SCHNORR RANDOMNESS AND THE LEBESGUE DIFFERENTIATION THEOREM

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ABSTRACT. We exhibit a close correspondence between L_1-computable functions and Schnorr tests. Using this correspondence, we prove that a point $x \in [0,1]^d$ is Schnorr random if and only if the Lebesgue Differentiation Theorem holds at x for all L_1-computable functions $f \in L_1([0,1]^d)$.

1. INTRODUCTION

Throughout mathematics there are many measure-theoretic theorems of the form "property P holds for almost all x." An important component of the theory of algorithmic randomness has been to prove that random points satisfy such theorems.

Recently, there has been interest in the converse problem, namely, to characterize notions of randomness in terms of classical theorems which hold almost everywhere. An example of such a classical theorem is the Birkhoff Ergodic Theorem.

Theorem 1.1 (Birkhoff's Ergodic Theorem)**.** Given a probability space (X, μ), an ergodic[1] transformation $T : X \to X$, and a function $f \in L_1(X, \mu)$, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i<n} f(T^i(x)) = \int f \, d\mu \tag{1}$$

for almost all $x \in X$.

A connection between Birkhoff's theorem and algorithmic randomness appeared in [16], where it was shown (see also [9]) that (1) holds for every L_1-computable function f and every Martin-Löf random point x.

In ergodic theory, a point x is called *typical*[2] for a given transformation T if (1) holds for every bounded continuous function f. In [7], a characterization of Schnorr randomness in terms of dynamical typicalness was given. Here we state a slightly improved version, obtained using a result from [1] (see also [8]) which concerns the computability of the rate of convergence of ergodic averages.

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[1]A transformation $T : X \to X$ is said to be *ergodic* (with respect to a probability measure μ on X) if for every measurable set A satisfying $T^{-1}(A) = A$ either $\mu(A) = 1$ or $\mu(A) = 0$.

[2]The set of typical points has full measure. We remark that, if in the definition of typical point we relax the functions f to be integrable only (or even characteristic functions of measurable sets), then the resulting set of typical points would be empty.

Theorem 1.2. ([1], [8]) Let X be a computable probability space. A point $x \in X$ is Schnorr random if and only if x is typical for every computable ergodic transformation $T : X \to X$.

The question of whether a similar characterization would hold for Martin-Löf randomness was raised. A positive answer to this question was given independently by Franklin, Greenberg, Miller, and Ng [6] and Bienvenu, Day, Hoyrup, Mezhirov and Shen [2], who proved the following.

Theorem 1.3 ([6], [2])**.** Let X be a computable probability space, and let $T : X \to X$ be a computable ergodic transformation. Then, a point $x \in X$ is Martin-Löf random if and only if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i<n} \chi_A(T^i(x)) = \mu(A).$$

for all effectively closed sets A.

Similarly but in a somewhat different direction, Brattka, Miller and Nies [3] have obtained some interesting equivalences between randomness and differentiability.

Theorem 1.4 ([3])**.** For $x \in [0, 1]$ we have

(1) x is computably random if and only if every nondecreasing computable function $f : [0, 1] \to \mathbb{R}$ is differentiable at x.
(2) x is Martin-Löf random if and only if every computable function $f : [0, 1] \to \mathbb{R}$ of bounded variation is differentiable at x.
(3) x is weakly 2-random if and only if every almost everywhere differentiable computable function $f : [0, 1] \to \mathbb{R}$ is differentiable at x.

We now turn to the subject of the present paper, an analysis of the Lebesgue Differentiation Theorem. The classical theorem reads as follows.

Theorem 1.5 (Lebesgue Differentiation Theorem)**.** For each $f \in L_1([0, 1]^d)$ we have

$$f(x) \quad = \quad \lim_{Q\to x} \frac{\int_Q f d\mu}{\mu(Q)} \tag{2}$$

for almost all $x \in [0, 1]^d$. The limit is taken over all cubes Q containing x as the diameter of Q tends to 0.

In [14] it was shown that, for each $f \in L_1([0, 1]^d)$ which is L_1-computable in the sense of Definition 2.6 below, the equation (2) holds for all $x \in [0, 1]^d$ which are random in the sense of Martin-Löf. At the end of [14], the question of the converse was posed. The purpose of the present paper is to answer this question by sharpening the results of [14]. Roughly speaking, our main result is as follows:

Theorem 1.6. A point $x \in [0, 1]^d$ is Schnorr random if and only if (2) holds for all L_1-computable functions $f \in L_1([0, 1]^d)$.

In other words, the Lebesgue Differentiation Theorem characterizes Schnorr randomness. Moreover, our proof of Theorem 1.6 establishes certain relationships between Schnorr tests and L_1-computable functions. In Lemma 3.15 below we associate to each L_1-computable function $f \in L_1([0, 1]^d)$ a Schnorr test such that (2) holds for all $x \in [0, 1]^d$ which pass the test. Consequently, by Theorem 3.16

below, (2) holds for all Schnorr random x. In Lemma 3.14 we obtain a computationally motivated estimate of the rate of convergence in (2). In Theorem 4.7 below, we associate to each Schnorr test an L_1-computable $f \in L_1([0,1]^d)$ such that for all $x \in [0,1]^d$ which fail the test, the limit in (2) does not exist. Combining these results, we have a close correspondence between L_1-computable functions and Schnorr tests.

Methodologically, our proofs are perhaps somewhat novel. In verifying our Schnorr tests, we use Tarski's quantifier elimination theorem for the real number system (see Lemma 3.3 below) as well as some ideas from computable measure theory [12, 15] (see Lemmas 2.12 and 3.5 below). So far as we know, this is the first time that quantifier elimination has been applied in randomness theory.

