ON THE STRENGTH OF KÖNIG'S DUALITY THEOREM FOR COUNTABLE BIPARTITE GRAPHS

STEPHEN G. SIMPSON

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Abstract. Let CKDT be the assertion that, for every countably infinite bipartite graph G, there exist a vertex covering C of G and a matching M in G such that C consists of exactly one vertex from each edge in M. (This is a theorem of Podewski and Steffens [12].) Let ATR_0 be the subsystem of second order arithmetic with arithmetical transfinite recursion and restricted induction. Let RCA_0 be the subsystem of second order arithmetic with recursive comprehension and restricted induction. We show that CKDT is provable in ATR_0. Combining this with a result of Aharoni, Magidor and Shore [2], we see that CKDT is logically equivalent to the axioms of ATR_0, the equivalence being provable in RCA_0.

1. Introduction

A *bipartite graph* is an ordered triple $G = (X, Y, E)$ such that X and Y are sets, $X \cap Y = \emptyset$, and $E \subseteq \{\{x, y\} : x \in X, y \in Y\}$. The *vertices* of G are the elements of $X \cup Y$. The *edges* of G are the elements of E.

A *covering of* G is a set $C \subseteq X \cup Y$ such that every edge of G has a vertex in C, *i.e.* we have $C \cap e \neq \emptyset$ for all $e \in E$.

A *matching* in G is a pairwise disjoint set $M \subseteq E$. Here pairwise disjointness means that no two edges in M have a common vertex, *i.e.* we have $e_1 \cap e_2 = \emptyset$ for all $e_1, e_2 \in M$ such that $e_1 \neq e_2$.

For any set S we use $|S|$ to denote the cardinality of S, *i.e.* the number of elements in S. If G is any bipartite graph and C is any covering of G and M is any matching in G, then clearly $|C| \geq |M|$. The *König duality theorem* [7] asserts that, for any

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finite bipartite graph G, there exist a covering C of G and a matching M in G such that $|C| = |M|$. In other words,

$$\min\{|C| : C \text{ is a covering of } G\} = \max\{|M| : M \text{ is a matching in } G\}\,.$$

Definition 1.1. *For any bipartite graph G, a* König covering of G *is an ordered pair (C, M) such that C is a covering of G, M is a matching in G, and C consists of exactly one vertex from each edge of M. (The last condition means that $C \subseteq \bigcup M$ and $|C \cap e| = 1$ for each $e \in M$.)*

Clearly if (C, M) is a König covering of G then $|C| = |M|$. König [7] showed that every finite bipartite graph has a König covering. From this the König duality theorem follows immediately.

Podewski, Steffens and Aharoni extended the König duality theorem to infinite bipartite graphs. In order to make such extensions meaningful, they considered König coverings. Podewski and Steffens [12] showed that every countably infinite bipartite graph has a König covering. Aharoni [1] showed that every uncountable bipartite graph has a König covering. We refer to the Podewski–Steffens theorem (respectively Aharoni's theorem) as *the König duality theorem for countable* (respectively *uncountable*) *bipartite graphs*.

Aharoni, Magidor and Shore [2] considered the following logical or foundational question: *Which set existence axioms are needed to prove the König duality theorem for countable bipartite graphs?* Aharoni, Magidor and Shore obtained a partial answer to this question, but they did not answer it completely. The purpose of this paper is to finish the work which was begun by Aharoni, Magidor and Shore.

The general question of which set existence axioms are needed to prove specific mathematical theorems is of basic importance for the foundations of mathematics. This general question has been studied fruitfully in the context of subsystems of second order arithmetic. For this purpose, five of the most important subsystems of second order arithmetic are RCA_0, WKL_0, ACA_0, ATR_0, and Π^1_1-CA_0. It is known that these five systems are of strictly increasing strength as regards their ability to prove mathematical theorems. Moreover, for many particular mathematical theorems, it turns out that one can determine the weakest natural subsystem of second order arithmetic in which the given mathematical theorem is provable. Such results are established by showing that the given mathematical theorem is logically equivalent to the axioms of the specified subsystem of second order arithmetic, the equivalence being proved in a weaker system. Consider for example the Bolzano–Weierstrass theorem: every bounded sequence of real numbers has a convergent subsequence. It is known that the weakest subsystem of second order arithmetic in which the Bolzano–Weierstrass theorem is provable is ACA_0. This is established by showing that the Bolzano-Weierstrass theorem is logically equivalent to the axioms of ACA_0, the equivalence being proved in the weaker system RCA_0.

For a survey of subsystems of second order arithmetic and their role in foundational studies, see my article [16]. A fuller treatment will appear in [17]. For additional results and open problems concerning logical and foundational aspects of combinatorics, see the articles in *Logic and Combinatorics* [8], especially [3].

