Notes on Undecidability and Incompleteness

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Abstract

Let Q be Robinson's weak theory of arithmetic. We use recursion-theoretical methods to show that Q is essentially undecidable. Consequently, any recursively axiomatizable theory in which Q is interpretable is undecidable and incomplete. This is a strengthening of theorems of Gödel, Rosser and Tarski. We also present proofs of Gödel's First and Second Incompleteness Theorems. In these proofs, the role of Q is perhaps a bit unusual.

1 Undecidable Theories

This section is based on a talk which I gave on November 18, 2008 in the Penn State Logic Seminar. Sankha Basu took notes, and this section is essentially a polished version of those notes.

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Recall Chapter 2 of my Math 558 notes [1], where we proved that all recursive functions are definable over $(\mathbb{N}, +, \cdot, 0, 1, =)$. We now begin with a refinement of that result.

Definition 1.1. The Δ_0 *formulas* are the smallest class of number-theoretical formulas containing all *atomic* formulas (i.e., of the form $t_1 = t_2$ where t_1, t_2 are polynomials with coefficients from $\mathbb{N}$) and closed under $\wedge$, $\vee$, $\neg$, $\Rightarrow$, $\Leftrightarrow$ and *bounded quantifiers* $(\forall x < t)$, $(\exists x < t)$ where t is a term not involving x. Note that $(\forall x < t)\, \Phi \equiv \forall x\, (x < t \Rightarrow \Phi)$ and $(\exists x < t)\, \Phi \equiv \exists x\, (x < t \wedge \Phi)$.

Definition 1.2. A Δ_0 *predicate* is a number-theoretical predicate $P \subseteq \mathbb{N}^k$ which is defined over $(\mathbb{N}, +, \cdot, 0, 1, =, <)$ by a Δ_0 formula.

Remark 1.3. Obviously all Δ_0 predicates are primitive recursive. It can be shown that the Δ_0 predicates are a small subclass of the primitive recursive predicates. Trivially, the class of Δ_0 predicates is closed under $\wedge$, $\vee$, $\neg$, $\Rightarrow$, $\Leftrightarrow$ and bounded quantification.

Definition 1.4. A Σ_1 *formula* is a formula of the form $\exists x\, \Phi$ where x is a number variable and Φ is a Δ_0 formula. A *generalized* Σ_1 *formula* is a formula of the form $\exists x_1 \cdots \exists x_k\, \Phi$ where $x_1, \ldots, x_k$ are number variables and Φ is a Δ_0 formula. A Σ_1 *predicate* is a number-theoretical predicate which is defined over $\mathbb{N}$ by a Σ_1 formula. Equivalently, it is defined over $\mathbb{N}$ by a generalized Σ_1 formula.

Lemma 1.5. The class of Σ_1 predicates is closed under $\wedge$, $\vee$, $\exists x$, $\exists x < t$, $\forall x < t$.

Proof.

$$\begin{aligned} \exists x\, \exists y\, \underbrace{P(x, y, -)}_{\Delta_0} &\equiv \exists z\, \underbrace{(\exists x < z)\, (\exists y < z)\, P(x, y, -).}_{\Delta_0} \\ (\exists x\, P(x, -)) \wedge (\exists y\, Q(y, -)) &\equiv \exists x\, \exists y\, (P(x, -) \wedge Q(y, -)). \\ (\forall x < t)\, \exists y\, \ldots &\equiv \exists z\, (\forall x < t)\, (\exists y < z)\, \ldots. \end{aligned}$$

□

Theorem 1.6. For every recursive function $f : \mathbb{N}^k \to \mathbb{N}$, the graph of f is a Σ_1 predicate.

Proof. We use the familiar characterization of recursive functions in terms of composition, primitive recursion and minimization.

Composition:

For example, suppose $f = g \circ h$. Then

$$f(x) = y \;\equiv\; \exists z\, (\underbrace{h(x) = z}_{\Sigma_1} \wedge \underbrace{g(z) = y}_{\Sigma_1})$$

so this is Σ_1.

Minimization:

Suppose $f(-) =$ least y such that $R(-, y)$, where R is recursive. Then

$$f(-) = y \;\equiv\; \underbrace{\chi_R(-, y) = 1}_{\Sigma_1} \wedge (\forall z < y)\; \underbrace{(\chi_R(-, z) = 1)}_{\Sigma_1}$$

so this is Σ_1.

Primitive Recursion:

Recall the Gödel beta-function, $\beta(a, r, i) = \mathrm{Rem}(r, a \cdot (i + 1) + 1)$. We have

$$\beta(a, r, i) = v \;\equiv\; \exists u\, ((r = u \cdot (a \cdot (i + 1) + 1) + v) \wedge (v < a \cdot (i + 1) + 1))$$

so the graph of the beta-function is Σ_1. Now suppose f is obtained by primitive recursion, $f(0,-) = g(-)$ and $f(x+1,-) = h(x, f(x,-), -)$ where the graphs of g and h are Σ_1. Then

$$\begin{aligned} f(x,-) = w \quad \equiv \quad & \exists a\, \exists r\, (\beta(a,r,0) = g(-) \wedge \beta(a,r,x) = w \\ & \wedge (\forall i < x)\, \beta(a,r,i+1) = h(i, \beta(a,r,i), -)) \end{aligned}$$

so the graph of f is Σ_1. □

Corollary 1.7. The Σ_1 predicates are the same as the Σ_1^0 predicates.

