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Σ^1_1 AND Π^1_1 TRANSFINITE INDUCTION

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§0. Introduction.

In this paper we explore some technical questions related to the formal system ATR_0 of arithmetical transfinite recursion with quantifier free induction on the natural numbers. This system, and indeed all of the formal systems considered in this paper, are subsystems of second order arithmetic and use classical logic.

The specific system ATR_0 was introduced by Friedman [4] and was studied in some detail by Friedman, McAloon and Simpson [6]. (A stronger system ATR, consisting of ATR_0 plus full induction on the natural numbers, had been introduced earlier by Friedman [3] and had been studied by Friedman [1] and Steel [13].)

The interest of ATR_0 has by now been well established. On the one hand, it was shown in [3], [4], [6] and [13] that ATR_0 is just strong enough to formalize many mathematical theorems which depend on having a good theory of countable well orderings. Indeed, many such theorems turn out to be provably equivalent to ATR_0 over a relatively weak base thoery ACA_0. (As an example here we may cite the theorem that every uncountable Borel set contains a perfect subset.) On the other hand, it was shown in [6] that ATR_0 is proof theoretically not very strong, e.g. its proof theoretic ordinal is just the Feferman/Schütte ordinal Γ_0. (From recent work of Jäger [10] and Friedman (§5 below) it follows that the proof theoretic ordinal of ATR is Γ_{ε_0}.)

The purpose of this paper is to study the systems Σ^1_1-TI_0 and Π^1_1-TI_0 of Σ^1_1 and Π^1_1 transfinite induction along arbitrary well orderings of the natural numbers. These systems were defined in [4]. We show in §2 that Σ^1_1-TI_0 is equivalent to ATR_0 plus Σ^1_1 ordinary induction, or equivalently ATR_0 plus Π^1_1 ordinary induction. (Here "ordinary" means "along the usual well ordering of the natural numbers".) We also show that Σ^1_1-TI_0 is properly stronger than ATR_0. In §4 we show that Π^1_1-TI_0 is equivalent to the system Σ^1_1-DC_0 of Σ^1_1 dependent choices with quantifier free induction on the natural numbers (denoted HDC_0 in [4]). These results in §§2 and 4 answer questions which were naturally suggested by

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the results of [4] and [13].

In §3 we use the results of §2 to prove a conjecture from [6] concerning partition calculus in ATR_0. It was known from [6] that ATR_0 proves that Galvin/Prikry theorem for closed sets. We now show in §3 that ATR_0 does not prove the Galvin/Prikry theorem for finite sequences of closed sets.

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§1. Preliminaries

All of the formal systems considered in this paper are in the language of second order arithmetic which consists of $+,\cdot,0,1,=,<,\in$, number variables $k,m,n,\ldots,$ set variables $X,Y,\ldots,$ propositional connectives, number quantifiers, and set quantifiers. Number variables are intended to range over natural numbers and set variables are intended to range over sets of natural numbers. For general background see Kreisel [11].

A formula is said to be arithmetical if it contains no set quantifiers. The weakest system we shall consider is ACA_0 which consists of the usual ordered semiring axioms for the natural numbers, the quantifier free induction axiom

$$0\in X \& \forall k(k\in X\rightarrow k+1\in X) \rightarrow \forall k(k\in X),$$

and arithmetical comprehension axioms

$$\exists X\ \forall m(m \in X \leftrightarrow \theta(m))$$

where θ is arithmetical and does not contain X. It is easy to see that ACA_0 is finitely axiomatizable. All systems are assumed to include ACA_0.

Within ACA_0 we have the arithmetical pairing function

$$(m,n) = \frac{1}{2}(m + n + 1)(m + n) + m.$$

A binary relation R on the natural numbers is identified with a set $X = \{(m,n) : m\ R\ n\}$. A well ordering is a binary relation $\prec$ which is a linear ordering of the natural numbers such that

$$\forall X[\forall n(\forall m\prec n(m\in X)\rightarrow n\in X)\rightarrow\forall n(n\in X)].$$

We write $WO(\prec)$ to mean that $\prec$ is a well ordering of the natural numbers. Thus $WO(\prec)$ is a Π^1_1 formula with a free set variable $\prec$. The scheme of transfinite induction (TI_0) consists of all instances of

$$WO(\prec)\&\forall n(\forall m\prec n\varphi(m) \rightarrow \varphi(n)) \rightarrow \forall n\varphi(n)$$

where φ is an arbitrary formula.

A Σ_n^1 (respectively Π_n^1) formula is one consisting of n set quantifiers beginning with an existential (respectively universal) one followed by an arithmetical matrix. By Σ_n^1-TI_0 (respectively Π_n^1-TI_0) we mean the system consisting of ACA_0 plus the transfinite induction scheme TI_0 restricted to Σ_n^1 (respectively Π_n^1) formulas φ. It is known that the system Π_∞^1-TI_0 ($= \bigcup_{n\in\omega}\Pi_n^1$-$TI_0$) is not finitely axiomatizable. (See the beginning of §4 below.) The main purpose of this paper is to study the systems Σ_1^1-TI_0 and Π_1^1-TI_0.