2. Preliminary definitions and notation

Notation 2.1. Fix a positive integer d, the *dimension*. We consider real-valued, Lebesgue measurable functions f and Lebesgue measure μ on the unit cube $[0,1]^d$ in d-dimensional Euclidean space. The L_1-norm is defined by

$$\|f\|_1 \;=\; \int_{[0,1]^d} |f| \;=\; \int_{x\in[0,1]^d} |f(x)|\, d\mu(x)\,.$$

Recall that $L_1([0,1]^d)$ is the space of all f such that $\|f\|_1$ is finite. Moreover, for all $f, g \in L_1([0,1]^d)$ we have $\|f-g\|_1 = 0$ if and only if $\mu(\{x \mid f(x) \neq g(x)\}) = 0$.

Notation 2.2. We use Q as a variable ranging over *cubes* in $[0,1]^d$. Thus Q denotes a set of the form

$$Q \;=\; \bigl\{\langle x_1,\ldots,x_d\rangle \bigm| |x_i - a_i| \leq r \text{ for all } i = 1,\ldots,d\bigr\} \tag{3}$$

where $a_1,\ldots,a_d,r$ are real numbers with $0 \leq a_i - r < a_i + r \leq 1$. If $a_1,\ldots,a_d,r$ are rational, we say that Q is a *rational cube*. Throughout this paper, letters such as $i,j,k,l,m,n,\ldots$ range over the natural numbers.

Remark 2.3. The classical Lebesgue Differentiation Theorem (see for instance [17]) reads as follows. Given $f \in L_1([0,1]^d)$ we can find a set S depending on f such that $\mu(S) = 0$ and

$$f(x) \;=\; \lim_{Q\to x} \frac{\int_Q f\, d\mu}{\mu(Q)}$$

for all $x \notin S$. The limit is taken over all cubes Q containing x as the diameter of Q tends to 0.

Definition 2.4. A *finite step function* is a function of the form

$$f(x) = \sum_{i=1}^{k} c_i \chi_{Q_i}(x)$$

where χ_{Q_i} is the characteristic function of a cube Q_i in $[0,1]^d$. If $c_1,\ldots,c_k$ and $Q_1,\ldots,Q_k$ are rational we say that f is a *finite rational step function*.

Remark 2.5. It is well known that, given $f \in L_1([0,1]^d)$ and $\epsilon > 0$, we can find a polynomial f_ϵ with rational coefficients such that $\|f - f_\epsilon\|_1 < \epsilon$. Such polynomials are describable by finite strings of symbols which are amenable to computation. In the following definition and throughout this paper, we view such polynomials as computable approximations of f. Moreover, instead of rational polynomials, we could equally well use finite rational step functions.

Definition 2.6. A function $f \in L_1([0,1]^d)$ is said to be L_1-*computable* if there exists a computable sequence of polynomials with rational coefficients, denoted f_n, such that

$$\|f - f_n\|_1 \;\leq\; \frac{1}{2^n} \tag{4}$$

for all n. Equivalently, $f \in L_1([0,1]^d)$ is L_1-computable if and only if there exists a computable sequence of finite rational step functions f_n such that (4) holds.

Remark 2.7. The idea behind Definition 2.6 is that we are endowing $L_1([0,1]^d)$ with the structure of a computable metric space. The L_1-computable functions are then the computable points of that space. For more on the theory of computable metric spaces, see [12, §2.4].

Definition 2.8.

(1) An *open ball* in $[0,1]^d$ is a set of the form
$$B(a,r) \;=\; \left\{x \in [0,1]^d \;\middle|\; |x-a| < r\right\}$$
where $a = \langle a_1,\ldots,a_d\rangle \in [0,1]^d$ and $|x-a|$ denotes Euclidean distance. If $a_1,\ldots,a_d,r$ are rational, we say that $B(a,r)$ is a *rational open ball* or a *basic open set*. Note that rational open balls, like polynomials with rational coefficients amd finite rational step functions, are amenable to computation.

(2) A set $U \subseteq [0,1]^d$ is said to be Σ^0_1 if
$$U \;=\; \bigcup_{i=0}^{\infty} B(a_i,r_i)$$
where $B(a_i,r_i)$, $i = 0,1,\ldots$ is a computable sequence of rational balls. A Σ^0_1 set is also known as an *effectively open* set, because it is the union of a computable sequence of basic open sets.

(3) A sequence of sets $U_n \subseteq [0,1]^d$, $n = 0,1,2,\ldots$ is *uniformly* Σ^0_1 if
$$U_n \;=\; \bigcup_{i=0}^{\infty} B(a_{n,i},r_{n,i})$$
for all n, where $B(a_{n,i},r_{n,i})$, $n = 0,1,2,\ldots$, $i = 0,1,2,\ldots$ is a computable double sequence of rational balls.

(4) A set $P \subseteq [0,1]^d$ is *effectively closed* or Π^0_1 if its complement is effectively open.

The next two definitions can be found in [13, §3.1]. See also [5].

Definition 2.9. A *Martin-Löf test* is a uniformly Σ^0_1 sequence of sets $U_n \subseteq [0,1]^d$, $n = 0,1,2,\ldots$ such that $\mu(U_n) \leq 1/2^n$ for all n. A point $x \in [0,1]^d$ is said to *pass the test* if $x \notin \bigcap_{n=0}^{\infty} U_n$. We say that x is *Martin-Löf random* if it passes every Martin-Löf test.

Definition 2.10. A *Schnorr test* is a Martin-Löf test U_n, $n = 0,1,2,\ldots$ such that $\mu(U_n)$ is uniformly computable for all n. We say that x is *Schnorr random* if it passes every Schnorr test.

Remark 2.11. In [14] it was shown that if x is Martin-Löf random, the Lebesgue Differentiation Theorem holds at x for all L_1-computable functions. We now prove, in Section 3 below, that the same result holds if x is Schnorr random. The converse is proved in Section 4.

In order to construct Schnorr tests, we shall use the following lemma.

Lemma 2.12. Let $U, V \subseteq [0,1]^d$ be Σ^0_1 sets. Then $U \cap V$ and $U \cup V$ are Σ^0_1 sets. If in addition $\mu(U)$ and $\mu(V)$ are computable real numbers, then $\mu(U \cap V)$ and $\mu(U \cup V)$ are computable real numbers. Moreover, these statements hold uniformly.