Aharoni, Magidor and Shore [2] made a major contribution to the foundational program of [16]. They obtained two important results. First, the König duality theorem for countable bipartite graphs (*i.e.* CKDT) is provable in Π^1_1-CA_0. Second, CKDT logically implies the axioms of ATR_0, this implication being provable in the weak system RCA_0. (Aharoni, Magidor and Shore also obtained results concerning logical aspects of some other infinitistic variants of the König duality theorem.)

The main result of the present paper is that the König duality theorem for countable bipartite graphs is provable in ATR_0. This is established in Section 3 below. Combining this with the results of Aharoni, Magidor and Shore, we see that CKDT is logically equivalent to the axioms of ATR_0, the equivalence being provable in RCA_0. Thus ATR_0 is the weakest natural subsystem of second order arithmetic in which CKDT is provable.

2. Subsystems of Second Order Arithmetic

In this section we present some background material concerning ATR_0 and related systems. We present little more than what is needed for our main result, the provability of CKDT in ATR_0. For a broad survey of subsystems of second order arithmetic, see [16]. For detailed information on ATR_0, see [6], [14], [15], [16], and [17].

All of the systems which we shall consider are first-order theories in *the language of second order arithmetic*. This is a first-order language with two sorts of variables: number variables i, j, k, m, n, ..., and set variables U, V, W, X, Y, Z, ... Number variables are intended to range over the set of natural numbers $\omega = \{0, 1, 2, \dots\}$, while set variables are intended to range over subsets of ω. *Numerical terms* are built up as usual from number variables, the constant symbols 0 and 1, and the binary operations of addition and multiplication. The *atomic formulas* of the language are $t_1 = t_2$, $t_1 < t_2$, and $t_1 \in X$, where t_1 and t_2 are numerical terms and X is any set variable. *Formulas* are built up from atomic formulas by means of propositional connectives, number quantifiers $\forall n$ and $\exists n$ where n is any number variable, and set quantifiers $\forall X$ and $\exists X$ where X is any set variable. A *sentence* is a formula with no free variables. The *universal closure* of a formula is the sentence obtained from the formula by prefixing it with universal quantifiers on all of its free number variables and free set variables. Note that $X = Y$ is not a formula of our language. Rather, equality for sets is defined by extensionality:

$$X = Y \quad \equiv \quad \forall n\, (n \in X \leftrightarrow n \in Y)\,.$$

All of the systems which we shall consider include the Basic Arithmetical Axioms and the Restricted Induction Axiom, expressing elementary properties of the natural number system. The *Basic Arithmetical Axioms* are the universal closures of the formulas $n+1 \neq 0$, $m+1 = n+1 \rightarrow m = n$, $m+0 = m$, $m+(n+1) = (m+n)+1$, $m \cdot 0 = 0$, $m \cdot (n+1) = m \cdot n + m$, $\sim m < 0$, and $m < n+1 \leftrightarrow (m < n \vee m = n)$. The *Restricted Induction Axiom* is the universal closure of

$$(0 \in X \wedge \forall n \, (n \in X \rightarrow n+1 \in X)) \rightarrow \forall n \, (n \in X) \, .$$

The *Comprehension Scheme* consists of the universal closures of all formulas of the form

$$\exists X \, \forall n \, (n \in X \leftrightarrow \varphi(n)) \tag{1}$$

where $\varphi(n)$ is any formula in which X does not occur freely. The idea here is that the given instance of the Comprehension Scheme asserts the existence of an explicitly defined set $X = \{n : \varphi(n)\}$ consisting of all natural numbers n such that $\varphi(n)$ holds. *Second order arithmetic*, also called Z_2, is the first-order theory whose axioms are the Basic Arithmetical Axioms, the Restricted Induction Axiom, and the Comprehension Scheme.

A *theorem of* Z_2 is any sentence which is deducible from the axioms of Z_2. A *subsystem of* Z_2 is any first-order theory T in the language of Z_2 whose axioms are included in the theorems of Z_2. A *theorem of* T is any sentence which is deducible from the axioms of T. Theorems of T are also said to be *provable in* T. At all times we employ the usual axioms and deduction rules of classical first-order logic, with equality for the numerical sort. The *intended model* of the language of Z_2 is

$$(P(\omega), \omega, +, \cdot, 0, 1, <, =)$$

where $(\omega, +, \cdot, 0, 1, <, =)$ is the standard natural number system and $P(\omega)$ is the power set of ω. Clearly all of the axioms of Z_2 are true in the intended model. If T is any subsystem of Z_2, a *model of* T is any structure $\mathcal{M}$ such that all of the axioms of T are true in $\mathcal{M}$. Here we are employing the well known Tarski truth definition for models of a first-order theory. By the Gödel completeness theorem for first-order logic, the theorems of T are precisely the sentences which are true in all models of T.

An *ω-model of* T is a model $\mathcal{M}$ of T whose numerical part is the standard natural number system. Thus we have

$$\mathcal{M} = (\mathcal{S}, \omega, +, \cdot, 0, 1, <, =)$$

where $\mathcal{S} \subseteq P(\omega)$. We shall sometimes identify $\mathcal{M}$ with $\mathcal{S}$.