We shall have occasion to consider certain comprehension and choice principles. By Π_n^1-CA_0 we mean ACA_0 plus all comprehension axioms

$$\exists X\ \forall m(m \in X \leftrightarrow \varphi(m))$$

in which φ is Π_n^1 and does not contain X. By Δ_n^1-CA_0 we mean ACA_0 plus all instances of

$$\forall m(\varphi(m) \leftrightarrow \sim\psi(m)) \rightarrow \exists X\ \forall m(m \in X \leftrightarrow \varphi(m))$$

where φ and ψ are Π_n^1 and do not contain X. Write $(Y)_k = \{y : (y,k) \in Y\}$. By Σ_n^1-AC_0 we mean ACA_0 plus all instances of the countable choice scheme

$$\forall k\exists X\varphi(k,X) \rightarrow \exists Y\forall k\varphi(k,(Y)_k)$$

where φ is Σ_n^1 and does not contain Y or bound occurrences of k. By Σ_n^1-DC_0 we mean ACA_0 plus all instances of the Σ_n^1 dependent choice scheme

$$\forall X\exists Y\varphi(X,Y) \rightarrow \exists Z\forall k\varphi((Z)_k,(Z)_{k+1})$$

where φ is Σ_n^1 and does not contain Z or k. It is well known and easy to see that Σ_{n+1}^1-DC_0 includes Σ_{n+1}^1-AC_0 which includes Δ_{n+1}^1-CA_0 which includes Π_n^1-CA_0. Obviously Π_1^1-CA_0 implies Σ_1^1-TI_0 and Π_1^1-TI_0. It is known from [2] and [9] that Π_1^1-CA_0 is proof theoretically stronger than Π_∞^1-TI_0.

An important role in this paper will be played by the system ATR_0. ATR_0 consists of ACA_0 plus the scheme of arithmetical transfinite recursion

$$WO(\prec) \rightarrow$$
$$\exists X\forall y\forall n[(y,n)\in X \leftrightarrow \theta(y,\{(x,m):m\prec n\ \&\ (x,m) \in X\})]$$

where $\theta(y,X)$ is arithmetical. Intuitively, X is a set obtained by iterating arithmetical comprehension along the well ordering $\prec$. It is easy to see that ATR_0 is finitely axiomatizable: the axioms are those of ACA_0 plus a Π_2^1 sentence asserting that the Turing jump operator can be iterated along any well ordering starting at any set.

For background information on ATR_0 see [1], [3], [6]. One may also consult [4] and [13], but see the comment just before Lemma 2.7 below. Two important facts which we shall need are

(i) ATR_0 proves Σ^1_1-AC_0;

(ii) ATR_0 proves comparability of well orderings, i.e. if $\prec_1$ and $\prec_2$ are well orderings of ω then they are isomorphic or one is isomorphic to a proper initial segment of the other. (We say that $\prec_1$ and $\prec_2$ are <u>isomorphic</u> if there exists a binary relation of isomorphism between them.)

Each of the systems above may be strengthened by adding the <u>induction scheme</u>

$$\varphi(0)\&\forall k(\varphi(k) \to \varphi(k+1)) \to \forall k\varphi(k)$$

where φ is arbitrary. This scheme formalizes the principle of proof by induction on the natural numbers. If S_0 denotes one of our systems with only the quantifier free induction axiom, then S will denote S_0 plus the induction scheme. For instance, $ATR = ATR_0$ + induction scheme, and Σ^1_1-$TI = \Sigma^1_1$-TI_0 + induction scheme.

§2. Σ^1_1 <u>transfinite induction</u>.

In this section, restrictions of the ordinary induction scheme will play an important role. By Σ^1_1 (respectively Π^1_1) <u>induction</u> we mean the induction scheme restricted to formulas φ which are Σ^1_1 (respectively Π^1_1). Note that the induction scheme restricted to arithmetical φ is provable in ACA_0.

In addition to Σ^1_1 and Π^1_1 induction on the natural numbers, it will be convenient to consider the following finite form of Π^1_1-CA_0 which we call <u>finite</u> Π^1_1-CA_0:

$$\forall n\exists X\forall i \le n(i \in X \leftrightarrow \varphi(i))$$

where φ is a Π^1_1 formula not containing X.

2.1 <u>Lemma</u>. <u>Over</u> ACA_0 <u>the following are pairwise equivalent</u>:

1. Π^1_1 <u>induction</u> (<u>on the natural numbers</u>);
2. Σ^1_1 <u>induction</u> (<u>on the natural numbers</u>);
3. <u>finite</u> Π^1_1-CA_0.

<u>Proof</u>. $1 \leftrightarrow 2$: Assume Π^1_1 induction. Suppose $\forall k(\varphi(k) \to \varphi(k+1))$ and $\sim\varphi(n)$ where φ is Σ^1_1. Prove by induction on $i \le n$ that $\sim\varphi(n-i)$. In particular $\sim\varphi(0)$. This proves Σ^1_1 induction from Π^1_1 induction. The converse is similar.

