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SET THEORETIC ASPECTS OF ATR_0

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§1. Introduction.

In this paper we study certain fairly weak formal systems which are nevertheless just strong enough to formalize certain aspects of mathematical practice. All of the systems we consider use classical logic.

By ATR_0 we mean the formal system of arithmetical transfinite recursion with quantifier free induction on the natural numbers. This is an interesting finitely axiomatizable subsystem of second order arithmetic. It was first isolated by H. Friedman [10], [11]. Detailed studies of it have appeared in Steel [28], Friedman/McAloon/Simpson [13], and Simpson [27]. A precise description of the language and axioms of ATR_0 is given in §2 below.

The interest of ATR_0 has by now been well established. On the one hand, it was shown [10], [11], [13], [26], [27], [28] that ATR_0 is just strong enough to prove many mathematical theorems which depend on having a good theory of countable well orderings. Indeed many such theorems, when stated in the language of second order arithmetic, turn out to be provably equivalent to ATR_0 over a weak base theory. (As an example here we may cite the theorem that every uncountable Borel set contains a perfect subset. Statements involving Borel sets, perfect sets, and countable well orderings are formalized in the language of second order arithmetic by means of codes.) On the other hand, it was shown in [13] that ATR_0 is proof theoretically not very strong; e.g. its proof theoretic ordinal is just the Feferman/Schütte ordinal Γ_0.

Although ATR_0 is a subsystem of second order arithmetic, the purpose of this paper is to examine ATR_0 from a set theoretic viewpoint. To this end we isolate in §2 a certain finitely axiomatizable system of set theory, ATR_0^s, whose key axiom asserts that every well ordering is isomorphic to a von Neumann ordinal. The system ATR_0^s appears to be a very natural and interesting fragment of ZF = Zermelo/Fraenkel set theory. We show in §3 that ATR_0^s is a conservative extension of ATR_0. This is done by showing that ATR_0 is strong enough to carry

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out the usual arguments in which hereditarily countable sets are encoded by countable well founded trees.

In §4 we study models of ATR_0^s. We show that any countable model of ATR_0^s has a proper $\in$-transitive submodel which is again a model of ATR_0^s. This amounts to showing that every countable model of ATR_0 has a proper Σ_1^1 elementary submodel. Throughout this paper we employ mainly model theoretic methods, although proof theoretic results are mentioned briefly from time to time.

In §5 we compare ATR_0^s with the better known system KP = Kripke/Platek set theory, i.e. the theory of admissible sets [1], [2]. KP is also a natural and interesting subsystem of ZF. Intuitively speaking, ATR_0^s is different from KP because Barwise compactness is internal to the system, rather than being a property of models of the system as it is for KP.

Independently of our work, McAloon and Ressayre [23] defined a system of set theory which is similar to ATR_0^s, and stated a result similar to our Theorem 3.6. Our Theorem 4.10 was proved in answer to a question raised by McAloon and Ressayre [23]. Friedman [12], after learning about our results in §3, devised an elegant theory of sets and classes which is also a conservative extension of ATR_0.

§2. The systems ATR_0 and ATR_0^s.

The language of second order arithmetic consists of $+$, $\cdot$, 0, 1, $=$, $<$, $\in$, number variables i, j, k, m, n, ..., set variables X, Y, Z, ... , propositional connectives, number quantifiers $\forall n$, $\exists n$, and set quantifiers $\forall X$, $\exists X$. Number variables are intended to range over the set ω of natural numbers, and set variables are intended to range over subsets of ω. A formula in the language of second order arithmetic is said to be arithmetical if it contains no set quantifiers. The weakest formal system we shall consider is ACA_0 which consists of the usual ordered semiring axioms for ω, the quantifier free induction axiom

$$0\in X \& \forall k(k\in X \rightarrow k+1\in X) \rightarrow \forall k(k\in X)$$

and arithmetical comprehension axioms

$$\exists X\forall m(m\in X \leftrightarrow \theta(m))$$

where $\theta(m)$ is arithmetical and does not mention X. It can be shown that ACA_0 is finitely axiomatizable: the principal axiom asserts that for any set X, the Turing jump of X exists.

Within ACA_0 we have the arithmetical pairing function

$$(m,n) = \frac{1}{2}(m+n+1)(m+n) + m.$$

Binary relations R on the natural numbers are identified with sets $X = \{(m,n) : m\ R\ n\}$. A well ordering is a binary relation $\prec$ on the natural numbers which is a linear ordering of its field, such that

$$\forall X[\forall n(\forall m \prec n(m\in X) \to n\in X) \to \forall n(n\in X)].$$

We write $WO(\prec)$ to mean that $\prec$ is a well ordering in the sense just described. Thus $WO(\prec)$ is a Π^1_1 formula with a free set variable $\prec$.

The system ATR_0 consists of ACA_0 plus a scheme of arithmetical transfinite recursion which asserts that arithmetical comprehension can be iterated along any well ordering. ATR_0 includes all axioms of the form

$$WO(\prec) \to \exists X\forall j\forall n[(j,n)\in X \leftrightarrow \theta(j,\{(i,m) : m\prec n \& (i,m)\in X\})]$$

where $\theta(j,Y)$ is arithmetical. It can be shown that ATR_0 is finitely axiomatizable: the axioms are those of ACA_0 plus a Π^1_2 sentence asserting that the Turing jump operator can be iterated along any well ordering starting at any set.