An *arithmetical formula* is a formula which contains no set quantifiers. Note that an arithmetical formula may contain free set variables, as well as free and bound number variables and number quantifiers. A Σ^1_1 (respectively Π^1_1) formula is one of the form $\exists X \, \theta$ (respectively $\forall X \, \theta$) where X is any set variable and θ is any arithmetical formula. More generally, for $k \in \omega$, a formula is said to be Σ^1_k (respectively Π^1_k) if it is of the form $\exists X_1 \, \forall X_2 \ldots X_k \; \theta$ (respectively $\forall X_1 \, \exists X_2 \ldots X_k \; \theta$) where θ is arithmetical.

Thus a Σ^1_k or Π^1_k formula consists of k alternating set quantifiers followed by a formula containing no set quantifiers. In a Σ^1_k formula the initial set quantifier is existential, while in a Π^1_k formula it is universal (assuming $k > 0$).

The *Arithmetical Comprehension Scheme* consists of all instances of the Comprehension Scheme (1) in which the formula $\varphi(n)$ is arithmetical.

Definition 2.1. ACA_0 *is the subsystem of* Z_2 *whose axioms are the Basic Arithmetical Axioms, the Restricted Induction Axiom, and the Arithmetical Comprehension Scheme.*

The letters ACA stand for *arithmetical comprehension axiom*. More generally, for $k \in \omega$, we define Π^1_k-CA_0 to be the subsystem of Z_2 consisting of ACA_0 plus all instances of the Comprehension Scheme (1) in which the formula $\varphi(n)$ is Π^1_k. One could define Σ^1_k-CA_0 similarly, but nothing new is obtained, since Σ^1_k-CA_0 is easily seen to be logically equivalent to Π^1_k-CA_0. Note also that Π^1_0-CA_0 is the same as ACA_0. It can be shown that, for all $k \in \omega$, Π^1_{k+1}-CA_0 is stronger than Π^1_k-CA_0. In particular, Π^1_1-CA_0 is stronger than ACA_0.

Π^1_1-CA_0 and ACA_0 are two of the most important subsystems of Z_2. There are at least two other important subsystems, RCA_0 and WKL_0, both of which are weaker than ACA_0. Although RCA_0 and WKL_0 are of great interest, we shall not define these systems here because they are not essential to our purpose.

When reasoning within a subsystem of Z_2, we use the symbol $\mathbb{N}$ to denote the set of natural numbers within the system, *i.e.* $\mathbb{N} = \{n : n = n\}$. Thus $\forall n\,(n \in \mathbb{N})$ is provable in ACA_0. We introduce the *numerical pairing function*

$$(m, n) = (m + n)^2 + m\,.$$

The usual properties such as

$$\forall i\,\forall j\,\forall m\,\forall n\ ((i, j) = (m, n) \leftrightarrow (i = m \wedge j = n))$$

can be proved as theorems of ACA_0. We shall also need a *set pairing function*,

$$(X, Y) = X \oplus Y = \{2n : n \in X\} \cup \{2n + 1 : n \in Y\}$$

and again the usual properties can be proved in ACA_0.

Reasoning within ACA_0 and using the numerical pairing function, we may view any set $Y \subseteq \mathbb{N}$ as encoding a countable sequence of sets $\langle (Y)_n : n \in \mathbb{N}\rangle$ where

$$(Y)_n = \{m : (m, n) \in Y\}\,.$$

The *Countable Choice Scheme* consists of the universal closures of all formulas of the form

$$(\forall n\,\exists X\,\varphi(n, X)) \rightarrow \exists Y\,\forall n\ \varphi(n, (Y)_n) \tag{2}$$

where $\varphi(n, X)$ is any formula in which Y does not occur freely.

Definition 2.2. Σ^1_1-AC_0 *is the subsystem of* Z_2 *consisting of* ACA_0 *plus all instances of the Countable Choice Scheme* (2) *in which the formula* $\varphi(n, X)$ *is* Σ^1_1.

The letters AC stand for *axiom of choice*. It can be shown that the system Σ^1_1-AC_0 is intermediate in strength between ACA_0 and Π^1_1-CA_0.

Still reasoning within ACA_0 and using the numerical pairing function, we may view any set $X \subseteq \mathbb{N}$ as encoding a binary relation $R \subseteq \mathbb{N} \times \mathbb{N}$, where $(i\,R\,j) \equiv (i, j) \in X$. We therefore say that X is a *linear ordering of* $\mathbb{N}$, abbreviated $\mathrm{LO}(X)$, if $\forall i\,\forall j\,\forall k\,(((i, j) \in X \wedge (j, k) \in X) \rightarrow (i, k) \in X)$ and $\forall i\,(i, i) \notin X$ and $\forall i\,\forall j\,(i = j \vee (i, j) \in X \vee (j, i) \in X)$. We say that X is a *well ordering of* $\mathbb{N}$, abbreviated $\mathrm{WO}(X)$, if $\mathrm{LO}(X)$ and

$$\forall Y\,((\forall j\,(\forall i\,((i, j) \in X \rightarrow i \in Y) \rightarrow j \in Y)) \rightarrow \forall j\,(j \in Y))\,. \tag{3}$$