Proof. A proof can be found in [15, Lemma 2.3.1.2]. See also [12]. □

Remark 2.13. Given a Martin-Löf test or a Schnorr test U_n, $n = 0, 1, 2, \ldots$, we may safely assume (by taking intersections and applying Lemma 2.12) that $U_{n+1} \subseteq U_n$ holds for all n.

3. Schnorr points are Lebesgue for L_1-computable functions

Thr purpose of this section is to prove Theorem 3.16. Essentially, Theorem 3.16 says that the Lebesgue Differentiation Theorem 1.5 holds for all Schnorr random points $x \in [0,1]^d$ and all L_1-computable functions $f \in L_1([0,1]^d)$.

Remark 3.1. The key lemmas in this section are Lemmas 3.6 and 3.13. The idea of these lemmas is to associate Schnorr tests V_k and V_k^* to each L_1-computable function f. The V_k's insure the existence of the limit $\widehat{f}(x) = \lim_{n\to\infty} f_n(x)$, and the V_k^*'s insure that x is a Lebesgue point for f. In order to construct the V_k's and the V_k^*'s, we employ the method of *effective quantifier elimination* as embodied in the following well known theorem, due originally to Tarski.

Theorem 3.2. The theory of real closed ordered fields is complete, decidable, and admits elimination of quantifiers.

Proof. See for instance [10, Theorem 8.4.4, page 385]. □

Lemma 3.3. Let S be a set in the d-dimensional unit cube $[0,1]^d$ such that S is first-order definable over the real number system. Then, the d-dimensional Lebesgue measure of S is a computable real number. Moreover, this holds uniformly in the given first-order definition of S.

Proof. We use the following well known fact: given a non-zero polynomial $f \in \mathbb{Z}[x_1, \ldots, x_d]$, the set $\{x \in [0,1]^d \mid f(x) = 0\}$ is of measure 0. Now, given a first-order definition of a set S in $[0,1]^d$ as above, apply effective quantifier elimination to obtain a quantifier-free definition of S. Let $f_1, ..., f_n \in \mathbb{Z}[x_1, \ldots, x_d]$ be a list of the nonzero polynomials which occur in the quantifier-free definition of S. For each $i = 1, ..., n$ let

$$\begin{aligned} U_i &= \{x \in [0,1]^d \mid f_i(x) > 0\}, \\ V_i &= \{x \in [0,1]^d \mid f_i(x) < 0\}, \\ C_i &= \{x \in [0,1]^d \mid f_i(x) = 0\}. \end{aligned}$$

Then each C_i is of measure 0. Thus $[0,1]^d$ is the disjoint union of a set of measure 0 plus at most 2^n-many sets of the form $W_1 \cap \cdots \cap W_n$ where each W_i is either U_i or V_i. Moreover, each of the sets $W_1 \cap \cdots \cap W_n$ is effectively open, and S and $[0,1]^d \setminus S$ may be written as the union of some of these sets plus a set of measure 0. Since the measure of an effectively open set is left recursively enumerable, it follows that the measure of S is recursive, i.e., computable. The same argument holds uniformly. □

The following simple lemma is extremely useful in probability theory.

Lemma 3.4 (Chebyshev Inequality)**.** Given $f \in L_1$ and $\epsilon > 0$, let

$$S(f, \epsilon) = \{x \mid |f(x)| > \epsilon\}.$$

Then $\mu(S(f, \epsilon)) \leq \|f\|_1/\epsilon$.

Proof. We have $\|f\|_1 = \int |f| d\mu \geq \int_{S(f,\epsilon)} |f| d\mu \geq \epsilon\mu(S(f, \epsilon))$. □

Lemma 3.5. Let $U = \bigcup_{n=1}^{\infty} U_n$ where U_n is uniformly Σ_1^0 and $\mu(U_n)$ is uniformly computable and $\mu(U_n) \leq 1/2^n$ for all n. Then U is Σ_1^0 and $\mu(U)$ is computable. Moreover, this holds uniformly.

Proof. For all $n \in \mathbb{N}$ we have $U_n = \bigcup_{k=1}^{\infty} B_{nk}$ where B_{nk}, $k = 0, 1, 2, \ldots$, is a computable sequence of rational balls. Thus, by diagonalization, we can write U as the union of a computable sequence of rational balls. Thus U is Σ_1^0. By Lemma 2.12 $\bigcup_{n=1}^{k} U_n$ is Σ_1^0 and has computable measure uniformly in k. In addition, $\mu(\bigcup_{n=k}^{\infty} U_n) \leq 1/2^{k-1}$. Thus, letting $c_k = \mu\left(\bigcup_{n=1}^{k} U_n\right)$, we have a computable sequence of real numbers which effectively approximates $\mu(U)$. □

Lemma 3.6 (see [4, Proposition 4.1])**.** Let $f \in L_1([0, 1]^d)$ be L_1-computable with polynomial approximations f_n as in Definition 2.6. Then, we can find a uniformly Σ_1^0 sequence of sets V_k, $k = 0, 1, 2, \ldots$, such that the following statements hold:

(1) $\mu(V_k) \leq (2 + \sqrt{2})/2^{k-1}$.
(2) The sequence $\mu(V_k)$, $k = 1, 2, \ldots$ is uniformly computable.
(3) For all $x \notin V_k$ and $n \geq k$ we have

$$|f_i(x) - f_{2n}(x)| \leq \frac{2 + \sqrt{2}}{2^n}$$

for all $i \geq 2n$.

Proof. Let $V_k = \bigcup_{i=2k}^{\infty} S_i$ where $S_i = S(f_i - f_{i+1}, 1/2^{i/2})$. By Lemma 3.4 we have

$$\mu(V_k) \leq \sum_{i=2k}^{\infty} \mu(S_i) \leq \sum_{i=2k}^{\infty} 2^{i/2}\|f_i - f_{i+1}\|_1 \leq \sum_{i=2k}^{\infty} 2^{i/2} \cdot \frac{2}{2^i} \leq \sum_{i=2k}^{\infty} \frac{2}{2^{i/2}} = \frac{2(2 + \sqrt{2})}{2^k}.$$

Moreover, as in [14], V_k is uniformly Σ_1^0.