In §5 we shall briefly consider the scheme of transfinite induction which consists all instances of

$$WO(\prec) \& \forall n(\forall m\prec n\varphi(m) \to \varphi(n)) \to \forall n\varphi(n)$$

where φ is an arbitrary formula. A formula is said to be Σ^1_k (respectively Π^1_k) if it consists of a string of k set quantifiers beginning with an existential (respectively universal) one, followed by an arithmetical formula. By Σ^1_k-TI_0 (respectively Π^1_k-TI_0) we mean the formal system consisting of ACA_0 plus the transfinite induction scheme restricted to formulas φ which are Σ^1_k (respectively Π^1_k). More information on Σ^1_1-TI_0 and Π^1_1-TI_0 can be found in Simpson [27].

The set theoretic language is just the first order language with $=$, $\in$, and set theoretic variables $u,v,w,x,y,\ldots$. We employ without comment a number of abbreviations which are familiar from textbooks on ZF set theory. Consider the following set theoretic axioms:

1. Axiom of extensionality: $\forall u(u\in x \leftrightarrow u\in y) \to x = y$.

2. Axiom of regularity: $x \neq \phi \to \exists u\in x(u\cap x = \phi)$.

3. Axiom of infinity: $\exists x(\phi\in x \& \forall u\in x(u\cup\{u\} \in x))$.

4. Axioms asserting that the universe is closed under primitive recursive set functions (Jensen/Karp [22]).

5. A set r of ordered pairs is said to be regular if $\forall x(x\neq\phi \to \exists u\in x\forall v\in x\ \langle v,u\rangle \notin r)$. Our key axiom asserts that if r is regular then

there exists a function f with field(r) $\subseteq$ domain(f) and, for all $u \in$ domain(f), $f(u) = \{f(v) : \langle v,u \rangle \in r\}$.

6. Axiom of choice: $\forall x \exists r$ (r is a well ordering of x).

7. Axiom of countability: $\forall x$ (x is countable).

Our basic system ATR_0^s consists of axioms 1 through 5. Axioms 6 and 7 are regarded as optional extra axioms. Actually, our main interest is in the full system consisting of axioms 1 through 7, i.e. ATR_0^s plus the axiom of countability.

Note that ATR_0^s is a subsystem of ZF. We shall begin §3 by showing that ATR_0^s is finitely axiomatizable. Hence ATR_0^s plus the axiom of countability is finitely axiomatizable.

§3. Conservative extension result.

In this section we present some basic results about ATR_0^s and about the relationship between ATR_0^s and ATR_0. Keep in mind that ATR_0^s is a system of set theory while ATR_0 is a system of second order arithmetic.

3.1 Theorem. *ATR_0^s is finitely axiomatizable.*

Proof. We claim that the scheme of closure under primitive recursive set functions can be replaced by an axiom asserting closure under rudimentary functions F_0-F_8 (Jensen [21] p. 239), plus an axiom asserting that for any transitive set t and ordinal α, the constructible hierarchy $L_\alpha(t)$ starting at t exists, plus an axiom $\forall x \exists t(x \in t$ & t is transitive). To prove the claim, note first that axiom 5 (applied to linear orderings r) yields closure under primitive recursive ordinal functions. Now apply Theorem 2.5 of Jensen/Karp [22].

3 2 Remark. We remark on some alternative simplified axiomatizations of ATR_0^s. One may add the axiom of choice to ATR_0^s. If this is done, then axiom 5 can be replaced by its special case in which r is a linear ordering. (The special case just says that every well ordering is isomorphic to an ordinal.) Clearly the axiom of countability implies the axiom of choice. In the presence of the axiom of countability, a different simplification is possible: the axiom about $L_\alpha(t)$ can be replaced by its special case in which $\alpha = \omega$ and $t = \phi$. We omit proof of these remarks. We conjecture that either of the mentioned simplifications can be made even without the axiom of choice.

In any case we have:

3.3 Lemma. *In any model of ATR_0^s, the hereditarily countable sets form a model of ATR_0^s plus the axiom of countability.*

Proof. Obvious. (A set is said to be hereditarily countable if the smallest transitive set containing it is countable.)

We now exhibit a close relationship of mutual interpretability between ATR_0^s and ATR_0. Assume that the language of second order arithmetic has been interpreted into the set theoretic language in the usual way: number variables range over ω, set variables range over subsets of ω, etc. We then have:

3.4 Lemma. Each axiom of ATR_0 is a theorem of ATR_0^s.

Proof. Let $\prec$ be a well ordering of ω, let $\theta(j,X)$ be an arithmetical formula, and let $\varphi(n,Y)$ be an arithmetical formula which asserts that Y is the result of iterating θ along $\prec$ up to n. Thus $\varphi(n,Y)$ says

$$Y = \{(j,m) : m \prec n \ \& \ \theta(j,\{(i,k) : k \prec m \ \& \ (i,k) \in Y\})\}.$$

Reasoning in ATR_0^s, let t be the transitive closure of $\{\prec\}$, let α be the ordinal of $\prec$, and for each n let $|n| < \alpha$ be the ordinal of the restriction of $\prec$ to $\{m : m \prec n\}$. Using the axiom of regularity, prove by induction on $|n|$ that $L_{|n|+1}(t)$ contains a set Y such that $\varphi(n,Y)$. We omit details.