Let $\varphi(n, j, W)$ be any formula with two distinguished free number variables n and j and a distinguished free set variable W. If Z is a set and X is a well ordering of $\mathbb{N}$, we say that Z *is obtained by transfinite recursion along* X *via* $\varphi(n, j, W)$, abbreviated $\mathrm{Rec}(X, \varphi, Z)$, if

$$\forall j\,\forall n\,(n \in (Z)_j \leftrightarrow \varphi(n, j, (Z)^j_X)))$$

where

$$(Z)^j_X = \{(m, i) : m \in (Z)_i \wedge (i, j) \in X\}\,.$$

The idea here is that, for each j, the set $(Z)_j$ is defined recursively in terms of the sets $(Z)_i$ for all i preceding j in the well ordering X. The *Transfinite Recursion Scheme* consists of the universal closures of all formulas of the form

$$\forall X\,(\mathrm{WO}(X) \rightarrow \exists Z\,\mathrm{Rec}(X, \varphi, Z)) \tag{4}$$

where Z does not occur freely in $\varphi(n, j, W)$. Thus the Transfinite Recursion Scheme asserts the existence of sets defined by transfinite recursion along arbitrary well orderings of $\mathbb{N}$.

Definition 2.3. ATR_0 *is the subsystem of* Z_2 *consisting of* ACA_0 *plus all instances of the Transfinite Recursion Scheme* (4) *in which the formula* φ *is arithmetical.*

The letters ATR stand for *arithmetical transfinite recursion*. It can be shown that ATR_0 is intermediate in strength between Σ^1_1-AC_0 and Π^1_1-CA_0. The system ATR_0 was introduced by Friedman ([5], [4]), who also emphasized its importance for the foundations of mathematics. It is known [16] that many mathematical theorems are provable in ATR_0 and indeed logically equivalent to ATR_0, the equivalence being provable in ACA_0 (in fact in RCA_0). For example, this is the case for the open Ramsey theorem (see [6] and [17]).

An important technique for proving mathematical theorems within ATR_0 is the use of inner models ([6], [10], [17]). Within ATR_0, any subset Z of $\mathbb{N}$ determines a countable set $\mathcal{S} = \{(Z)_n : n \in \mathbb{N}\}$ of subsets of $\mathbb{N}$. This set of sets $\mathcal{S}$ may be identified with a countable ω-model $\mathcal{M} = (\mathcal{S}, \mathbb{N}, +, \cdot, 0, 1, <, =)$ and in this way Z may be regarded as a *code* of the inner model $\mathcal{M}$. In particular, for any set $W \subseteq \mathbb{N}$, we have $W \in \mathcal{M}$ if and only if $\exists n(W = (Z)_n)$. Given such a *countable coded* ω*-model* $\mathcal{M}$, we can carry out the Tarski truth definition within ATR_0 to obtain a full satisfaction predicate for $\mathcal{M}$. Here formulas of the language of Z_2 are identified with their Gödel numbers. Thus within ATR_0 we may speak of countable coded ω-models of T, where T is any recursively axiomatized subsystem of Z_2.

The following result from [17] will be used to prove our main theorem, in Section 3 below.

Lemma 1. *The following is provable in* ATR_0*. For any set* $W \subseteq \mathbb{N}$*, there exists a countable coded* ω*-model* $\mathcal{M}$ *of* Σ^1_1-AC_0 *such that* $W \in \mathcal{M}$*.*

Proof. We shall use the formalization within ATR_0 of some facts and techniques from recursive function theory and hyperarithmetical theory [13]. For details of the formalization within ATR_0, see [4], [6], and [17].

We shall use the arithmetical formula

$$\mathrm{WO}(X, Z) \;\equiv\; \mathrm{LO}(X) \wedge \forall Y\,(Y \text{ Turing reducible to } Z \rightarrow \mathrm{Ind}(X, Y))\,,$$

where $\forall Y\,\mathrm{Ind}(X, Y)$ is the formula (3). Trivially we have

$$\forall X\,(\mathrm{WO}(X) \leftrightarrow \forall Z\,\mathrm{WO}(X, Z))\,.$$

Reasoning within ATR_0, fix a set $W \subseteq \mathbb{N}$. Consider the arithmetical formula

$$\eta(W, X, Z) \;\equiv\; \mathrm{WO}(X, Z) \,\wedge\, \forall j\;((Z)_j = \text{Turing jump of } (W \oplus X) \oplus (Z)^j_X)\,.$$

By arithmetical transfinite recursion we have

$$\forall X\,(\mathrm{WO}(X) \rightarrow \exists Z\,\eta(W, X, Z))\,.$$

On the other hand, the formula $\mathrm{WO}(X)$ is complete Π^1_1 and hence not equivalent to any Σ^1_1 formula (see [13], Chapter 16). In particular, $\mathrm{WO}(X)$ is not equivalent to the Σ^1_1 formula $\exists Z\,\eta(W, X, Z)$. These considerations imply that there exist sets X and Z such that $\eta(W, X, Z) \wedge {\sim}\mathrm{WO}(X)$. Fix such an X and Z.