$2 \to 3$: Note first that Σ_1^1 induction implies finite Σ_1^1-AC_0, i.e. the scheme

$$\forall k \exists X \theta(k,X) \to \forall n \exists Y \forall k<n\, \theta(k,(Y)_k)$$

where θ is arithmetical and Y does not occur in θ. Now let φ be a Π_1^1 formula for which Π_1^1-CA_0 fails, i.e. for some fixed n, there is no set X such that $\forall i \le n\ (i \in X \leftrightarrow \varphi(i))$. Let $\psi(k)$ say that there exists a finite set $s \subseteq \{0,1,\ldots,n\}$ of cardinality k such that $\forall i(i \in s \to \varphi(i))$. Clearly $\psi(0)$ and $\forall k(\psi(k) \to \psi(k+1))$, and by finite Σ_1^1-AC_0 the formula $\psi(k)$ is equivalent to a Π_1^1 formula. Hence by Π_1^1 induction we have in particular $\psi(n+2)$ which is absurd.

$3 \to 1$: Suppose $\varphi(0)$ and $\forall k(\varphi(k) \to \varphi(k+1))$ where φ is Π_1^1. Given n, by finite Π_1^1-CA_0 there exists X such that $\forall k \le n(k \in X \leftrightarrow \varphi(k))$. Then $0 \in X$ and $\forall k < n(k \in X \to k+1 \in X)$ so by quantifier free induction we have $n \in X$, i.e. $\varphi(n)$. This completes the proof.

For technical reasons we consider the following weak form of Σ_1^1-AC_0 which we call weak Σ_1^1-AC_0:

$$\forall k \exists! X \theta(k,X) \to \exists Y \forall k\, \theta(k,(Y)_k)$$

where θ is arithmetical, Y does not occur in θ, and $\exists! X$ abbreviates "there exists a unique X such that." Weak Σ_1^1-AC_0 is of course not to be confused with finite Σ_1^1-AC_0 which was introduced in the proof of Lemma 2.1.

2.2 Lemma. Δ_1^1-TI_0 plus finite Σ_1^1-AC_0 implies weak Σ_1^1-AC_0.

Proof. By ACA_0 there exist Skolem functions for any arithmetical formula. Thus to prove weak Σ_1^1-AC_0 it suffices to prove

$$\forall k \exists! f \forall n \theta(k,f[n]) \to \exists g \forall k \forall n \theta(k,g_k[n])$$

where θ is arithmetical, f and g are function variables (intended to range over one-place functions from natural numbers to natural numbers), $f[n] = \langle f(0),\ldots,f(n-1)\rangle$, and $g_k(m) = g((m,k))$. Assume the hypothesis $\forall k \exists! f \forall n \theta(k,f[n])$ and let T be the tree of unsecured finite sequences for the conclusion, i.e. $t \in T$ if and only if $\forall k \le \ell h(t) \forall n \le \ell h(t_k) \theta(k,t_k[n])$. If the conclusion fails, then T has no path, i.e. the Kleene/Brouwer ordering of T is a well ordering. Define $t \in T$ to be good if and only if

$$\exists g(\forall k \le \ell h(t) \forall n \theta(k,g_k[n]) \ \&\ g[\ell h(t)]=t),$$

i.e.

$$\forall g(\forall k \le \ell h(t) \forall n \theta(k,g_k[n]) \to g[\ell h(t)]=t).$$

By hypothesis and finite Σ^1_1-AC_0 these two definitions of goodness are equivalent. Thus the property of goodness is Δ^1_1. Trivially the empty sequence is good, and the hypothesis easily implies that each good $t \in T$ has a good immediate extension in T. Thus we have a failure of Δ^1_1-TI_0 along the Kleene/Brouwer ordering of T. This completes the proof.

Remark. The scheme of weak Σ^1_1-AC_0 is perhaps of some independent interest. It is easy to see that Δ^1_1-CA_0 implies weak Σ^1_1-AC_0 and that every ω-model of weak Σ^1_1-AC_0 is closed under relative hyperarithmeticity. Hence the hyperarithmetic sets form the minimum ω-model of weak Σ^1_1-AC_0. Another easy observation is that given any descending sequence of Turing degrees separated by Turing jump, the reals recursive in all degrees in the sequence form an ω-model of weak Σ^1_1-AC_0. For more information about such sequences see Friedman [5] and Steel [12]. Van Wesep [14] has shown that there exists an ω-model of weak Σ^1_1-AC_0 which is not a model of Δ^1_1-CA_0.

The next lemma expresses the well known fact that number variables in Π^1_1 predicates can be uniformized, provably in ATR_0.

2.3 Lemma. Let $\varphi(n)$ be a Π^1_1 formula. There exists a Π^1_1 formula $\varphi^*(n)$ such that ATR_0 proves $\forall n(\varphi^*(n) \to \varphi(n))$ and $\exists n\varphi(n) \to \exists! n\varphi^*(n)$.

Proof. Let $\prec_n$ be the Kleene/Brouwer ordering of the tree of unsecured sequences for $\varphi(n)$. Thus by ACA_0 we have that $\varphi(n)$ holds if and only if $\prec_n$ is a well ordering. Put $\varphi^*(n)$ if and only if $\varphi(n)$ & $\sim\exists p<n(\prec_p$ is isomorphic to $\prec_n)$ & $\sim\exists p(\prec_p$ is isomorphic to a proper initial segment of $\prec_n)$. This works because ATR_0 proves comparability of well orderings.

