Baire categoricity and Σ^0_1 induction

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First draft: February 24, 2012
This draft: October 23, 2013

Abstract

We investigate the reverse-mathematical status of a version of the Baire Category Theorem known as BCT. In a 1993 paper Brown and Simpson showed that BCT is provable in RCA_0. We now show that BCT is equivalent to RCA_0 over RCA_0^*.

Keywords: Reverse mathematics, second-order arithmetic, Baire Category Theorem, RCA_0, RCA_0^*, Σ^0_1 induction.

2010 Mathematics Subject Classification: Primary 03B30; Secondary 03F25, 54E52.

Notre Dame Journal of Formal Logic, 55, 2014, pp. 75–78.

Consider the following version of the Baire Category Theorem.

Definition 1. Let BCT be the statement that, in any complete separable metric space, the intersection of any countable sequence of dense open sets is dense. Thus BCT is essentially the usual statement of the Baire Category Theorem for complete separable metric spaces.

The purpose of this paper is to prove a new result concerning the reverse-mathematical status of BCT. From Brown/Simpson [1, Theorem 2.1] we already know that BCT is provable in RCA_0. Here of course RCA_0 is the usual base theory for reverse mathematics [6], consisting of Δ^0_1 comprehension plus Σ^0_1 induction. We now prove that BCT is logically equivalent to RCA_0 over the weaker base theory RCA_0^*. The system RCA_0^* was first introduced in [7]. Two recent papers making use of RCA_0^* are [9] and [11].

In addition to BCT itself, we consider the following special case of BCT.

Definition 2. The *Cantor space* is the space $\{0,1\}^{\mathbb{N}}$ of infinite sequences of 0's and 1's. We endow $\{0,1\}$ with the discrete topology and $\{0,1\}^{\mathbb{N}}$ with the product topology. Let $\mathrm{BCT}(\{0,1\}^{\mathbb{N}})$ be the statement that BCT holds for the Cantor space.

More precisely, let $\{0,1\}^*$ be the set of finite sequences of 0's and 1's. For $\sigma \in \{0,1\}^*$ and $X \in \{0,1\}^{\mathbb{N}}$ we write $\sigma \subset X$ to mean that σ is an initial segment of X. We also write $[\![\sigma]\!] = \{X \in \{0,1\}^{\mathbb{N}} \mid \sigma \subset X\}$. Note that the sets $[\![\sigma]\!]$ where $\sigma \in \{0,1\}^*$ form a basis for the topology of $\{0,1\}^{\mathbb{N}}$. For $\sigma, \tau \in \{0,1\}^*$ we write $\sigma \subset \tau$ to mean that σ is a proper initial segment of τ. We say that $D \subseteq \{0,1\}^*$ is *dense in* $\{0,1\}^*$ if for all $\sigma \in \{0,1\}^*$ there exists $\tau \in D$ such that $\sigma \subset \tau$. Thus the dense open sets in $\{0,1\}^{\mathbb{N}}$ are just the sets of the form $[\![D]\!] = \bigcup_{\tau \in D} [\![\tau]\!]$ where D is dense in $\{0,1\}^*$. We say that $X \in \{0,1\}^{\mathbb{N}}$ *meets* D if there exists $\tau \in D$ such that $\tau \subset X$. Within RCA_0^* let $\mathrm{BCT}(\{0,1\}^{\mathbb{N}})$ be the statement that for all sequences of dense sets $D_i \subseteq \{0,1\}^*$, $i \in \mathbb{N}$, and all $\sigma \in \{0,1\}^*$ there exists $X \in \{0,1\}^{\mathbb{N}}$ such that $\sigma \subset X$ and X meets D_i for each $i \in \mathbb{N}$.

Theorem 1. The following are pairwise equivalent over RCA_0^*.

1. RCA_0.
2. BCT.
3. $\mathrm{BCT}(\{0,1\}^{\mathbb{N}})$.
4. For all finite sequences of dense sets $D_i \subseteq \{0,1\}^*$, $1 \le i \le n$, there exists $X \in \{0,1\}^{\mathbb{N}}$ such that X meets D_i for each $i = 1, \ldots, n$.

Proof. We reason in RCA_0^*. The implication $1 \Rightarrow 2$ is already known [1, Theorem 2.1]. The implications $2 \Rightarrow 3$ and $3 \Rightarrow 4$ are obvious.