3.5 Lemma. Any model of ATR_0 can be expanded to a model of ATR_0^s plus the axiom of countability.

Proof. Within ATR_0 we make the following definitions. A tree is a non-empty set T of (codes for) finite sequences of natural numbers such that $s \subseteq t \ \& \ t \in T \rightarrow s \in T$. A tree T is said to be well founded if T has no path, i.e. there is no function f such that $\forall n\, f[n] \in T$ where $f[n] = \langle f(0),\dots,f(n-1)\rangle$. Trees T and T' are said to be isomorphic, written $T \cong T'$, if there exists an isomorphism between them, i.e. an order preserving bijection of T onto T'. If s and t are finite sequences of natural numbers, $s^\frown t$ is the concatenation of s followed by t. If T is a tree and $s \in T$, we write $T_s = \{t : s^\frown t \in T\}$. A tree T is said to be suitable if it is well founded and, for all $s \in T$, if $s^\frown\langle m\rangle \in T$ and $s^\frown\langle n\rangle \in T$ and $T_{s^\frown\langle m\rangle} \cong T_{s^\frown\langle n\rangle}$ then $m = n$.

Clearly the class of suitable trees is Π^1_1. The point of the definition is that if T and T' are suitable then there is at most one order preserving bijection of T onto T'. Hence the relation $T \cong T'$ of isomorphism between suitable trees is Δ^1_1 on Π^1_1. If T and T' are suitable trees we write $T \,\tilde{\in}\, T'$ to mean $\exists n(\langle n\rangle \in T' \ \& \ T \cong T'_{\langle n\rangle})$. The relation $\tilde{\in}$ is again Δ^1_1 on Π^1_1. We are using the Σ^1_1 axiom of choice, a consequence of ATR_0.

We interpret the set theoretic language into the language of second order arithmetic as follows. Set theoretic variables are interpreted as ranging over suitable trees. The equality relation = between set theoretic variables is interpreted as $\cong$, and $\in$ is interpreted as $\tilde{\in}$. The idea here is that a well founded tree T is to be identified with a hereditarily countable set

$$|T| = \{|T_{\langle n\rangle}| : \langle n\rangle \in T\}.$$

The restriction to suitable trees is for convenience only. Note that for suitable trees we have $|T| = |T'|$ if and only if $T \cong T'$, and $|T| \in |T'|$ if and only if $T \tilde{\in} T'$.

We must verify that the suitable tree interpretations of axioms 1 through 7 are theorems of ATR_0. Recall that a set theoretic formula is Δ_0 if it is built up using only bounded quantifiers $\forall u\in x$, $\exists u\in x$. Note that for each Δ_0 formula $\varphi(x_1,\ldots,x_n)$ the corresponding suitable tree formula $\tilde{\varphi}(T_1,\ldots,T_n)$ is Δ^1_1 on the Π^1_1 class of suitable trees. Hence we may apply Δ^1_1 comprehension (a consequence of ATR_0) to get closure under rudimentary functions. Furthermore, given suitable trees corresponding to a transitive set t and an ordinal α, we can use arithmetical transfinite recursion along a well ordering of type $\omega\cdot\alpha$ to define a suitable tree corresponding to $L_\alpha(t)$. The rest of the verification is routine.

The above discussion implies that any model M of ATR_0 can be expanded to a model M^s of ATR_0^s plus the axiom of countability. The elements of M^s are the equivalence classes of suitable trees in M under $\cong$ in M. This completes the proof of Lemma 3.5.

Combining Lemmas 3.3, 3.4 and 3.5 we obtain immediately the following conservative extension result (one may compare Theorem 4.6 of Feferman [6]):

3.6 <u>Theorem</u>. <u>Let</u> σ <u>be a sentence in the language of second order arithmetic</u>. <u>The following are equivalent</u>.

(i) ATR_0^s <u>plus the axiom of countability proves</u> σ;

(ii) ATR_0^s <u>proves</u> σ;

(iii) ATR_0 <u>proves</u> σ.

§4. <u>Models of</u> ATR_0^s.

Let $M = (|M|, \in^M)$ and $N = (|N|, \in^N)$ be models of ATR_0^s. We say that N is an <u>$\in$-transitive submodel of</u> M if $|N| \subseteq |M|$ and, for all $a \in |M|$ and $b \in |N|$, $a \in^M b$ if and only if $a \in |N|$ and $a \in^N b$. The purpose of this section is to prove that every model of ATR_0^s has a proper $\in$-transitive submodel which is again a model of ATR_0^s. This answers a question which was raised by McAloon and Ressayre [23].

The main part of our argument consists in showing that the proofs of certain well known theorems from hyperarithmetic theory can be pushed through in ATR_0. Our notation for hyperarithmetic theory is as in §3 of [13]. We say that X is <u>hyperarithmetic in</u> Y if there exists $e \in \mathcal{O}^Y$ such that X is recursive in H_e^Y. The principal axiom of ATR_0 is equivalent to the assertion that $\forall Y\ \forall e(e \in \mathcal{O}^Y \to H_e^Y$ exists).