2.4 Lemma. Let $\psi(m)$ and $\varphi(m,n)$ be Π^1_1 formulas. Then ATR_0 plus Π^1_1 induction (on the natural numbers) proves

$$\forall m[\psi(m) \to \exists n[\psi(n)\&\varphi(m,n)]] \to$$
$$\forall m[\psi(m) \to \exists f[f(0)=m\&\forall i[\psi(f(i))\&\varphi(f(i),f(i+1))]]].$$

Proof. Assume $\forall m[\psi(m) \to \exists n[\psi(n)\&\varphi(m,n)]]$. By ATR_0 and the previous lemma we may also assume $\forall m[\psi(m) \to \exists! n\varphi(m,n)]$. Fix m such that $\psi(m)$ holds. Let $\theta(k,s)$ say that s encodes a finite sequence of length $k+1$ such that $s(0) = m$ and $\forall i<k[\psi(s(i))\&\varphi(s(i),s(i+1))]$. By Σ^1_1-AC_0 (a consequence of ATR_0), the statement $\exists s\theta(k,s)$ is Π^1_1 so we can use Π^1_1 induction to prove that this statement holds for all k. Thus we have $\forall k\exists! s\theta(k,s)$. Hence, by Δ^1_1-CA_0 (a consequence of Σ^1_1-AC_0), there exists f such that $\forall k\theta(k,f[k+1])$, i.e. $f(0) = m$ and $\forall i[\psi(f(i))\&\varphi(f(i),f(i+1))]$. This completes the proof.

2.5 Theorem. The following are pairwise equivalent:

1. ATR_0 plus Π_1^1 induction (along the natural numbers);
2. ATR_0 plus Σ_1^1 induction (along the natural numbers);
3. ATR_0 plus finite Π_1^1-CA_0.
4. Σ_1^1-TI_0.

Proof. The pairwise equivalence of 1, 2, and 3 is by Lemma 2.1. Let $\prec$ be a linear ordering of the natural numbers on which Σ_1^1-TI_0 fails, i.e. we have a Π_1^1 formula $\psi(m)$ such that $\exists m\psi(m)$ and $\forall m[\psi(m) \rightarrow \exists n \prec m\psi(n)]$. By Lemma 2.4 we obtain a function f such that $\forall k[\psi(k) \& f(k+1) \prec f(k)]$, i.e. f is an infinite descending sequence through $\prec$. This proves $1 \rightarrow 4$.

Obviously Σ_1^1-TI_0 includes Σ_1^1 induction on the natural numbers so it remains only to prove that Σ_1^1-TI_0 implies ATR_0. Assume Σ_1^1-TI_0. By Lemma 2.2 we have weak Σ_1^1-AC_0. Let $\prec$ be a well ordering and suppose we are given an arithmetical formula $\theta(y,X)$. Let $\varphi(n,X)$ be the arithmetical formula which asserts that X is the result of iterating θ along $\prec$ up to n, i.e.

$$X = \{(y,m): m \prec n \ \& \ \theta(y,\{(x,k): k \prec m \ \& \ (x,k) \in X\})\}.$$

It is easy to see that for each n there is at most one X such that $\varphi(n,X)$. In order to prove ATR_0 we must prove $\forall n\exists X\varphi(n,X)$. Let n be fixed. By Σ_1^1-TI_0 we may assume $\forall m \prec n\exists X\varphi(m,X)$. Hence $\forall m \prec n\exists! X\varphi(m,X)$ so by weak Σ_1^1-AC_0 there exists Y such that $\forall m \prec n\varphi(m,(Y)_m)$. Then clearly $\varphi(n,X)$ if we put $X = \{(y,m): m \prec n \ \& \ \theta(y,(Y)_m)\}$. This completes the proof.

2.6 Corollary. (Friedman [3], Steel [13]). The systems ATR and Σ_1^1-TI (both with full induction on the natural numbers) are equivalent.

We shall now show that Σ_1^1-TI_0 is properly stronger than ATR_0. This result contradicts a claim which was made in Theorem 8 of [4] and on page 22 of [13].

2.7 Lemma. Over Σ_1^1-AC_0 the following are equivalent:

1. Σ_1^1 induction (on the natural numbers);
2. Π_3^1 soundness of ACA_0, i.e. the assertion that any Π_3^1 sentence provable in ACA_0 is true.

Proof. $2 \rightarrow 1$: Suppose that we have a failure of Σ_1^1 induction, i.e. $\varphi(0)$ and $\forall k(\varphi(k) \rightarrow \varphi(k+1))$ and $\sim\varphi(n)$ for some fixed n. By Π_3^1 soundness of ACA_0 let M be a model of ACA_0 plus $\varphi(0)$ plus $\forall k(\varphi(k) \rightarrow \varphi(k+1))$ plus $\sim\varphi(n)$.

The standard integers are canonically identified with an initial segment of the integers of M. Let $Z = \{m: M \models \sim\varphi(m)\}$. Then Z contains the standard integer n yet has no least element. This is absurd.