We claim that that $\mu(V_k)$ is uniformly computable. By Lemma 3.4 we have $\mu(S_i) \leq 1/2^{i/2}$, so by Lemma 3.5 it suffices to show that $\mu(S_i)$ is uniformly computable. But S_i is uniformly first-order definable, so by Lemma 3.3 S_i has computable measure uniformly in i. This proves our claim.

Finally, for all $x \notin V_k$ and $n \geq k$ and $i \geq 2n$ we have

$$\begin{aligned} |f_i(x) - f_{2n}(x)| &\leq \sum_{l=2n}^{i-1} |f_l(x) - f_{l+1}(x)| \\ &\leq \sum_{l=2n}^{\infty} |f_l(x) - f_{l+1}(x)| \\ &\leq \sum_{l=2n}^{\infty} \frac{1}{2^{l/2}} \\ &= \frac{2 + \sqrt{2}}{2^n} \end{aligned}$$

and this completes the proof. □

Lemma 3.7 (see [4, Remark 4.3])**.** Let $f \in L_1([0,1]^d)$ be L_1-computable with polynomial approximations f_n as in Definition 2.6. Then $\lim_{n\to\infty} f_n(x)$ exists for all Schnorr random x.

Proof. Let $x \in [0,1]^d$ be Schnorr random. The sets V_k of Lemma 3.6 form a Schnorr test. Since x is Schnorr random, we can find k such that $x \notin V_k$. Moreover, for all $x \notin V_k$ and $n \geq k$ we have $|f_i(x) - f_{2n}(x)| \leq (2+\sqrt{2})/2^n$ for all $i \geq 2n$. Thus $f_n(x)$ converges uniformly for all $x \notin V_k$. In particular $\lim_{n\to\infty} f_n(x)$ exists. □

Definition 3.8. Given an L_1-computable function $f \in L_1([0,1]^d)$, define

$$\widehat{f}(x) \;=\; \begin{cases} \lim\limits_{n\to\infty} f_n(x) & \text{if } x \text{ is Schnorr random,} \\ 0 & \text{otherwise.} \end{cases}$$

where f_n is a computable sequence of approximations as in Definition 2.6. The following theorem implies that $\widehat{f}$ does not depend on the choice of f_n.

Theorem 3.9.

(1) If $f \in L_1([0,1]^d)$ is L_1-computable, then $\|f - \widehat{f}\|_1 = 0$.
(2) Given L_1-computable functions $f, g \in L_1([0,1]^d)$, we have $\|f - g\|_1 = 0$ if and only if $\widehat{f}(x) = \widehat{g}(x)$ for all x.

Thus $\widehat{f}$ is a canonical representative of the equivalence class of f modulo the equivalence relation $\|f - g\|_1 = 0$.

Proof. For part 1, suppose $\|f - \widehat{f}\|_1 > 0$, i.e., $\mu(\{x \mid |f(x) - \widehat{f}(x)| > 0\}) > 0$. Let $\epsilon > 0$ be so small that $\mu(\{x \mid |f(x) - \widehat{f}(x)| > \epsilon\}) > \epsilon$. By Lemma 3.6, we have $|\widehat{f}(x) - f_{2n}(x)| \leq (2+\sqrt{2})/2^n$ for all Schnorr random $x \notin V_n$, where $\mu(V_n) \leq (2+\sqrt{2})/2^{n-1}$, for all n. It follows that

$$\mu(\{x \mid |f(x) - f_{2n}(x)| > \epsilon - (2+\sqrt{2})/2^n\}) > \epsilon - (2+\sqrt{2})/2^{n-1}$$

for all n. Thus

$$\|f - f_{2n}\|_1 > (\epsilon - (2+\sqrt{2})/2^n)(\epsilon - (2+\sqrt{2})/2^{n-1})$$

for all n, contradicting the fact that $\|f - f_{2n}\|_1$ goes to 0 as n goes to infinity.

For part 2, note that $\widehat{f}(x) = \widehat{g}(x)$ for all x, implies $\|f - g\|_1 = \|\widehat{f} - \widehat{g}\|_1 = 0$ in view of part 1. It remains to prove that if $\|f - g\|_1 = 0$ then $\widehat{f}(x) = \widehat{g}(x)$ for all x. By the definition of $\widehat{f}$, it suffices to prove $\widehat{f}(x) = \widehat{g}(x)$ for all Schnorr random x. Let

$$W_k = \{x \mid (\exists n \geq k)\,(|f_{2n}(x) - g_{2n}(x)| > 1/2^n)\} = \bigcup_{n=k}^{\infty} S_n$$

where $S_n = S(f_{2n} - g_{2n}, 1/2^n)$. Clearly W_k is uniformly Σ^0_1. Moreover, $\|f - g\|_1 = 0$ implies $\|f_{2n} - g_{2n}\|_1 \leq 1/2^{2n-1}$, so by Lemma 3.4 we have

$$\mu(W_k) \leq \sum_{n=k}^{\infty} \mu(S_n) \leq \sum_{n=k}^{\infty} \frac{2^n}{2^{2n-1}} = \sum_{n=k}^{\infty} \frac{1}{2^{n-1}} = \frac{1}{2^{k-2}}.$$

Moreover S_n is uniformly first-order definable, hence by Lemma 3.3 $\mu(S_n)$ is uniformly computable, and by Lemma 3.4 we have $\mu(S_n) \leq 1/2^{n-1}$. Thus by Lemma

3.5 $\mu(W_k)$ is uniformly computable and the sets W_k form a Schnorr test. In particular, if x is Schnorr random we have $x \notin W_k$ for some k, hence $|f_{2n}(x) - g_{2n}(x)| \leq 1/2^n$ for all $n \geq k$, hence $\widehat{f}(x) = \widehat{g}(x)$. □

Remark 3.10. Part 1 of the above theorem follows from the fact that, for an L_1-computable function f, the representative $\widehat{f}$ restricted to Martin-Löf random points, equals the *layerwise computable* representative [11, Proposition 4.2,Theorem 4.3]. The second part of the above theorem may be viewed as a refinement of [11, Theorem 4.3].