We say that X <u>is Σ_1^1 in</u> Y if there exists a Σ_1^1 formula $\varphi(m,Y)$, with no free set variables other than Y, such that $\forall m(m \in X \leftrightarrow \varphi(m,Y))$. We say that X <u>is Δ_1^1 in</u> Y if both X and $\omega \setminus X$ are Σ_1^1 in Y. A well known theorem of Kleene [17] asserts that X is hyperarithmetic in Y if and only if X is Δ_1^1 in Y. The following lemma entails that Kleene's theorem is provable in ATR_0.

4.1 <u>Lemma</u>. <u>The following is provable in</u> ACA_0. <u>Let</u> Y <u>be a set such that</u> $\forall e(e \in \mathcal{O}^Y \to H_e^Y$ <u>exists</u>). <u>Then for all</u> X, X <u>is hyperarithmetic in</u> Y <u>if and only if</u> X <u>is Δ_1^1 in</u> Y.

<u>Proof</u>. Recall (from §3 of [13]) that there is an arithmetical formula $H(Y,e,Z)$ such that if $e \in \mathcal{O}^Y$ then H_e^Y is defined as the unique Z such that $H(Y,e,Z)$. Suppose first that X is hyperarithmetic in Y. Then $X = (H_e^Y)_i$ for some $e \in \mathcal{O}^Y$ and some i. Thus we have

$$m \in X \leftrightarrow \exists Z(H(Y,e,Z)\ \&\ m \in (Z)_i)$$

$$\leftrightarrow \forall Z(H(Y,e,Z) \to m \in (Z)_i)$$

so X is Δ_1^1 in Y.

Conversely, suppose that X is Δ_1^1 in Y. In ACA_0 alone we can prove that $\mathcal{O}^Y$ is complete Π_1^1 in Y (although we cannot prove that $\mathcal{O}^Y$ exists as a set). Hence we can find a recursive function f such that $\forall m(m \in X \leftrightarrow f(m) \in \mathcal{O}^Y)$. For $i,j \in \mathcal{O}^Y$ write $|i|^Y \le |j|^Y$ to mean that there exists an order isomorphism of $\{e : e <_{\mathcal{O}}^Y i\}$ onto a proper initial segment of $\{e : e \le_{\mathcal{O}}^Y j\}$. Such an isomorphism is called a <u>comparison map</u>. Under the given hypothesis on Y, we can prove in ACA_0 that either $|i|^Y \le |j|^Y$ or $|j|^Y \le |i|^Y$ since the appropriate comparison map is recursive in H_i^Y. We claim that there exists $e \in \mathcal{O}^Y$ such that $|f(m)|^Y \le |e|^Y$ for all $m \in X$. If not, then for all i we have that $i \in \mathcal{O}^Y$ if and only if $i \in \mathcal{O}_+^Y$ and $\exists m(m \in X\ \&\ |i|^Y \le |f(m)|^Y)$. Hence $\mathcal{O}^Y$ is Σ_1^1 in Y, contradicting the fact that $\mathcal{O}^Y$ is complete Π_1^1 in Y. This proves the claim.

We now see that for all m, $m \in X$ if and only if $|f(m)|^Y \leq |e|^Y$ via a comparison map which is recursive in H_e^Y. Thus X is arithmetical in H_e^Y. It follows that X is hyperarithmetic in Y. This completes the proof of Lemma 4.1.

We now proceed to show that the proof of a result of Gandy, Kreisel, and Tait [14] can be pushed through in ATR_0. For sets of integers X and Y write $X \in Y$ to mean that $\exists i(X = (Y)_i)$ where $(Y)_i = \{m : (m,i) \in Y\}$.

4.2 Lemma. The following is provable in ATR_0. Let A and Y be sets such that A is not hyperarithmetic in Y. Let $\varphi(X,Y)$ be a Σ_1^1 formula with no free set variables other than X and Y. If $\exists X\varphi(X,Y)$ then $\exists X(\varphi(X,Y) \,\&\, A \notin X)$.

Proof. Let Σ_1^1-AC_0 be the Σ_1^1 axiom of choice, i.e.

$$\forall i\exists X\theta(i,X) \rightarrow \exists Y\forall i\theta(i,(Y)_i)$$

for arithmetical θ. We shall make use of the result of Friedman [7], [11] that ATR_0 proves Σ_1^1-AC_0.

If $X,Y \in Z$ and if $\varphi(X,Y)$ is a formula in the language of second order arithmetic, write $Z \models_\omega \varphi(X,Y)$ to mean that Z encodes a countable ω-model of $\varphi(X,Y)$, i.e. $\varphi(X,Y)$ is true when the bound set variables in it are interpreted as ranging over $\{(Z)_i : i \in \omega\}$. A formula is said to be essentially Σ_1^1 if it is in the smallest class of formulas containing the arithmetical formulas and closed under existential set quantification and universal number quantification. In view of Σ_1^1-AC_0 it is easy to see that ATR_0 proves the following instance of an ω-model reflection principle: for essentially Σ_1^1 formulas $\varphi(X,Y)$, if $\varphi(X,Y)$ is true then $\exists Z(Z \models_\omega ACA_0 + \varphi(X,Y))$.