Using $\mathrm{WO}(X, Z)$ and the fact that X is Turing reducible to Z, it is easy to see that the linear ordering X has the following properties: there is a least element, and any element other than the greatest element (if there is one) has an immediate successor. Using ${\sim}\mathrm{WO}(X)$, let $J \subseteq \mathbb{N}$ be such that $\mathrm{Ind}(X, J)$ fails, and put $I = \{j : \forall i\,((i, j) \in X \rightarrow i \in J)\}$. Then clearly I is a *cut* in X, *i.e.* we have $\exists i\,\exists j\,(i \in I \wedge j \notin I)$ and $\forall i\,\forall j\,((i \in I \wedge j \notin I) \rightarrow (i, j) \in X))$ and $\forall i\,(i \in I \rightarrow \exists k\,((i, k) \in X \wedge k \in I))$ and $\forall j\,(j \notin I \rightarrow \exists k\,((k, j) \in X \wedge k \notin I))$.

By arithmetical comprehension, there exists a countable coded ω-model $\mathcal{M}$ consisting of all sets A such that A is Turing reducible to $(Z)_i$ for some $i \in I$. Clearly

$W \in \mathcal{M}$ and $X \in \mathcal{M}$. It is also clear that $\mathcal{M}$ is closed under $\oplus$ and Turing reducibility and the Turing jump operator. From this it follows by Post's theorem ([13], Chapter 14) that $\mathcal{M}$ is an ω-model of ACA_0.

We claim that $\mathcal{M}$ is an ω-model of Σ^1_1-AC_0. To see this, let $\varphi(n, U)$ be any Σ^1_1 formula. Let $n_1, \ldots, n_k, U_1, \ldots, U_m$ be the free variables of $\varphi(n, U)$ other than n and U. Fix $a_1, \ldots, a_k \in \mathbb{N}$ and $A_1, \ldots, A_m \in \mathcal{M}$ and suppose that $\mathcal{M}$ satisfies $\forall n\, \exists U\, \bar{\varphi}(n, U)$, where

$$\bar{\varphi}(n, U) \;\equiv\; \varphi(n, U)[n_1/a_1, \ldots, n_k/a_k, U_1/A_1, \ldots, U_m/A_m]\,.$$

Let us write

$$\varphi(n, U) \;\equiv\; \exists V\, \theta(n, U, V)$$

where $\theta(n, U, V)$ is arithmetical, and put

$$\bar{\theta}(n, U, V) \;\equiv\; \theta(n, U, V)[n_1/a_1, \ldots, n_k/a_k, U_1/A_1, \ldots, U_m/A_m]\,.$$

Then $\mathcal{M}$ satisfies $\forall n\, \exists U\, \exists V\, \bar{\theta}(n, U, V)$. It follows that for each $n \in \mathbb{N}$ there exists $i \in I$ such that

$$\exists U\, \exists V\, (\bar{\theta}(n, U, V) \wedge U \text{ and } V \text{ are Turing reducible to } (Z)_i)\,. \tag{5}$$

Hence by $\mathrm{WO}(X, Z)$ we have that for each $n \in \mathbb{N}$ there exists a least such i with respect to the linear ordering of $\mathbb{N}$ given by X. Define $f : \mathbb{N} \to \mathbb{N}$ by $f(n) =$ the least $i \in \mathbb{N}$ with respect to the linear ordering X such that (5) holds. Since $f(n) \in I$ for all $n \in \mathbb{N}$, it follows that f is Turing reducible to $(Z)_j$ for any $j \notin I$. Hence by $\mathrm{WO}(X, Z)$ there exists $k \in \mathbb{N}$ such that k is the least upper bound, with respect to the linear ordering X, of the range of f. Since $f(n) \in I$ for all $n \in \mathbb{N}$, it follows that $k \in I$. Thus we have a set $(Z)_k \in \mathcal{M}$ such that $\forall n\, \exists U\, \exists V\, (\bar{\theta}(n, U, V) \wedge$ U and V are Turing reducible to $(Z)_k$). We can now use arithmetical comprehension within $\mathcal{M}$ to find a set $T \in \mathcal{M}$ such that $\forall n\, \bar{\theta}(n, ((T)_n)_0, ((T)_n)_1)$. Putting $Y = \{(m, n) : ((m, 0), n) \in T\}$, we obtain $Y \in \mathcal{M}$ such that $\forall n\, \bar{\theta}(n, (Y)_n, ((T)_n)_1)$. Thus $\mathcal{M}$ satisfies $\exists Y\, \forall n\, \exists V\, \bar{\theta}(n, (Y)_n, V)$, *i.e.* $\exists Y\, \forall n\, \bar{\varphi}(n, (Y)_n)$. Thus $\mathcal{M}$ is an ω-model of Σ^1_1-AC_0 and the proof of Lemma 1 is complete.