The following lemma is the key ingredient in the classical proof of the Lebesgue Differentiation Theorem.

Lemma 3.11 (Hardy/Littlewood Inequality)**.** We can find a positive constant c depending only on the dimension d such that the following holds. Given $f \in L_1([0,1]^d)$ and $\epsilon > 0$, let $S^*(f,\epsilon)$ be the union of all cubes Q such that

$$\frac{\int_Q |f| d\mu}{\mu(Q)} \;>\; \epsilon \tag{5}$$

holds. Then $\mu(S^*(f,\epsilon)) \leq c\|f\|_1/\epsilon$.

Proof. See [14, Lemma 4.5]. □

Remark 3.12. If Q is a cube as in (3), note that $\int_Q f d\mu$ and $\mu(Q)$ depend continuously on $a_1,\ldots,a_d,r$ since μ is absolutely continuous. Therefore, it is often possible to restrict attention to rational cubes. For instance, in the classical statements of the Lebesgue Differentiation Theorem and the Hardy/Littlewood Inequality, it makes no difference whether we consider arbitrary cubes or rational cubes. The advantage of rational cubes is that they are amenable to computation.

Lemma 3.13. Let $f \in L_1([0,1]^d)$ be L_1-computable with polynomial approximations f_n as in Definition 2.6. Let c be the constant from Lemma 3.11. Then, we can find uniformly Σ^0_1 sets V^*_k, $k = 1,2,\ldots$, such that the following statements hold:

(1) $\mu(V^*_k) \;\leq\; c\dfrac{2+\sqrt{2}}{2^{k-1}}$.
(2) The sequence $\mu(V^*_k)$ is uniformly computable.
(3) For all $x \notin V^*_k$ and $n \geq k$ we have

$$\frac{\int_Q |f - f_{2n}| d\mu}{\mu(Q)} \;\leq\; \frac{2+\sqrt{2}}{2^n}$$

for all $Q \ni x$.

Proof. We imitate the proof of Lemma 3.6 replacing the Chebyshev inequality by the Hardy-Littlewood inequality. Let $V^*_k = \bigcup_{i=2k}^\infty S^*_i$ where $S^*_i = S^*(f_i - f_{i+1}, 1/2^{i/2})$. By Remark 3.12 the sets V^*_k are uniformly Σ^0_1. By Lemma 3.11 we have $\mu(V^*_k) \leq \sum_{i=2k}^\infty \mu(S^*_i) \leq \sum_{i=2k}^\infty 2^{i/2} c\|f_i - f_{i+1}\|_1 \leq \sum_{i=2k}^\infty 1/2^{i/2} = 2c(2+\sqrt{2})/2^k$. Moreover, by definition we have

$$S^*_i = \left\{ x \in [0,1]^d \;\middle|\; (\exists Q \ni x)\left(\frac{1}{\mu(Q)}\int_Q |f_i - f_{i+1}| \geq \frac{1}{2^{i/2}}\right)\right\}$$

where Q ranges over cubes in $[0,1]^d$. Thus S_i^* is first-order definable, so by Lemma 3.3 $\mu(S_i^*)$ is computable, uniformly in i. Since $\mu(S_i^*) \leq 2c/2^{i/2}$ it follows by Lemma 3.5 that $\mu(V_k^*)$ is computable, uniformly in k.

Suppose now that $x \notin V_k^*$. Then for all rational cubes Q containing x and all $n \geq k$ and $i \geq 2n$ we have

$$\frac{1}{\mu(Q)} \int_Q |f_i - f_{i+1}| \leq \frac{1}{2^{i/2}}.$$

Thus

$$\frac{1}{\mu(Q)} \int_Q |f - f_{2n}| \leq \sum_{i=2n}^{\infty} \frac{1}{2^{i/2}} \leq \frac{2+\sqrt{2}}{2^n}$$

and this completes the proof. □

Lemma 3.14. Let $f \in L_1([0,1]^d)$ be L_1-computable with polynomial approximations f_n as in Definition 2.6. Then, we can find a computable sequence of rational numbers D_n such that the following holds. For all k and all $n \geq k$ and all $x \notin V_k \cup V_k^*$ we have

$$\left| \lim_{m\to\infty} f_m(x) - \frac{\int_Q f}{\mu(Q)} \right| \;\leq\; \frac{2+\sqrt{2}}{2^{n-1}} + D_n \cdot (\text{diameter of } Q)$$

for all $Q \ni x$. Here V_k and V_k^* are as in Lemmas 3.6 and 3.13 respectively. In particular, if $x \in [0,1]^d$ is Schnorr random, we have

$$\left| \widehat{f}(x) - \frac{\int_Q f}{\mu(Q)} \right| \;\leq\; \frac{2+\sqrt{2}}{2^{n-2}} + D_n \cdot (\text{diameter of } Q).$$

Proof. Since f_{2n} is a polynomial with rational coefficients, we can compute a positive rational number D_n which is an upper bound of the maximum gradient $\max\{|\nabla f_{2n}(x)| \mid x \in [0,1]^d\}$. It follows by the Mean Value Theorem that

$$\left| f_{2n}(x) - \frac{\int_Q f_{2n}}{\mu(Q)} \right| \;\leq\; D_n \cdot (\text{diameter of } Q)$$

for all $x \in Q$. By Lemmas 3.6 and 3.13 we have

$$|\lim_{n\to\infty} f_n(x) - f_{2n}(x)| \;\leq\; \frac{2+\sqrt{2}}{2^{n-1}}$$

and

$$\left| \frac{\int_Q f}{\mu(Q)} - \frac{\int_Q f_{2n}}{\mu(Q)} \right| \;\leq\; \frac{\int_Q |f - f_{2n}|}{\mu(Q)} \;\leq\; \frac{2+\sqrt{2}}{2^{n-1}}$$

for all $n \geq k$ whenever $Q \ni x \notin V_k \cup V_k^*$. Combining these inequalities we obtain the desired conclusion. □

Lemma 3.15. Given an L_1-computable function $f \in L_1([0,1]^d)$ with polynomial approximations f_n as in Definition 2.6, there exists a Schnorr test U_n, $n = 1, 2, \ldots$ such that for all $x \notin \bigcap_n U_n$,

$$\lim_{n\to\infty} f_n(x) \;=\; \lim_{Q\to x} \frac{\int_Q f}{\mu(Q)}$$

where the limit is taken over all $Q \ni x$ as the diameter of Q tends to 0.