With these observations in mind, we now proceed to the proof of Lemma 4.2. Let $f,g,h,\ldots$ be function variables intended to range over unary functions from ω into ω. We assume that such variables have been introduced into the language of second order arithmetic in the usual way. We write $(f)_i(m) = f((m,i))$ and $f[n] = \langle f(0),\ldots,f(n-1)\rangle$. In view of ACA_0 and the Kleene normal form theorem for Σ_1^1 formulas, it will suffice to prove the following assertion in ATR_0:

Let g be a function which is not hyperarithmetic in Y. Let $\theta(t,Y)$ be an arithmetical formula with no free set (or function) variables other than Y. If $\exists f\forall n\theta(f[n],Y)$ then $\exists f(\forall n\theta(f[n],Y) \,\&\, \forall i\; g \neq (f)_i)$.

Assume the hypotheses. The following true statements are essentially Σ_1^1: $\forall e(e\in\mathcal{O}^Y \rightarrow H_e^Y$ exists); g is not hyperarithmetic in Y; $\exists f\forall n\theta(f[n],Y)$. We can therefore find Z such that $Z \models_\omega ACA_0 +$ these statements. Say that a finite sequence t is good if

$$Z \models_\omega \exists f(\forall n\theta(f[n],Y) \ \&\ f[\ell h(t)] = t).$$

Clearly the empty sequence is good. We claim that if s is good then for all i we can find a good $t \supseteq s$ such that $g[\ell h((t)_i)] \neq (t)_i$. If not, then for all m and n we would have that $g(m) = n$ if and only if $Z \models_\omega \exists$ good t ($t \supseteq s$ & $(t)_i(m) = n$). Hence $Z \models_\omega g$ is Δ^1_1 in Y. Hence by Lemma 4.1 it would follow that $Z \models_\omega g$ is hyperarithmetic in Y. This contradiction proves the claim.

Now standing outside Z and applying the claim repeatedly, we can find good sequences $t_0 \subseteq t_1 \subseteq \ldots \subseteq t_i \subseteq \ldots$ so that for all i, $g[\ell h((t_i)_i)] \neq (t_i)_i$. (The sequence of good sequences $t_0, t_1, \ldots$ is recursive in the satisfaction set for the ω-model Z.) Putting $f = \bigcup_{i\in\omega} t_i$ we get $\forall n\theta(f[n],Y)$ and $\forall i\ g \neq (f)_i$. This completes the proof of Lemma 4.2.

Write

$$X \oplus Y = \{2n : n\in X\} \cup \{2n+1 : n\in Y\}.$$

4.3 Lemma. The following is provable in ATR_0. Let A and Y be sets such that A is not hyperarithmetic in Y. Let $\varphi(X,Y)$ be a Σ^1_1 formula with no free set variables other than X and Y. If $\exists X\varphi(X,Y)$ then $\exists X[\varphi(X,Y)$ & A not hyperarithmetic in $X \oplus Y]$.

Proof. Let $\psi(X,Y,Z)$ be a Σ^1_1 formula saying that $X,Y \in Z$ and

$$Z \models_\omega ACA_0 + \varphi(X,Y) + \forall e(e\in\mathcal{O}^{X\oplus Y} \rightarrow H_e^{X\oplus Y} \text{ exists}).$$

Since the statement in question is essentially Σ^1_1, we can find X and Z such that $A \notin Z$ and, for the given Y, $\psi(X,Y,Z)$ holds. Then clearly $\forall W(W$ hyperarithmetic in $X \oplus Y \rightarrow W \in Z)$. Thus we have $\varphi(X,Y)$ and A is not hyperarithmetic in $X \oplus Y$. This proves Lemma 4.3.

4.4 Lemma. The following is provable in ATR_0. Let $\varphi(X,Y)$ be a Σ^1_1 formula with no free set variables other than X and Y. If $\exists X\varphi(X,Y)$ and if A is such that $\forall i[(A)_i$ is not hyperarithmetic in $Y]$, then $\exists X[\varphi(X,Y)$ & $\forall i[(A)_i$ not hyperarithmetic in $X \oplus Y]]$.

Proof. Straightforward generalization of the proofs of Lemmas 4.2 and 4.3.

Note that one cannot strengthen the conclusion to say that for all A there exists X such that $\varphi(X,Y)$ and $\forall i[(A)_i$ hyperarithmetic in $X \oplus Y \rightarrow (A)_i$ hyperarithmetic in $Y]$.

Let M be a model of ATR_0 and let N be a submodel of M. We say that N is a Σ^1_1 elementary submodel of M if N has the same integers as M and, for all Σ^1_1 formulas φ with parameters from N, N satisfies φ if and only if M satisfies φ. Note that any such N is again a model of ATR_0, since ATR_0 is axiomatized by Π^1_2 sentences.

4.5 Theorem. Let M be a countable model of ATR_0, and let $A,Y \in M$ be such that $M \models \forall i[(A)_i$ is not hyperarithmetic in $Y]$. Then M has a Σ^1_1 elementary submodel N such that $Y \in N$ and $\forall i (A)_i \notin N$.