$1 \rightarrow 2$: Reasoning in Σ^1_1-ACA_0, let σ be a true Σ^1_3 sentence. We shall use Σ^1_1 induction to prove consistency of ACA_0 plus σ. Let L be the language of second order arithmetic augmented by set constants C_i, $i \in \omega$. Write $\sigma \equiv \exists U \forall X \exists Y \theta(U,X,Y)$ where θ is arithmetical and let $\varphi(U,X,Y)$ be the arithmetical formula

$$\forall i \exists j \exists k[\theta(U,(X)_i,(Y)_j) \;\&\; W^X_i = (Y)_k]$$

where W^X_i denotes the i^{th} set recursively enumerable in X. Let T be the L-theory consisting of RCA_0 ($\equiv$ ordered semiring axioms plus recursive comprehension plus quantifier free induction) together with axioms $\varphi(C_0,C_i,C_{i+1})$, $i \in \omega$. We shall prove consistency of T. Let T_k be the restriction of T to C_i, $i \leq k$. Fix a set U_0 such that $\forall X \exists Y \theta(U_0,X,Y)$. By Σ^1_1-AC_0 we have $\forall X \exists Y \varphi(U_0,X,Y)$. Hence by Σ^1_1 induction we have

$$\forall k \exists Z \forall i < k[(Z)_0 = U_0 \;\&\; \varphi(U_0,(Z)_i,(Z)_{i+1})].$$

It follows by cut elimination that $\forall k(T_k$ is consistent). Hence by the compactness theorem T is consistent. But from any model of T we can easily extract a model of ACA_0 plus σ by throwing away all sets except those which are recursive in C_i for some $i \in \omega$. Thus ACA_0 plus σ is consistent. This completes the proof.

2.8 Theorem. Σ^1_1-TI_0 proves Π^1_3 soundness of ATR_0, i.e. the assertion that any Π^1_3 sentence provable in ATR_0 is true. In particular Σ^1_1-TI_0 proves consistency of ATR_0.

Proof. We reason in Σ^1_1-TI_0. By Theorem 2.5 we have ATR_0 and hence Σ^1_1-AC_0. Let σ be a true Σ^1_3 sentence. We know that ATR_0 consists of ACA_0 plus a Π^1_2 sentence so we may as well assume that σ includes this Π^1_2 sentence. Now apply Lemma 2.7 to conclude that ACA_0 plus σ is consistent, i.e. ATR_0 plus σ is consistent.

2.9 Corollary. ATR_0 does not prove Σ^1_1-TI_0.

Proof. Immediate from Theorem 2.8 plus Gödel's second incompleteness theorem.

2.10 Corollary. ATR_0 does not prove Π_1^1 induction, Σ_1^1 induction, or finite Π_1^1-CA_0.

Proof. Immediate from the previous Corollary plus Theorem 2.5.

§3. Partition calculus in Σ_1^1-TI_0.

In this section we use the notation of §3 of [6]. We study closed sets in the space $[\omega]^\omega$ of infinite sets of natural numbers. It was shown in [6] that ATR_0 is equivalent to ACA_0 plus the Galvin/Prikry theorem for closed sets, i.e. the assertion that for every closed set $C \subseteq [\omega]^\omega$ there exists $A \in [\omega]^\omega$ such that either $[A]^\omega \subseteq C$ or $[A]^\omega \cap C = \phi$. The purpose of this section is to prove a similar result in which ATR_0 is replaced by the stronger theory Σ_1^1-TI_0.

A set $U \subseteq \omega$ is said to be hyperarithmetic if U is recursive in H_b for some $b \in \mathcal{O}$.

3.1 Lemma (ATR_0). Let C_i, $i < n$ be a recursively coded finite sequence of closed sets in $[\omega]^\omega$. If there is no hyperarithmetic $U \in [\omega]^\omega$ such that $\exists i < n[U]^\omega \cap C_i = \phi$ then there exists $A \in [\omega]^\omega$ such that $\forall i < n[A]^\omega \subseteq C_i$.

Proof. The proof of Theorem 3.8 of [6] actually establishes this stronger result.

3.2 Theorem. Over ACA_0 the following are equivalent:

1. Σ_1^1-TI_0;
2. For any finite sequence of closed sets $C_i \subseteq [\omega]^\omega$, $i < n$, there exists $A \in [\omega]^\omega$ such that for each $i < n$ either $[A]^\omega \subseteq C_i$ or $[A]^\omega \cap C_i = \phi$.

Proof. $1 \rightarrow 2$: By relativization we may safely assume that the given sequence of closed sets C_i, $i < n$, is recursively coded.

We claim that there exists a hyperarithmetic set $U \in [\omega]^\omega$ such that for each $i < n$ either $[U]^\omega \cap C_i = \phi$ or there is no hyperarithmetic $V \in [U]^\omega$ such that $[V]^\omega \cap C_i = \phi$. Suppose not. Let $\psi(k)$ be the assertion that there exists a hyperarithmetic $V \in [\omega]^\omega$ and a finite set s of cardinality k such that $\forall i \in s\ (i<n\ \&\ [V]^\omega \cap C_i = \phi)$. Clearly $\psi(0)$ and $\forall k(\psi(k) \rightarrow \psi(k+1))$. By Σ_1^1-AC_0 (a consequence of ATR_0) the formula $\psi(k)$ is equivalent to a Π_1^1 formula. Hence by Π_1^1 induction we have $\psi(n+1)$ which is absurd. This proves the claim.