Proof. This follows from Lemma 3.14. □

Theorem 3.16. Let $f \in L_1([0,1]^d)$ be L_1-computable. Then for all Schnorr random x we have

$$\widehat{f}(x) \;=\; \lim_{Q \to x} \frac{\int_Q f}{\mu(Q)} \tag{6}$$

where the limit is taken over all cubes $Q \ni x$ as the diameter of Q tends to 0.

Proof. The sets V_k and V_k^* form Schnorr tests. Hence, for any Schnorr random $x \in [0,1]^d$ we can find k such that $x \notin V_k \cup V_k^*$. Given $\epsilon > 0$ let $n \geq k$ be so large that

$$\frac{2+\sqrt{2}}{2^{n-2}} \;<\; \frac{\epsilon}{2}$$

and let D_n be as in Lemma 3.14. We then have

$$\left| \widehat{f}(x) - \frac{\int_Q f}{\mu(Q)} \right| \;<\; \frac{2+\sqrt{2}}{2^{n-2}} + \frac{\epsilon}{2} \;<\; \epsilon$$

for all Q of diameter $< \epsilon/2D_n$. This completes the proof. □

Remark 3.17. Lemma 3.15, combined with the fact that every Schnorr test admits computable points passing the test (see for instance [15, Theorem 5.1.0.4]), implies the following interesting observation.

Corollary 3.18. Given a computable sequence of L_1-computable functions $f_i \in L_1([0,1]^d)$ with approximating sequences f_{in} as in Definition 2.6, and given an effectively closed set $P \subseteq [0,1]^d$ of computable positive measure, we can effectively find a computable point $x \in P$ such that for each i we have

$$\lim_{n \to \infty} f_{in}(x) \;=\; \lim_{Q \to x} \frac{\int_Q f}{\mu(Q)}$$

where the limit is taken over all $Q \ni x$ as the diameter of Q tends to 0.

Remark 3.19. The classical Lebesgue Differentiation Theorem (see Remark 2.3) follows from Theorem 3.16 by relativization to an arbitrary Turing oracle. Thus, Lemma 3.14 and Theorem 3.16 may be viewed as computationally motivated refinements or generalizations of the Lebesgue Differentiation Theorem. Such results were first obtained by Pathak in [14] which was based on her undergraduate research project performed under Simpson's supervision.

4. Lebesgue points for L_1-computable functions are Schnorr

Remark 4.1. In this section we shall prove a converse to Theorem 3.16. Namely, if $x \in [0,1]^d$ is such that the limit in (6) exists for all L_1-computable functions $f \in L_1([0,1]^d)$, then x is random in the sense of Schnorr. In fact, we shall associate a particular f to each Schnorr test, as stated in Lemmas 4.5 and 4.6.

Definition 4.2. Two cubes Q_1 and Q_2 are said to be *almost disjoint* if their intersection is entirely contained in the boundary of Q_1.

Lemma 4.3. Let $Q_1, \ldots, Q_n$ be a finite sequence of pairwise almost disjoint rational cubes, and let R be a rational cube such that $R \not\subseteq Q_1 \cup \cdots \cup Q_n$. Then, we can effectively extend $Q_1, \ldots, Q_n$ to a longer finite sequence of pairwise almost disjoint rational cubes $Q_1, \ldots, Q_n, Q_{n+1}, \ldots, Q_{n+k}$ such that

$$Q_1 \cup \cdots \cup Q_n \cup Q_{n+1} \cup \cdots \cup Q_{n+k} = Q_1 \cup \cdots \cup Q_n \cup R.$$

Proof. Let $m \in \mathbb{N}$ be the common denominator of all of the coordinates of all of the vertices of $Q_1, \ldots, Q_n, R$. We can then break up each of these cubes into almost disjoint cubes with edge length $1/m$. That is, we can write each of $Q_1, \ldots, Q_n, R$ as a finite union of pairwise almost disjoint cubes of the form

$$\left\{ \langle x_1, \ldots, x_d \rangle \;\middle|\; x_i \in \left[\frac{l_i}{m}, \frac{l_i + 1}{m} \right], i = 1, \ldots, d \right\}$$

where $l_1, \ldots, l_d$ are natural numbers less than m. Let $Q_{n+1}, \ldots, Q_{n+k}$ be a list of the cubes of this form that are contained in R and not contained in $Q_1, \ldots, Q_n$. This gives our desired conclusion. □

Lemma 4.4. Given a nonempty Σ^0_1 set $U \subseteq [0,1]^d$, we can effectively find a computable sequence of pairwise almost disjoint rational cubes Q_i such that $U = \bigcup_{i=1}^\infty Q_i$.