Proof. Let $\varphi(e,X,Y)$ be a universal Σ^1_1 formula and let $\{e_n : n \in \omega\}$ be an enumeration of the integers of M. Fix $A,Y \in M$ as in the hypothesis of the theorem. Use Lemma 4.4 repeatedly to define a sequence of sets $X_0,X_1,\ldots,X_n,\ldots \in M$ such that $X_0 = Y$ and for all $n, M \models \forall i[(A)_i$ is not hyperarithmetic in $X_0 \oplus \ldots \oplus X_n]$, and if $M \models \exists X\, \varphi(e_n,X,X_0 \oplus \ldots \oplus X_n)$ then X_{n+1} is such an X. Let N be the submodel of M consisting of $\{X_n : n \in \omega\}$. It is clear that M satisfies the desired conclusions.

4.6 Corollary. Let M be a countable model of ATR_0. Then

$$\bigcap \{N : N \text{ is a } \Sigma^1_1 \text{ elementary submodel of } M\}$$

$$= \{A : M \models A \text{ is hyperarithmetic}\}.$$

Proof. Immediate from the previous theorem.

4.7 Remark. An ω-model M is said to be a β-model if M is a Σ^1_1 elementary submodel of the ω-model consisting of all subsets of ω. A special case of Corollary 4.6 is that if M is a β-model then

$$\bigcap \{N : N \text{ is a } \Sigma^1_1 \text{ elementary submodel of } M\}$$

$$= \{A : A \text{ is hyperarithmetic}\}.$$

This special case had been proved earlier by Simpson [25].

4.8 Corollary. Let M be a countable model of ATR_0. Then M has a Σ^1_1 elementary submodel N such that $N \neq M$.

Proof. Immediate from the previous corollary in view of the well known fact that ATR_0 proves $\exists A$ (A is not hyperarithmetic).

4.9 Remark. A consequence of Corollary 4.8 is that every ω-model of ATR_0 has a proper ω-submodel which is again a model of ATR_0. This result had been proved earlier by Quinsey (Chapter 6 of [24]) using a completely different method. Actually Quinsey proved the following generalization. Let $T \supseteq ATR_0$ be a recursively axiomatizable theory in the language of second order arithmetic. Then any ω-model of T has a proper ω-submodel which is again a model of T. The ω-submodels produced by Quinsey [24] are not in general Σ^1_1 elementary.

We now use the results of §3 to reformulate Theorem 4.5 in set theoretic terms.

4.10 Theorem. Let $M = (|M|, \in^M)$ be a countable model of ATR^s_0. Let A and Y be sets of integers in M such that $M \models \forall i \in \omega[(A)_i$ is not hyperarithmetic in Y]. Then M has an $\in$-transitive submodel $N=(|N|, \in^N)$ such that $Y \in N$, $\forall i (A)_i \notin N$, and N is again a model of ATR^s_0.

Proof. In view of Theorem 3.3 we may assume that $M \models \forall x$ (x is countable). Let $\mathcal{M}$ be the model of ATR_0 consisting of the integers and sets of integers in M. The proof of Lemma 3.5 shows that, conversely, M is canonically isomorphic to $\mathcal{M}^s$ = {suitable trees in $\mathcal{M}$}. Now by Theorem 4.5 let $\mathcal{N}$ be a Σ^1_1 elementary submodel of $\mathcal{M}$ such that $Y \in \mathcal{N}$ and $\forall i (A)_i \notin \mathcal{N}$. Let N be the submodel of M consisting of those elements which are represented by suitable trees in $\mathcal{N}$. Note that since $\mathcal{N}$ is a Σ^1_1 elementary submodel of $\mathcal{M}$, for $T \in \mathcal{N}$ we have $\mathcal{N} \models T$ is suitable if and only if $\mathcal{M} \models T$ is suitable. Using this remark it is easy to check that N is an $\in$-transitive submodel of M and is canonically isomorphic to $\mathcal{N}^s$. By the proof of Lemma 3.5 it follows that $N \models ATR^s_0$.

The next corollary answers a question of McAloon and Ressayre [23].

4.11 Corollary. Let M be a countable model of ATR^s_0. Then M has a proper $\in$-transitive submodel N which is again a model of ATR^s_0.

Proof. Immediate from Theorem 4.10.

The next corollary is anticipated by Friedman [8].

4.12. Corollary. ATR^s_0 has a well founded model of height ω_1^{CK}.

Proof. Let M be the well founded model consisting of all hereditarily countable sets. Apply Theorem 4.10 with $Y = \phi$, A = Kleene's $\mathcal{O}$. We get a well founded model N of ATR^s_0 which does not contain $\mathcal{O}$. Hence N has height ω_1^{CK}.

4.13 Remark. The well known fact that

$$L_{\omega_1^{CK}} = \{|T| : T \text{ is a hyperarithmetic suitable tree}\}$$

is provable in ATR_0^s. Therefore, in view of Corollary 4.6, it is natural to conjecture that for every countable $M \models ATR_0^s$,

$$\bigcap \{N : N \text{ is an } \in\text{-transitive submodel of } M \text{ and } N \models ATR_0^s\} = \{x : M \models x \in L_{\omega_1^{CK}}\}.$$

We have been unable to prove this conjecture, even in the special case when M is well-founded of height ω_1^{CK}

§5. Comparison with KP.