Remark. The assertion considered in the previous lemma ("for all W there exists a countable coded ω-model of Σ^1_1-AC_0 containing W") is in fact equivalent to ATR_0 over RCA_0. This is shown in [17].

3. Proof of the Main Theorem

The purpose of this section is to prove our main result:

Theorem 1. *The König duality theorem for countable bipartite graphs* (i.e. CKDT) *is provable in* ATR_0*.*

In order to prove Theorem 1, the following notions will be useful. Let $G = (X, Y, E)$ be a bipartite graph. For $y \in Y$, the *neighborhood* of y in G is

$$N_G(y) = \{x \in X : \{x, y\} \in E\}\,.$$

For $A \subseteq X$, the *demand* of A with respect to G is

$$D_G(A) = \{y \in Y : N_G(y) \subseteq A\}\,.$$

For $A \subseteq X$ and $B \subseteq Y$, a *matching of A into B* is a matching M such that $X \cap (\bigcup M) = A$ and $Y \cap (\bigcup M) \subseteq B$. In this case we write

$$M : A \to B$$

and, for $x \in A$, $M(x) =$ the unique y such that $\{x, y\} \in M$. Thus

$$M = \{\{x, M(x)\} : x \in A\}\,.$$

If M is any matching in G and if v and w are vertices of G, an *M-alternating path* from v to w is a sequence of vertices $v = v_0,\ v_1,\ \ldots\ ,\ v_n = w$ such that $\{v_i, v_{i+1}\} \in E$ for all $i < n$, $\{v_i, v_{i+1}\} \in M$ for all odd $i < n$, and $\{v_i, v_{i+1}\} \notin M$ for all even $i < n$.

We now begin the proof of Theorem 1. We reason in ATR_0. Let $G = (X, Y, E)$ be a countable bipartite graph. We shall prove in ATR_0 that a König covering of G exists.

By Lemma 1, there exists a countable coded ω-model $\mathcal{M}$ of Σ^1_1-AC_0 such that $G \in \mathcal{M}$. Fix such an $\mathcal{M}$. Let A^* be the union of all sets $A \subseteq X$ such that $A \in \mathcal{M}$ and in $\mathcal{M}$ there is a matching $F : A \to D_G(A)$. Note that A^* is definable over $\mathcal{M}$. Hence A^* exists by arithmetical comprehension, using a code of $\mathcal{M}$ as a parameter.

Lemma 2. *There exists a matching $F^* : A^* \to D_G(A^*)$.*

Proof. By arithmetical comprehension using a code of $\mathcal{M}$ as a parameter, we can find an enumeration $\langle (A_n, F_n) : n \in \mathbb{N}\rangle$ of all pairs $(A, F) \in \mathcal{M}$ such that F is a matching of A into $D_G(A)$. Then $A^* = \bigcup\{A_n : n \in \mathbb{N}\}$. For $x \in A^*$ define $F^*(x) = F_n(x)$ where $n =$ the least n such that $x \in A_n$. To see that F^* is one-to-one, suppose $F^*(x_1) = F^*(x_2) = y$. For $i = 1, 2$ put $n_i =$ the least n such that $x_i \in A_n$. Then $F_{n_1}(x_1) = F_{n_2}(x_2) = y$. Hence $y \in D_G(A_{n_1}) \cap D_G(A_{n_2})$. Hence $x_1, x_2 \in A_{n_1} \cap A_{n_2}$. It follows that $n_1 = n_2$. Hence $x_1 = x_2$. Thus F^* is a matching, and clearly $F^* : A^* \to D_G(A^*)$. This proves the lemma.

Put $X^* = X - A^*$ and $Y^* = Y - D_G(A^*)$. We shall need to consider certain subgraphs of G of the form

$$G' = G - \{x_0, \ldots, x_{n-1}, y_0, \ldots, y_{n-1}\}$$

where $x_0,\ \ldots\ ,x_{n-1} \in X^*$ and $y_0,\ \ldots,\ y_{n-1} \in Y^*$. For any such graph G' we shall use the notation $G' = (X', Y', E')$ where $X' = X - \{x_0, \ldots, x_{n-1}\}$, $Y' =$

$Y - \{y_0, \ldots, y_{n-1}\}$, and $E' = E \cap \{\{x, y\} : x \in X', y \in Y'\}$. Note that for any such graph G' we have $G' \in \mathcal{M}$.

Let G' be a subgraph of G as above. We say that G' is *good* if there is no set $A \subseteq X'$ such that $A \in \mathcal{M}$ and in $\mathcal{M}$ there is a matching $F : A \to D_{G'}(A)$ such that

$$\Big(D_{G'}(A) - \bigcup F\Big) \cap Y^* \neq \emptyset\,.$$

Lemma 3. *G is good.*

Proof. Let $A \subseteq X$ be such that $A \in \mathcal{M}$ and in $\mathcal{M}$ there is a matching $F : A \to D_G(A)$. Then $A \subseteq A^*$. Hence $D_G(A) \subseteq D_G(A^*)$. Hence by the definition of Y^* we have $D_G(A) \cap Y^* = \emptyset$. This shows that G is good.