Proof. Let R_i, $i = 1, 2, \ldots$ be a computable sequence of rational cubes such that $U = \bigcup_{i=1}^\infty R_i$. We shall refine this to a pairwise almost disjoint sequence. Assume inductively that we have found a pairwise disjoint sequence of rational cubes $Q_1, \ldots, Q_{n_k}$ such that $\bigcup_{i=1}^k R_i = \bigcup_{j=1}^{n_k} Q_j$. We may safely assume that $R_{k+1} \not\subseteq \bigcup_{i=1}^k R_i$. Apply Lemma 4.3 to effectively find a longer pairwise disjoint sequence of rational cubes $Q_1, \ldots, Q_{n_{k+1}}$ with $n_{k+1} > n_k$ such that $\bigcup_{i=1}^{k+1} R_i = \bigcup_{j=1}^{n_{k+1}} Q_j$. Letting k go to infinity we obtain a computable sequence of pairwise disjoint rational cubes Q_j, $j = 1, 2, \ldots$ such that $\bigcup_{i=1}^\infty R_i = \bigcup_{j=1}^\infty Q_j$. □

Lemma 4.5. Given a Schnorr test U_n, $n = 1, 2, \ldots$, we can construct a bounded (in fact $0,1$-valued) L_1-computable function $f \in L_1([0,1]^d)$ such that

$$\limsup_{Q \to x} \frac{\int_Q f d\mu}{\mu(Q)} \;\geq\; \frac{3}{4} \qquad \text{and} \qquad \liminf_{Q \to x} \frac{\int_Q f d\mu}{\mu(Q)} \;\leq\; \frac{1}{4}$$

for all $x = \langle x_1, \ldots, x_d \rangle \in \bigcap_n U_n$ such that $x_1, \ldots, x_d$ are irrational.

Proof. Let Seq be the set of finite sequences of natural numbers. For $\sigma = \langle i_1, \ldots, i_n \rangle \in$ Seq we write $|\sigma| = n =$ the *length* of σ. We use $\langle\rangle$ to denote the *empty sequence*, i.e., the unique member of Seq of length 0. For $\sigma, \tau \in$ Seq let $\sigma^\frown\tau$ be their concatenation, i.e., σ followed by τ.

To each $\sigma \in$ Seq we effectively associate a rational cube Q_σ by induction on $|\sigma|$. We begin with $Q_{\langle\rangle} = [0,1]^d$. Given Q_σ, we effectively find an integer n_σ so large that $\mu(Q_\sigma \cap U_{n_\sigma}) < \mu(Q_\sigma)/4$. Then we apply Lemma 4.4 to effectively obtain a pairwise almost disjoint computable sequence of rational cubes $Q_{\sigma^\frown\langle i\rangle}$, $i = 0, 1, 2, \ldots$ such that

$$U_{n_\sigma} \cap (\text{interior of } Q_\sigma) \;=\; \bigcup_{i=0}^{\infty} Q_{\sigma^\frown\langle i\rangle} \;.$$

In this way we construct Q_σ for all $\sigma \in$ Seq.

Similarly we assign values to f. For all $x \in Q_\sigma \setminus U_{n_\sigma}$ let

$$f(x) = \begin{cases} 1 & \text{if } |\sigma| \text{ is odd,} \\ 0 & \text{if } |\sigma| \text{ is even.} \end{cases}$$

In particular $f(x)$ is defined for all $x \in [0,1]^d \setminus \bigcap_n U_n$. Since f is $0,1$-valued and $\mu(\bigcap_n U_n) = 0$, we clearly have $f \in L_1([0,1]^d)$.

Now let $x = \langle x_1, \ldots, x_d \rangle \in \bigcap_n U_n$ be such that $x_1, \ldots, x_d$ are irrational. Let $h : \mathbb{N} \to \mathbb{N}$ be such that $x \in Q_{h\restriction k}$ for all k. (Note that the existence of h is not guaranteed if even a single coordinate of x is rational, because then x could be on the boundary of a cube Q_σ, in which case $x \notin \bigcup_i Q_{\sigma^\frown\langle i\rangle}$ even though $x \in Q_\sigma \cap U_{n_\sigma}$.) If k is odd we have $f = 1$ on $Q_{h\restriction k} \setminus U_{n_{h\restriction k}}$, hence

$$\frac{1}{\mu(Q_{h\restriction k})} \int_{Q_{h\restriction k}} f d\mu \;\geq\; \frac{\mu(Q_{h\restriction k} \setminus U_{n_{h\restriction k}})}{\mu(Q_{h\restriction k})} \;>\; \frac{3}{4}$$

so $\limsup_{Q\to x} \int_Q f d\mu/\mu(Q) \geq 3/4$. If k is even we have $f = 0$ on $Q_{h\restriction k} \setminus U_{n_{h\restriction k}}$, hence

$$\frac{1}{\mu(Q_{h\restriction k})} \int_{Q_{h\restriction k}} f d\mu \;\leq\; \frac{\mu(Q_{h\restriction k} \cap U_{n_{h\restriction k}})}{\mu(Q_{h\restriction k})} \;<\; \frac{1}{4}$$

so $\liminf_{Q\to x} \int_Q f d\mu/\mu(Q) \leq 1/4$.

It remains to show that f is L_1-computable. We shall construct a computable sequence of finite rational step functions f_m which approximates f. In order to construct f_m, we shall first construct a finite sequence of integers $l_{m,1}, \ldots, l_{m,m}$. Assume inductively that we have defined $l_{m,1}, \ldots, l_{m,k}$ where $0 \leq k < m$. Let

$$T_{m,k} = \{\langle i_1, \ldots, i_k \rangle \mid 0 \leq i_1 \leq l_{m,1}, \ldots, 0 \leq i_k \leq l_{m,k}\}\,.$$

For each $\sigma \in T_{m,k}$ we know that $U_{n_\sigma} \cap (\text{interior of } Q_\sigma) = \bigcup_{i=0}^\infty Q_{\sigma^\frown\langle i\rangle}$ and $\mu(U_{n_\sigma} \cap Q_\sigma)$ is effectively computable. Hence, we can effectively find $l_{m,k+1}$ so large that

$$\sum_{\sigma \in T_{m,k}} \mu(W_\sigma) < \frac{1}{2^{m+k}} \qquad \text{where} \qquad W_\sigma = U_{n_\sigma} \cap Q_\sigma \setminus \bigcup_{i=0}^{l_{m,k+1}} Q_{\sigma^\frown\langle i\rangle}\,.$$

This completes the definition of $l_{m,1}, \ldots, l_{m,m}$. We now define f_m as follows. For all $x \in (\text{interior of } Q_\sigma) \setminus \bigcup_{i=0}^{l_{m,k+1}} Q_{\sigma^\frown\langle i\rangle}$ where $\sigma \in T_{m,k}$ and $0 \leq k < m$, let

$$f_m(x) = \begin{cases} 1 & \text{if } |\sigma| \text{ is odd,} \\ 0 & \text{if } |\sigma| \text{ is even.} \end{cases}$$

For all other x let $f_m(x) = 0$.