In this section we compare ATR_0^s to another, much better known, fragment of ZF, namely KP = Kripke/Platek set theory, which we take to include the axiom of infinity. For background material on KP see Barwise [1], [2]. For our purposes we take KP to consist of axioms 1 through 4 (see §2 above) plus the Δ_0 collection scheme

$$\forall u \exists v \varphi(u,v) \rightarrow \forall x \exists y \forall u \in x \exists v \in y \varphi(u,v)$$

where φ is Δ_0, plus the foundation scheme

$$\forall u(\forall v \in u \psi(v) \rightarrow \psi(u)) \rightarrow \forall u \psi(u)$$

where ψ is arbitrary. A Δ_0 formula is by definition a set theoretic formula in which all quantifiers are bounded, i.e. of the form $\forall u \in x$ or $\exists u \in x$.

Let KP_0^- be KP minus the foundation scheme. Of course KP_0^- implies foundation for Δ_0 formulas, and KP_0^- has the same well founded models as KP, viz. the admissible sets [1], [2]. However, models which are not well founded will play a role in our work, so we shall insist on the distinction between KP_0^- and KP. We shall also consider intermediate systems such as KP_0^- + Σ_k foundation (respectively KP_0^- + Π_k foundation) in which the foundation scheme is restricted to formulas ψ which are Σ_k (respectively Π_k). A set theoretic formula is said to be Σ_k (respectively Π_k) if it consists of a string of k quantifiers beginning with an existential (respectively universal) one, followed by a Δ_0 formula.

We begin by pointing out that there exist well founded models of ATR_0^s which are not models of KP_0^-, and vice versa. For $X \subseteq \omega$ let ω_1^X be the least ordinal not recursive in X.

5.1 Lemma. ATR_0^s plus KP_0^- together prove that ω_1^X exists for all $X \subseteq \omega$. In particular, if M is a well founded model of $ATR_0^s + KP_0^-$, then $X \in M$

implies $\omega_1^X \in M$.

Proof. We reason in $ATR_0^s + KP_0^-$. Given $X \subseteq \omega$ let $\{\prec_m^X : m \in \omega\}$ be an enumeration of all binary relations $\prec$ on ω such that $\prec$ is recursive in X and $\prec$ is a linear ordering of its field. For all $m \in \omega$, either there exists $Y \subseteq \omega$ such that Y witnesses $\sim WO(\prec_m^X)$, or there exists f such that f witnesses $WO(\prec_m^X)$ by mapping the field of $\prec_m^X$ isomorphically onto an ordinal. (This follows from the principal axiom of ATR_0^s.) Hence by Δ_0 collection there exists a set y such that for each $m \in \omega$ some appropriate witness lies in y. Hence ω_1^X exists since it is just $\{\text{range}(f) : f \in y \;\&\; \exists m \in \omega(f \text{ witnesses } WO(\prec_m^X))\}$. This completes the proof.

Let $\omega_1^{CK} = \omega_1^{\phi}$ = the least non recursive ordinal. It is well known that KP_0^- has well founded models of height ω_1^{CK}. (Indeed, $L_{\omega_1^{CK}}$ is the smallest well founded model of KP_0^-.) We have also seen (in Corollary 4.12 above) that ATR_0^s has well founded models of height ω_1^{CK}. The next theorem was anticipated by Simpson [25].

5.2 Theorem. Any well founded model of ATR_0^s of height ω_1^{CK} is not a model of KP_0^-. Any well founded model of KP_0^- of height ω_1^{CK} is not a model of of ATR_0^s.

Proof. Immediate from the previous lemma.

Next we present a model theoretic argument showing that ATR_0^s is stronger than KP_0^-.

5.3 Theorem. ATR_0^s proves the existence of an ω-model of $KP_0^- + \Pi_1$ foundation.

Proof. Reasoning in ATR_0, let M_0 and $M_1 \subseteq M_0$ be countable ω-models of ACA_0 as constructed in the proof of Corollary 4.2 (ii) of [27]. Form an ω-structure for the set theoretic language as follows. Interpret set theoretic variables as ranging over trees $T \in M_1$ such that $M_0 \models T$ is suitable. Interpret set theoretic equality = as $\cong$ in M_1, and interpret $\in$ as $\tilde{\in}$ in M_1. (See the proof of Lemma 3.5 above.) It is not hard to see that this interpretation gives a model of $KP_0^- + \Pi_1$ foundation.

