □

Remark 3.8. Corollary 3.7 follows from the results of Enderton/Friedman [2] assuming $\mathrm{V} = \mathrm{L}$. We do not know of any proof of Corollary 3.7 in ZFC, other than the proof which we have given here.

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