Let U be as in the above claim. By finite Π_1^1-CA_0 let $X = \{i<n : [U]^\omega \cap C_i = \phi\}$. The claim tells us that there is no hyperarithmetic $V \in [U]^\omega$ such that $\exists i \in X\ [V]^\omega \cap C_i = \phi$. Hence by Lemma 3.1 there exists $A \in [U]^\omega$ such that $\forall i \in X\ [A]^\omega \subseteq C_i$. Hence for each $i < n$ either $[A]^\omega \subseteq C_i$

(if $i \in X$) or $[A]^\omega \cap C_i = \phi$ (if $i \notin X$).

$2 \to 1$: We already know (by Theorem 3.2 of [6]) that the partition theorem 3.2.2 implies ATR_0. By Theorem 2.5 it remains to show that the partition theorem also implies finite Π^1_1-CA_0. Let $\varphi(i)$ be Π^1_1 and let $T_i \subseteq \omega^{<\omega}$ be the associated tree of unsecured sequences, i.e. $\varphi(i)$ holds if and only if there is no path through T_i. For any $X \in [\omega]^\omega$ let $\pi_X \in \omega^\omega$ be the function which enumerates the elements of X in increasing order. Put $X \in C_i$ if and only if π_X majorizes a path through T_i, i.e. $\exists f \forall j (f(j) \leq \pi_X(j)\ \&\ f[j] \in T_i)$ or equivalently by König's lemma $\forall k \exists t (t \in T_i\ \&\ \ell h(t)=k\ \&\ \forall j<k\ t(j) \leq \pi_X(j))$. Clearly C_i is a closed set in $[\omega]^\omega$. Now given n, use the partition theorem 3.2.2 to get $A \in [\omega]^\omega$ such that for each $i < n$ either $[A]^\omega \subseteq C_i$ or $[A]^\omega \cap C_i = \phi$. Then for $i < n$ we have $\varphi(i)$ if and only if $\sim [A]^\omega \subseteq C_i$. The latter formula is arithmetical so by ACA_0 we have $\exists X \forall i<n (i \in n \leftrightarrow \varphi(i))$. This completes the proof of the theorem.

The following corollary establishes a conjecture which was stated after Theorem 3.9 in [6].

3.3 Corollary. The partition theorem 3.2.2 is not provable in ATR_0.

Proof. Immediate from Theorems 3.2 and 2.9.

An argument similar to the above proof of Theorem 3.2 establishes the following result which was discovered jointly by S. Shelah and the author, long before the author's discovery of Theorem 3.2.

3.4 Theorem. Over ACA_0 the following are equivalent:

1. Π^1_1-CA_0;

2. For any infinite sequence of closed sets $C_i \subseteq [\omega]^\omega$, $i \in \omega$, there exists $A \in [\omega]^\omega$ such that for each $k \in A$ and $i < k$ either $[A/\{k\}]^\omega \subseteq C_i$ or $[A/\{k\}]^\omega \cap C_i = \phi$. (Here $A/\{k\} = \{n \in A : n>k\}$.)

§4. Π^1_1 transfinite induction.

Friedman [3] has shown that over ACA_0 the transfinite induction scheme Π^1_∞-$TI_0 = \bigcup_{n\in\omega} \Pi^1_n$-$TI_0$ is equivalent to the ω-model reflection scheme Σ^1_∞-$RFN_0 = \bigcup_{n\in\omega} \Sigma^1_n$-$RFN_0$. Here Σ^1_n-RFN_0 asserts that for any Σ^1_n sentence $\varphi(X_1,\ldots,X_m)$ with set parameters $X_1,\ldots,X_m$ there exists a countable ω-model M of ACA_0 such that $X_1,\ldots,X_m \in M$ and $M \models_\omega \varphi(X_1,\ldots,X_m)$. It is natural to ask how much transfinite induction is equivalent to how much ω-model reflection. As a rule, special cases of this question appear to be difficult. However, one special case is answered by the following theorem.

4.1 Theorem. Over ACA_0 the following are pairwise equivalent:

1. Π_1^1-TI_0.

2. Σ_1^1-DC_0.

3. Σ_3^1-RFN_0.

Remark. The equivalence of 2 and 3 is due to Friedman [1]. The equivalence of 1 and 2 may be derived from the appendix of Howard [8] together with the reduction of BI_1 to BI_0 in Howard/Kreisel [9]. The equivalence of 1 and 2 subsumes several results which have been stated by Friedman in Theorem 4.2 of [3] and Theorem 8 of [4].

Proof of Theorem 4.1. $1 \to 2$: Similar to the proof of Lemma 2.2. Recall that Σ_1^1-DC_0 says $\forall X \exists Y \varphi(X,Y) \to \exists Z \forall k \varphi((Z)_k,(Z)_{k+1})$ where φ is Σ_1^1. By ACA_0 there exist Skolem functions for the arithmetical matrix of φ, so to prove Σ_1^1-DC_0 it suffices to prove

$$\forall f \exists g \forall n \theta(f[n],g[n]) \to \exists h \forall k \forall m \theta(h_k[n],h_{k+1}[n])$$

where f,g,h are function variables, θ is arithmetical, $f[n] = \langle f(0),\ldots,f(n-1)\rangle$, and $h_k(m) = h((m,k))$. Assume the hypothesis and let T be the tree of unsecured sequences for the conclusion, i.e. $t \in T$ if and only if

$$\forall k < \ell h(t) \forall n \le \min(\ell h(t_k),\ell h(t_{k+1}))\theta(t_k[n],t_{k+1}[n]).$$

If the conclusion fails then T has no path, i.e. the Kleene/Brouwer ordering of T is a well ordering. Say that $t \in T$ is good if

$$\exists h(\forall k<\ell h(t) \forall n \theta(h_k[n],h_{k+1}[n]) \ \&\ h[\ell h(t)]=t).$$

Clearly the empty sequence is good, and the hypothesis $\forall f \exists g \forall n \theta(f[n],g[n])$ implies that each good t has a good immediate extension. The property of goodness is Σ_1^1 so we have a failure of Π_1^1-TI_0 along the Kleene/Brouwer ordering of T.