5.4 Remark. KP and related systems have been studied from a proof theoretic viewpoint. It is known from Howard [15], [16] and Jäger [20] that KP proves the same arithmetical sentences as $\Pi_\infty^1 - TI_0$ $(= \bigcup_{k\in\omega} \Pi_k^1\text{-}TI_0)$ or equivalently Feferman's system ID_1 [5] or equivalently ACA_0 + parameterless Π^1_1-CA_0, i.e. ACA_0 plus comprehension for Π^1_1 formulas with no free set variables. These results can also be proved model theoretically (cf. Friedman [8]). In any case, it follows that the proof theoretic ordinal of KP is the Howard ordinal $\theta\varepsilon_{\Omega+1}0$. Let KP^- be KP^-_0 plus full induction on the natural numbers. It is known from Friedman (unpublished, but see [7], [9], and footnote 8 of [6]) and Jäger [19] and Cantini [3] that the proof theoretic ordinal of $KP^- + \Pi_1$ foundation is $\theta\varepsilon_0 0$. On the other hand, by §4 of [13] together with Theorem 3.6 above, we know that the proof theoretic ordinal of ATR^s_0 is $\Gamma_0 = \theta\Omega 0$. Thus it emerges that ATR^s_0 is intermediate in strength between KP and $KP^- + \Pi_1$ foundation.

In the rest of this section we study what many would consider the canonical or obvious interpretation of KP^-_0 into ATR^s_0. It is well known that the smallest well founded model of KP^-_0 is $L_{\omega_1^{CK}}$. In ATR^s_0 we cannot prove that $L_{\omega_1^{CK}}$ exists as a set (see Corollary 4.12 above) but we can interpret the formula $x \in L_{\omega_1^{CK}}$ as an abbreviation for

$$\exists\alpha(x \in L_\alpha \;\&\; \sim\exists\beta \le \alpha\; L_\beta \models KP).$$

We then have:

5.5 Theorem. $ATR^s_0 \vdash (L_{\omega_1^{CK}} \models KP^-_0)$. <u>In other words,</u> ATR^s_0 <u>proves the axioms of</u> KP^-_0 <u>relativized to the transitive class</u> $L_{\omega_1^{CK}}$.

<u>Proof</u>. It is well known and easy to see that $L_{\omega_1^{CK}} = \{|T| : T$ is a hyperarithmetic suitable tree$\}$. The usual proof of this fact goes through in ATR^s_0. Then Δ_0 collection for $L_{\omega_1^{CK}}$ reduces to the well known Σ^1_1 bounding principle which is provable in ATR_0.

In a similar vein we have:

5.6 Theorem.

(i) $ATR^s_0 + \Sigma^1_1\text{-}TI_0 \vdash (L_{\omega_1^{CK}} \models KP^-_0 + \Sigma_1$ <u>foundation</u>).

(ii) $ATR^s_0 + \Pi^1_1\text{-}TI_0 \vdash (L_{\omega_1^{CK}} \models KP^-_0 + \Pi_1$ <u>foundation</u>).

(iii) $ATR^s_0 + \Pi^1_\infty\text{-}TI_0 \vdash (L_{\omega_1^{CK}} \models KP)$.

<u>Proof</u>. In ATR^s_0 one can prove as usual [2] that a property of hyperarithmetic sets is Π^1_1 if and only if it is Σ_1 over $L_{\omega_1^{CK}}$. This gives (i) and (ii),

and (iii) is also clear.

5.7 Corollary. KP_0^- does not prove Σ_1 or Π_1 foundation. In fact, KP_0^- does not prove

$$\varphi(0) \& \forall k \in \omega(\varphi(k) \to \varphi(k+1)) \to \forall k \in \omega \varphi(k)$$

for Σ_1 or Π_1 set theoretic formulas φ.

Proof. It follows from the proof of Corollary 2.10 of [27] that ATR_0 is consistent with the failure of some parameterless instances of Π_1^1 and Σ_1^1 induction on the natural numbers. As in Theorem 5.6 these failures of Π_1^1 and Σ_1^1 induction, interpreted in $L_{\omega_1^{CK}}$, become failures of Σ_1 and Π_1 foundation respectively.

5.8 Corollary. There is an ω-model of $KP^- + \Pi_1$ foundation which is not a model of Σ_1 foundation.

Proof. By the proof of Corollary 4.3 of [27] let M be an ω-model of $\Sigma_1^1\text{-}TI_0$ in which there is a failure of some parameterless instance of $\Pi_1^1\text{-}TI_0$. As in the proof of the previous Corollary, the $L_{\omega_1^{CK}}$ of M^S satisfies Π_1 foundation but not Σ_1 foundation.

We finish with some inconclusive remarks concerning the formalization of ordinal recursion theory. An occasionally useful theorem of KP is the Σ_1 recursion theorem ([1],[2]). The usual proof of Σ_1 recursion is formalizable in $KP_0^- + \Sigma_1$ foundation but apparently not in $KP_0^- + \Pi_1$ foundation. It would be interesting to know how much of the foundation scheme is needed to carry out the metarecursive priority arguments of Kreisel/Sacks [18] and Driscoll [4]. (For that matter, how much ordinary induction is needed for ordinary priority arguments on ω?) The most usual form of a metarecursive priority construction is that one defines a metarecursively enumerable set $A = \bigcup\{A^\sigma : \sigma<\omega_1^{CK}\}$ where the binary relation $\xi \in A^\sigma$ is explicitly primitive recursive. This can be carried out in KP_0^-. One then proves by induction on $n \in \omega$ that the n^{th} requirement is satisfield. This step seems to require instances of induction up to n which by Corollary 5.7 are not provable in KP_0^-. It may be necessary to use a nonrecursive indexing of requirements by integers less than some fixed (nonstandard) integer n.

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