Proof. Clearly Σ_1 implies Σ_1^0. Conversely, given a Σ_1^0 predicate $P(-) \equiv \exists x\, R(x,-)$ where R is recursive, we have

$$P(-) \;\equiv\; \exists x\, \underbrace{(\chi_R(x,-) = 1)}_{\Sigma_1}$$

and this is Σ_1. □

Definition 1.8. We consider theories $T = (L, S)$ where L is a finite language, the *language* of T, and S is a set of L-sentences, the *axioms* of T. Recall that an *L-sentence* is an L-formula with no free variables. Let B range over L-sentences.

1. $T \vdash B$ means that B is a theorem of T.
2. T is *consistent* if $T \nvdash \Phi \wedge \neg \Phi$.
3. T is *complete* if for all B either $T \vdash B$ or $T \vdash \neg B$.
4. T is *decidable* if $\mathrm{Thm}_T = \{\#(B) \mid T \vdash B\}$ is recursive.
5. T is *recursively axiomatizable* if $\mathrm{Ax}_T := \{\#(B) \mid B \in S\}$ is recursive.

Examples 1.9.

1. $\mathrm{Th}(\mathbb{N}, +, \cdot, 0, 1, =)$, the complete theory of the natural numbers, is not decidable and not recursively axiomatizable. In this case $L = \{+, \cdot, 0, 1, =\}$ and $S = \mathrm{TrueSnt}_{\mathbb{N}}$.
2. $\mathrm{Th}(\mathbb{R}, +, \cdot, 0, 1, =)$, the complete theory of the real numbers, is decidable. This is a consequence of quantifier elimination. In this case $S = \mathrm{TrueSnt}_{\mathbb{R}}$.

Theorem 1.10. If T is recursively axiomatizable, then Thm_T is recursively enumerable, i.e., Σ_1^0.

Sketch of proof. Use the tableau method or some other proof system. Given an L-sentence B, search for a finite proof of B from the axioms of T. □

Theorem 1.11. If T is recursively axiomatizable and complete, then T is decidable.

Proof. Let B range over L-sentences. By Theorem 1.10 we have that $\{\#(B) \mid T \vdash B\}$ is Σ^0_1. But then, by completeness of T, we also have that $\{\#(B) \mid T \nvdash B\} = \{\#(B) \mid T \vdash \neg B\}$ is Σ^0_1. Hence $\{\#(B) \mid T \vdash B\}$ is Δ^0_1, i.e., recursive. Thus T is decidable. $\square$

Definition 1.12. Consider the particular theory Q. Roughly speaking, Q is first-order arithmetic minus the induction scheme. Formally, the language of Q is $\{+, \cdot, 0, =, S\}$ where, $+$ and $\cdot$ are binary operations, S is a unary operation, 0 is a constant, and $=$ is a binary predicate. The axioms of Q are:

- $\forall x \, \forall y \, (Sx = Sy \Rightarrow x = y)$
- $\forall x \, (Sx \neq 0)$
- $\forall y \, (y \neq 0 \Rightarrow \exists x \, (Sx = y))$
- $\forall x \, (x + 0 = x)$
- $\forall x \, \forall y \, (x + Sy = S(x + y))$
- $\forall x \, (x \cdot 0 = 0)$
- $\forall x \, \forall y \, (x \cdot Sy = x \cdot y + x)$

The idea of using Q is due to Tarski/Mostowski/Robinson [2].

Remark 1.13. Q is a very weak fragment of first-order arithmetic. For instance $\mathsf{Q} \nvdash \forall x \, (0 + x = x)$, etc.

Notation 1.14.

1. We write $\underline{n} = \underbrace{S \cdots S}_{n \text{ times}} 0$. Or, inductively, $\underline{0} = 0$ and $\underline{n+1} = S\underline{n}$ for each $n \in \mathbb{N}$.
2. We introduce a 2-place predicate $\leq$ by $x \leq y \equiv \exists z \, (z + x = y)$.

Lemma 1.15. For each $m, n \in \mathbb{N}$ the following are provable in Q:

1. $\underline{m} + \underline{n} = \underline{m + n}$
2. $\underline{m} \cdot \underline{n} = \underline{m \cdot n}$
3. $\forall x \, (x \leq \underline{n} \Leftrightarrow (x = \underline{0} \lor x = \underline{1} \lor \cdots \lor x = \underline{n-1} \lor x = \underline{n}))$
4. $\underline{m} \neq \underline{n}$ if $m \neq n$
5. $\underline{m} \leq \underline{n}$ if $m \leq n$
6. $\forall x \, (x \leq \underline{n} \lor \underline{n} \leq x)$

Proof. By external induction on n.