Lemma 4. *Suppose G' is good. Suppose $x \in X' \cap X^*$ and $y \in Y' \cap Y^*$ are such that $G' - \{x, y\}$ is not good. Then there exists $A' \subseteq X'$ such that $x \in A'$ and $A' \in \mathcal{M}$ and in $\mathcal{M}$ there is a matching $F' : A' \to D_{G'}(A')$ such that $y \notin \bigcup F'$.*

Proof. Since $G' - \{x, y\}$ is not good, we can find a set $A \subseteq X' - \{x\}$, $A \in \mathcal{M}$, a matching $F : A \to D_{G'-\{x,y\}}(A)$, $F \in \mathcal{M}$, and a vertex $y^* \in (D_{G'-\{x,y\}}(A) - \bigcup F) \cap Y^*$.

We claim that there exists an F-alternating path in G' from y^* to x. To see this, let S be the set of all $x' \in X' - \{x\}$ such that there exists an F-alternating path in $G' - \{x, y\}$ from y^* to x', and let T be the set of all $y' \in Y' - \{y\}$ such that there exists an F-alternating path in $G' - \{x, y\}$ from y^* to y'. For any $x' \in S$ we clearly have $F(x') \in T$. Thus $F_S = \{\{x', F(x')\} : x' \in S\}$ is a matching of S into T. Note also that S, T, and F_S belong to $\mathcal{M}$. Moreover, for any $y' \in T$ we clearly have $N_{G'-\{x,y\}}(y') \subseteq S$. Thus $T \subseteq D_{G'-\{x,y\}}(S)$. However, since G' is good and $y^* \in (T - \bigcup F_S) \cap Y^*$, we cannot have $T \subseteq D_{G'}(S)$. Hence there must exist $y' \in T$ such that $\{x, y'\} \in E'$. Let $y^* = y'_0,\ x'_0,\ y'_1,\ x'_1,\ \ldots\ ,y'_n = y'$ be an F-alternating path in $G' - \{x, y\}$ from y^* to y'. Then $y^* = y'_0,\ x'_0,\ y'_1,\ x'_1,\ \ldots\ ,\ y'_n,\ x'_n = x$ is an F-alternating path in G' from y^* to x. This proves the claim.

Put $A' = A \cup \{x\}$. Then obviously $D_{G'-\{x,y\}}(A) \subseteq D_{G'}(A')$. Using our F-alternating path $y^* = y'_0,\ x'_0,\ y'_1,\ x'_1,\ \ldots\ ,\ y'_n,\ x'_n = x$ as above, put

$$F' = \Big(F - \{\{x'_i, y'_{i+1}\} : i < n\}\Big) \cup \{\{x'_i, y'_i\} : i \leq n\}\,.$$

Since F is a matching of A into $D_{G'-\{x,y\}}(A)$, and since $x, y^* \notin \bigcup F$ and $y^* \in D_{G'-\{x,y\}}(A)$, it follows that F' is a matching of A' into $D_{G'-\{x,y\}}(A)$. Therefore, F' is a matching of A' into $D_{G'}(A')$. It is also clear that $x \in A'$, $A' \in \mathcal{M}$, $F' \in \mathcal{M}$, and $y \notin \bigcup F'$. This completes the proof of Lemma 4.

Lemma 5. *Suppose that G' is good. Then for all $y \in Y' \cap Y^*$ there exists $x \in X' \cap X^*$ such that $\{x, y\} \in E'$ and $G' - \{x, y\}$ is good.*

Proof. Fix $y \in Y' \cap Y^*$ and assume for a contradiction that there is no $x \in X' \cap X^*$ such that $\{x, y\} \in E'$ and $G' - \{x, y\}$ is good.

We claim that for all $x \in N_{G'}(y)$ there exists $(A', F') \in \mathcal{M}$ such that $x \in A'$, $A' \subseteq X'$, F' is a matching of A' into $D_{G'}(A')$, and $y \notin \bigcup F'$. We prove this claim by considering two cases, $x \in A^*$ and $x \notin A^*$. If $x \notin A^*$, then $x \in X' \cap X^*$ and by assumption $G' - \{x, y\}$ is not good, so the claim follows by Lemma 4. If $x \in A^*$, then by the definition of A^* we can find a set $A \subseteq X$, $A \in \mathcal{M}$, $x \in A$, and a matching $F : A \to D_G(A)$, $F \in \mathcal{M}$. Then $A \subseteq A^* \subseteq X'$ and $D_G(A) \subseteq D_G(A^*) \subseteq Y'$, hence $D_G(A) \subseteq D_{G'}(A)$. Moreover $y \in Y^* = Y - D_G(A^*)$, hence $y \notin D_G(A)$, hence $y \notin \bigcup F$. Thus in this case our claim holds with $(A', F') = (A, F)$. This completes the proof of the claim.