$2 \to 3$: Similar to Lemma 2.7. Let $\varphi(U_0)$ be a true Σ_3^1 sentence with a set parameter U_0. Write $\varphi(U_0) \equiv \exists V \forall X \exists Y \theta(U_0,V,X,Y)$ where θ is arithmetical. Fix U_1 such that $\forall X \exists Y \theta(U_0,U_1,X,Y)$. Let $\varphi(X,Y)$ say that $(Y)_0 = U_0$ and $(Y)_1 = U_1$ and

$$\forall i \exists j \exists k[\theta(U_0,U_1,(X)_i,(Y)_j) \ \&\ W_i^X = (Y)_k]$$

where W_i^X is the i^{th} set recursively enumerable in X. By Σ_1^1-DC_0 there exists Z such that $\forall k \varphi((Z)_k,(Z)_{k+1})$. Clearly $M = \{((Z)_k)_i : k\in\omega \& i\in\omega\}$ is a countable ω-model of ACA_0 plus $\varphi(U_0)$. This proves Σ_3^1-RFN_0.

$3 \to 1$: Let $\prec$ be a linear ordering of the natural numbers and assume that we have a failure of Π^1_1-TI_0 on $\prec$, i.e. $\forall n(\forall m \prec n_\varphi(m) \to \varphi(n))$ and $\sim\varphi(p)$ where φ is Π^1_1. By Σ^1_3-RFN_0 there exists a countable ω-model M containing $\prec$ and satisfying.

$$\forall n(\forall m \prec n\varphi(m) \to \varphi(n)) \ \&\ \sim\varphi(p).$$

By ACA_0 let $Z = \{n: M \models_\omega \sim\varphi(n)\}$. Thus Z is nonempty and has no least element under $\prec$. Hence $\prec$ is not a well ordering. This completes the proof of Theorem 4.1.

4.2 Corollary. (i) Π^1_1-TI_0 plus ATR_0 proves the existence of an ω-model of Σ^1_1-TI. (ii) ATR_0 proves the existence of an ω-model of Π^1_1-TI.

Proof. The first part is immediate from Theorems 4.1 and 2.5 since ATR_0 consists of ACA_0 plus a Π^1_2 sentence. For the second part, reasoning in ATR_0, the proof of Theorem 3.7 of [6] gives $c \in \mathcal{O}_+ \setminus \mathcal{O}$ and a countable ω-model M_0 of ACA_0 satisfying "$c \in \mathcal{O}$ and H_c exists." Since $c \notin \mathcal{O}$ let $A \subseteq \{a: a <_0 c\}$ be such that $\forall b \leq_0 c(\forall a <_0 b(a \in A) \to b \in A)$. Put $M_1 = \{X: \exists a \in A(M_0 \models_\omega X$ is recursive in $H_a)\}$. It is not hard to see that M_1 is a countable ω-model of Σ^1_1-DC_0 and hence of Π^1_1-TI.

4.3 Corollary. Neither of Σ^1_1-TI and Π^1_1-TI implies the other, and there exist ω-models for the independence.

Proof. Both directions are immediate from Corollary 4.2 plus the ω-model form of Gödel's second incompleteness theorem (for which see Friedman [5] or Steel [12]).

Remark. The two previous corollaries are not really new since it is well known [7] that the hyperarithmetic sets satisfy Σ^1_1-DC_0. However, this fact is not provable in ATR_0, although it is provable in Σ^1_1-TI_0.

§5. Remark on a system considered by Jäger.

After the main part of this paper was written, Harvey Friedman pointed out that the idea of the proofs of Theorems 2.8 and 4.1 above can be used to settle the relationship between ATR and a related system ATR^J considered by Jäger [10]. With Friedman's permission we include this result here.

Let ATR^J_0 be just like ATR_0 except that arithmetical transfinite recursion is only assumed to hold for well orderings which are primitive recursive. ATR^J is ATR^J_0 plus full induction on the integers.

Say that $X \subseteq \omega$ is low if $\omega^X_1 = \omega^{CK}_1$, i.e. any well ordering of the natural numbers which is recursive in X is isomorphic to a primitive recursive well ordering of the natural numbers.