It remains to prove $4 \Rightarrow 1$. For this purpose we use the following lemma from [9]. Within RCA_0^* a set $C \subseteq \mathbb{N}$ is defined to be *infinite* if it is not finite, or equivalently, it is *unbounded*, i.e., $\forall n \, \exists c \, (n < c \in C)$.

Lemma 1. Over RCA_0^* the following are equivalent.

1. RCA_0.
2. Each infinite subset of $\mathbb{N}$ includes arbitrarily large finite subsets.

Proof. This is [9, Lemma 3.2]. □

We now prove $4 \Rightarrow 1$. Assume 4. By Lemma 1 it suffices to prove that each infinite subset of $\mathbb{N}$ includes arbitrarily large finite subsets. Given an infinite set $C \subseteq \mathbb{N}$, for each $i \in \mathbb{N}$ let D_i be the set of strings in $\{0,1\}^*$ of the form

$$\sigma^\frown\langle 1\rangle^\frown\underbrace{\langle 0,\ldots,0\rangle}_{c}{}^\frown\langle 1\rangle^\frown\underbrace{\langle 0,\ldots,0\rangle}_{i}{}^\frown\langle 1\rangle \tag{1}$$

where $c \in C$ and $c >$ the length of σ. The sequence of sets $\langle D_i \mid i \in \mathbb{N}\rangle$ exists by Δ^0_1 comprehension. Since C is infinite, each D_i is dense in $\{0,1\}^*$. Given $n \in \mathbb{N}$, apply 4 to obtain $X \in \{0,1\}^{\mathbb{N}}$ such that X meets D_i for each $i = 1, \ldots, n$. By Σ^0_1 bounding [7] plus Δ^0_1 comprehension, there exists a finite sequence of strings τ_i, $1 \le i \le n$, such that $\tau_i \in D_i$ and $\tau_i \subset X$ for each $i = 1, \ldots, n$. Consider the finite sequence $c_1, \ldots, c_n$ where τ_i is as in (1) with $c = c_i$. For $i \neq j$ we have $\tau_i \neq \tau_j$, hence $\tau_i \subset \tau_j$ or $\tau_j \subset \tau_i$, hence $c_i < c_j$ or $c_j > c_i$, hence $c_i \neq c_j$. Thus $\{c_1, \ldots, c_n\}$ is a finite subset of C of cardinality n, Q.E.D. □

Theorem 2. BCT is not Π^0_1-conservative over RCA^*_0.

Proof. Recall from [6, §X.4] and [7] that RCA^*_0 is RCA_0 with Σ^0_1 induction weakened to Σ^0_0 induction plus *natural number exponentiation*, i.e., the assertion that m^n exists for all $m, n \in \mathbb{N}$. It is known that RCA^*_0 is Π^0_2-equivalent to Elementary Function Arithmetic [7], hence much weaker than RCA_0 which is Π^0_2-equivalent to Primitive Recursive Arithmetic [6, §IX.3]. Since Primitive Recursive Arithmetic proves the consistency of Elementary Function Arithmetic (see for instance [4] or [6, Theorems II.8.11 and IX.3.16]), it follows that RCA_0 is not Π^0_1-conservative over RCA^*_0. This fact together with Theorem 1 gives Theorem 2. □

Remarks.

1. Beyond RCA^*_0 one may consider even weaker base theories for reverse mathematics. In this direction there is the following result of Fernandes [2]: BCT is Π^1_1-conservative over Σ^b_1-NIA+∇^b_1-CA. Note that Σ^b_1-NIA+∇^b_1-CA is "feasible," i.e., it does not include natural number exponentiation.

2. Actually Fernandes [2] showed that BCT is conservative over Σ^b_1-NIA+∇^b_1-CA not only for Π^1_1 sentences but also for sentences of the form $(\forall X)\,(\exists$ unique $Y)\,\Phi$ where Φ is arithmetical. And Yamazaki [10] showed that Π^0_∞-BCT is conservative over RCA_0 for this same class of sentences, which arose previously in connection with Tanaka's Conjecture [8, Theorem 4.18].

3. Our Theorems 1 and 2 were inspired by Fernandes [2, Proposition 1] and Hirschfeldt/Shore/Slaman [3, Theorem 4.3].

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