Note that $f(x) = f_m(x)$ for all x except possibly when $x \in W_\sigma$ for some $\sigma \in T_{m,k}$ and $0 \leq k < m$, or when $x \in Q_\sigma$ for some σ such that $|\sigma| = m$. We shall use this observation to show that $\|f - f_m\|_1$ is small. First, note that

$$\mu\left(\bigcup_{k<m} \bigcup_{\sigma \in T_{m,k}} W_\sigma\right) < \sum_{k<m} \frac{1}{2^{m+k}} < \frac{1}{2^{m-1}}\,.$$

In addition, by construction of the Q_σ's we have

$$\mu\left(\bigcup_{i=0}^{\infty} Q_{\sigma^\frown\langle i\rangle}\right) < \frac{\mu(Q_\sigma)}{4}$$

for each σ, hence by induction on m we have

$$\mu\left(\bigcup_{|\sigma|=m} Q_\sigma\right) \leq \frac{1}{4^m}$$

in view of almost disjointness. Since f and f_m are $0,1$-valued, it follows that

$$\|f - f_m\|_1 < \frac{1}{2^{m-1}} + \frac{1}{4^m} < \frac{1}{2^{m-2}}$$

for all m. Thus f is L_1-computable. □

Lemma 4.6. Let $x = \langle x_1, \dots, x_d \rangle \in [0,1]^d$ be such that at least one of $x_1, \dots, x_d$ is rational. Then, we can construct a bounded (in fact $0,1$-valued) L_1-computable function $f \in L_1([0,1]^d)$ such that

$$\limsup_{Q \to x} \frac{\int_Q f \, d\mu}{\mu(Q)} \ \geq \ \frac{3}{4} \qquad \text{and} \qquad \liminf_{Q \to x} \frac{\int_Q f \, d\mu}{\mu(Q)} \ \leq \ \frac{1}{4}.$$

Proof. We may safely assume that $x_1 = q$ is rational. For each n let $S_n \subseteq [0,1]^d$ be a slice of $[0,1]^d$ defined by $S_n = ([q - 1/2^{2n}, q + 1/2^{2n}] \cap [0,1]) \times [0,1]^{d-1}$. The width of this slice is $\mu(S_n) = \mu([q - 1/2^{2n}, q + 1/2^{2n}] \cap [0,1]) \leq 1/2^{2n-1}$. Moreover $S_0 = [0,1]^d$ and $x \in \bigcap_n S_n$. Define $f \in L_1([0,1]^d)$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in S_n \setminus S_{n+1} \text{ where } n \text{ is odd,} \\ 0 & \text{if } x \in S_n \setminus S_{n+1} \text{ where } n \text{ is even.} \end{cases}$$

Let Q_n be a cube such that $x \in Q_n \subseteq S_n$ and the edge length of Q_n is equal to the width of S_n, so that $\mu(Q_n) = \mu(S_n)^d$. For odd n we have $f = 1$ on $Q_n \setminus S_{n+1}$, hence

$$\frac{1}{\mu(Q_n)} \int_{Q_n} f \, d\mu \ \geq \ \frac{\mu(Q_n \setminus S_{n+1})}{\mu(Q_n)} \ \geq \ \frac{3}{4}$$

so $\limsup_{Q \to x} \int_Q f \, d\mu / \mu(Q) \geq 3/4$. For even n we have $f = 0$ on $Q_n \setminus S_{n+1}$, hence

$$\frac{1}{\mu(Q_n)} \int_{Q_n} f \, d\mu \ \leq \ \frac{\mu(Q_n \cap S_{n+1})}{\mu(Q_n)} \ \leq \ \frac{1}{4}$$

so $\liminf_{Q \to x} \int_Q f \, d\mu / \mu(Q) \leq 1/4$.

It remains to show that f is L_1-computable. Consider a computable sequence of finite rational step functions f_k defined by

$$f_k(x) = \begin{cases} 1 & \text{if } x \in S_n \setminus S_{n+1} \text{ where } n \text{ is odd and } n < k, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\|f - f_k\|_1 \leq \mu(S_k) \leq 1/2^{2k-1}$ and thus f is L_1-computable. □

Theorem 4.7. For all $x \in [0,1]^d$ the following are pairwise equivalent.

(1) x is Schnorr random.
(2) $\displaystyle\lim_{Q \to x} \frac{\int_Q f}{\mu(Q)}$ exists for all L_1-computable functions $f \in L_1([0,1]^d)$.
(3) For all L_1-computable functions $f \in L_1([0,1]^d)$ and approximating sequences f_n as in Definition 2.6, we have

$$\lim_{n \to \infty} f_n(x) \ = \ \lim_{Q \to x} \frac{\int_Q f}{\mu(Q)}$$

and both limits exist.

Proof. The implication $1 \Rightarrow 3$ has been proved in Theorem 3.16. The implication $3 \Rightarrow 2$ is trivial, and $2 \Rightarrow 1$ is obtained by combining Lemmas 4.5 and 4.6. □

Remark 4.8. Clearly the numbers $1/4$ and $3/4$ in Lemmas 4.5 and 4.6 are arbitrary and can be replaced by any pair $\epsilon, 1-\epsilon$ with $0 < \epsilon < 1$. Indeed, one can construct a $0, 1$-valued L_1-computable function f such that $\liminf_{Q \to x} \int_Q f d\mu/\mu(Q) = 0$ and $\limsup_{Q \to x} \int_Q f d\mu/\mu(Q) = 1$ for all $x \in \bigcap_n U_n$. Thus for any $x \in [0,1]^d$ which is not Schnorr random, the Lebesgue Differentiation Theorem for $0, 1$-valued L_1-computable functions fails as badly as possible. We thank the referee for pointing this out.

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