5.1 Lemma. Let $\varphi(X,Y)$ be arithmetical with no free set variables other than X and Y. Then ATR_0^J proves

$$X \text{ low } \& \ \exists Y\varphi(X,Y) \rightarrow \exists \text{ low } Y\varphi(X,Y).$$

Proof. We use the notation of §3 of [6]. In ACA_0 we can prove that for all X, $\mathcal{O}^X$ is complete Π_1^1 in X and hence not Σ_1^1 in X. Write

$$X\oplus Y = \{2n:n\in X\}\cup\{2n+1:n\in Y\}.$$

Assume now that X is low and $\exists Y\varphi(X,Y)$. By ATR_0^J we have that for each $e \in \mathcal{O}^X$ there exists Y such that $\varphi(X,Y)$ and $H_e^{X\oplus Y}$ exists. Since $\mathcal{O}^X$ is not Σ_1^1 in X it follows that there exist Y,e,Z such that $\varphi(X,Y)$, $e \in \mathcal{O}_+^X \setminus \mathcal{O}^X$, and $H(X\oplus Y,e,Z)$, i.e. Z is a pseudo-$H_e^{X\oplus Y}$. We claim now that $X \oplus Y$ is low. This follows from Theorem 4 of [5] relativized to $X \oplus Y$.

As in §4 of [6] write $X \ll Y$ to mean that there exists Z recursive in Y such that for all i, X and the Turing jump of $(Z)_{i+1}$ are recursive in $(Z)_i$.

5.2 Lemma. ATR_0^J proves $\forall$ low $X\exists Y(X \ll Y)$.

Proof. This a straightforward combination of the proofs of Lemma 5.1 above and Lemma 4.6 of [6].

5.3 Theorem (Friedman). ATR and ATR^J prove the same Π_1^1 sentences. For every model of ATR^J there is a model of ATR with the same integers.

Proof. Let M^J be a model of ATR^J plus σ where σ is a Σ_1^1 sentence. Write $\sigma \equiv \exists X\varphi(X)$ where φ is arithmetical. Within M^J apply Lemma 5.1 to get a low set X_0 such that $\varphi(X_0)$ holds. Consider the Σ_1^1 assertion

$$\exists Z\forall k[(Z)_0 = X_0 \ \& \ (Z)_k \ll (Z)_{k+1}].$$

We would like to find $Z \in M^J$ such that this holds in M^J. Unfortunately we cannot do this, but we shall find such a Z which is first order definable over M^J. By ACA_0 we can write our Σ_1^1 assertion in the form

$$\exists f\forall k\forall m\theta(f_k[m],f_{k+1}[m])$$

where θ is arithmetical. Disregarding Skolem functions, $\forall m\theta(f_k[m],f_{k+1}[m])$ says that $f_0 = X_0$ and $f_k \ll f_{k+1}$. Within M^J define a finite sequence t to be good if

$$(\exists \text{low } f)[\forall k<\ell h(t)\forall m\theta(f_k[m],f_{k+1}[m]) \ \& \ f[\ell h(t)]=t].$$

Clearly the empty sequence is good, and by Lemmas 5.1 and 5.2 each good sequence has a good immediate extension.

By induction on n we can prove that there exists a lexicographically leftmost good sequence of length n. Let f be the leftmost "path" through the "tree" of good sequences. (We use quotation marks to indicate that the objects in question are not elements of M^J but merely first order definable over M^J.) By Lemma 4.6 of [6] the "sets" which are recursive in f_k for some k form a model M of ATR_0 with the same integers as M. This model M is first order definable over M^J and therefore satisfies full induction since M^J does. Thus M is a model of ATR. Also M contains X_0 and hence satisfies σ. This proves the theorem.

5.4 Corollary. The proof theoretic ordinal of ATR is Γ_{ε_0}.

Proof. From Theorem 5.3 it follows that ATR has the same proof theoretic ordinal as ATR^J. Jäger [10] has shown that the proof theoretic ordinal of ATR^J is Γ_{ε_0}.

By a similar but easier argument one has:

5.5 Theorem (Friedman). ATR_0 and ATR_0^J prove the same Π_1^1 sentences. Every model of ATR_0^J has a submodel with the same integers which is a model of ATR_0.

Proof. Let M_0^J be a model of ATR_0^J plus σ where σ is a Σ_1^1 sentence. Write $\sigma \equiv \exists X\varphi(X)$ where φ is arithmetical. By Lemmas 5.1 and 5.2 we can find a sequence of sets $Z_k, k \in \omega$, such that M_0^J satisfies $\varphi(Z_0)$ and $Z_k << Z_{k+1}$. Here k ranges over standard integers. By Lemma 4.6 of [6] the sets which are recursive in Z_k for some k form a model M_0 of ATR_0. This model M_0 is a submodel of M_0^J.

The next corollary was proved earlier by Friedman [4], [6].

5.6 Corollary. The proof theoretic ordinal of ATR_0 is Γ_0.

Proof. From Theorem 5.5 it follows that ATR_0 and ATR_0^J have the same proof theoretic ordinal. Jäger [10] has shown that the proof theoretic ordinal of ATR_0^J is Γ_0.

We do not know the proof theoretic ordinal of $\Sigma_1^1\text{-TI}_0$ or of $\Sigma_1^1\text{-TI}_0 + \Pi_1^1\text{-TI}_0$ or of $\Sigma_1^1\text{-TI} + \Pi_1^1\text{-TI}$.

It is fairly clear that the proofs of Theorems 5.3 and 5.5 can be made to yield general results in the style of Theorems 2.8 and 4.1. We leave these general formulations to the reader.

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