Working within $\mathcal{M}$, let $\langle x'_n : n \in \mathbb{N}\rangle$ be an enumeration of the vertices in $N_{G'}(y)$. The above claim implies that for all $n \in \mathbb{N}$ there exists $(A', F') \in \mathcal{M}$ such that $x'_n \in A'$, $A' \subseteq X'$, F' is a matching of A' into $D_{G'}(A')$, and $y \notin \bigcup F'$. Applying the Σ^1_1 Countable Choice Scheme within $\mathcal{M}$, we obtain a sequence $\langle (A'_n, F'_n) : n \in \mathbb{N}\rangle \in \mathcal{M}$ such that for all $n \in \mathbb{N}$ we have $x'_n \in A'_n$, $A'_n \subseteq X'$, F'_n is a matching of A'_n into $D_{G'}(A'_n)$, and $y \notin \bigcup F'_n$.

Put $A = \bigcup\{A'_n : n \in \mathbb{N}\}$. Then $N_{G'}(y) = \{x'_n : n \in \mathbb{N}\} \subseteq A$, *i.e.* $y \in D_{G'}(A)$. Still working within $\mathcal{M}$, define $F(x)$ for all $x \in A$ by $F(x) = F'_n(x)$ where $n =$ the least n such that $x \in A'_n$. To see that F is one-to-one, suppose $F(x') = F(x'') = y'$. Let $n' =$ the least n such that $x' \in A'_n$, and let $n'' =$ the least n such that $x'' \in A'_n$. Then $F'_{n'}(x') = F'_{n''}(x'') = y'$. Hence $y' \in D_{G'}(A'_{n'}) \cap D_{G'}(A'_{n''})$. Hence $x', x'' \in A'_{n'} \cap A'_{n''}$. It follows that $n' = n''$. Hence $x' = x''$. Thus F is a matching. Clearly $F : A \to D_{G'}(A)$ and we also clearly have $A \in \mathcal{M}$, $F \in \mathcal{M}$, and $y \in (D_{G'}(A) - \bigcup F) \cap Y^*$. This contradicts the assumption that G' is good. The proof of Lemma 5 is complete.

We are now ready to finish the proof of Theorem 1. Still reasoning within ATR_0, fix a one-to-one enumeration $y_0, y_1, \ldots, y_n, \ldots$ of all the vertices in Y^*. The idea of this part of the proof is to apply Lemma 5 repeatedly to obtain a sequence of vertices $x_0, x_1, \ldots, x_n, \ldots$ in X^* so that

$$H = \{\{x_0, y_0\}, \{x_1, y_1\}, \ldots, \{x_n, y_n\}, \ldots\}$$

will be a matching. To begin, since G is good and $y_0 \in Y^*$, we can apply Lemma 5 with $G' = G$ and $y = y_0$ to obtain $x_0 \in X^*$ such that $\{x_0, y_0\} \in E$ and $G - \{x_0, y_0\}$ is good. Next, since $G - \{x_0, y_0\}$ is good and $y_1 \in Y^* - \{y_0\}$, we can apply Lemma 5 with $G' = G - \{x_0, y_0\}$ and $y = y_1$ to obtain $x_1 \in X^* - \{x_0\}$ such that $\{x_1, y_1\} \in E$ and $G - \{x_0, y_0, x_1, y_1\}$ is good. At stage n of the construction, we assume inductively that $G - \{x_0, y_0, \ldots, x_{n-1}, y_{n-1}\}$ is good. Since $y_n \in Y^* - \{y_0, \ldots, y_{n-1}\}$, we can apply Lemma 5 with $G' = G - \{x_0, y_0, \ldots, x_{n-1}, y_{n-1}\}$ and $y = y_n$ to obtain $x_n \in X^* - \{x_0, \ldots, x_{n-1}\}$ such that $\{x_n, y_n\} \in E$ and $G - \{x_0, y_0, \ldots, x_n, y_n\}$ is good. The inductive construction of the sequence $x_0, x_1, \ldots, x_n, \ldots$ is definable over $\mathcal{M}$. Thus H exists by arithmetical comprehension, using a code of $\mathcal{M}$ as a parameter.

Clearly H is a matching, $X \cap (\bigcup H) \subseteq X^*$, and $Y \cap (\bigcup H) = Y^*$. In addition, Lemma 2 provides a matching $F^* : (X - X^*) \to (Y - Y^*)$. Thus $F^* \cup H$ is again a matching. Since $D_G(X - X^*) = Y - Y^*$, it follows that $((X - X^*) \cup Y^*, F^* \cup H)$ is a König covering of G. This completes the proof of Theorem 1.

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Department of Mathematics, Pennsylvania State University, University Park, State College, PA 16802

E-mail address: simpson@math.psu.edu