1. E.g., $\underline{3}+\underline{2} = SSS0+SS0 = S(SSS0+S0) = SS(SSS0+0) = SSSSS0 = \underline{5}$.

2. Similar.

3. We prove this by external induction on n.

   Base step, $n = 0$: $x \leq \underline{0}$. So $\exists w\,(w + x = 0)$. This implies, $x = 0$ because otherwise $x = Su$ for some u. Then $0 = w + Su = S(w + u)$ which is a contradiction.

   Induction step: Suppose the claim holds for all $n \leq k$ where, $k \geq 0$. We now prove it for $n = k + 1$. Suppose $x \leq \underline{k+1}$. Then $w + x = \underline{k+1}$ for some w. If $x = 0$, we are done. If $x \neq 0$, $x = Su$ for some u, so $w + x = w + Su = \underline{k+1}$. Now $w + Su = S(w + u)$, so $w + u = \underline{k}$. Then by induction hypothesis $u = \underline{0} \vee u = \underline{1} \vee \cdots \vee u = \underline{n}$ which implies $x = \underline{1} \vee x = \underline{2} \vee \cdots \vee x = \underline{k+1}$, Q.E.D.

4. E.g., we prove $\underline{2} \neq \underline{3}$. Suppose $\underline{2} = \underline{3}$, i.e., $SS0 = SSS0$. This implies $S0 = SS0$ which in turn implies $0 = S0$, a contradiction.

5. Similar.

6. Suppose, $x \not\leq \underline{n}$. Then by part 3, $x \neq \underline{0}, \underline{1}, \ldots, \underline{n}$, so we can deduce from the axioms of Q that $x = \underbrace{SS\cdots S}_{n+1}w$ for some w. Then $x = \underbrace{SS\cdots S}_{n+1}w = Sw + \underbrace{SS\cdots S}_{n}0 = Sw + \underline{n}$, hence $\underline{n} \leq x$.

□

Theorem 1.16. For all Δ_0 sentences Φ we have:

1. $\mathsf{Q} \vdash \Phi$ if and only if Φ is true.

2. $\mathsf{Q} \vdash \neg\,\Phi$ if and only if Φ is not true.

Proof. The proof is by induction on the number of connectives and quantifiers in Φ. Suppose for instance that $\Phi \equiv (\forall x \leq t)\,\Psi$ where t is a variable-free term. By inductive hypothesis, $\mathsf{Q} \vdash \Psi[x/\underline{n}]$ for all $n \leq t$. Using part 3 of Lemma 1.15, we have $\mathsf{Q} \vdash \forall x\,(x \leq t \Rightarrow \Psi)$, i.e., $\mathsf{Q} \vdash \Phi$, Q.E.D. Next, suppose $\Phi \equiv (\exists x \leq t)\,\Psi$. By inductive hypothesis, there is $n \leq t$ such that $\mathsf{Q} \vdash \Psi[x/\underline{n}]$. Also, $\mathsf{Q} \vdash \underline{n} \leq t$, so $\mathsf{Q} \vdash (\exists x \leq t)\Psi$, i.e., $\mathsf{Q} \vdash \Phi$, Q.E.D. □

Theorem 1.17. For all generalized Σ_1 sentences S, $\mathsf{Q} \vdash S$ if and only if S is true.

Proof. Since the axioms of Q are true, all sentences provable in Q are true. For the converse, let $S \equiv \exists x_1 \cdots \exists x_k\,\Phi$ where Φ is Δ_0. If S is true, let $m_1, \ldots, m_k \in \mathbb{N}$ be such that $\Phi[x_1/\underline{m_1}, \ldots, x_k/\underline{m_k}]$ is true. Then, by the previous theorem, $\mathsf{Q} \vdash \Phi[x_1/\underline{m_1}, \ldots, x_k/\underline{m_k}]$. It follows that $\mathsf{Q} \vdash \exists x_1 \cdots \exists x_k\,\Phi$, i.e., $\mathsf{Q} \vdash S$, Q.E.D. □

Theorem 1.18. Q is undecidable.

Proof. Let H = the Halting Problem. Recall that H is Σ^0_1 but not recursive. Let Halt be a Σ_1 formula defining H. Then

$$\begin{aligned} H &= \{n \mid \mathrm{Halt}[x/\underline{n}] \text{ is true}\} \\ &= \{n \mid \mathsf{Q} \vdash \mathrm{Halt}[x/\underline{n}]\} \\ &= \{n \mid \mathsf{Q} \vdash \exists x\,(x = \underline{n} \wedge \mathrm{Halt})\}. \end{aligned}$$

Now $f(n) := \#(\exists x\,(x = \underline{n} \wedge \mathrm{Halt}))$ is a primitive recursive function. Moreover H is reducible to $\mathrm{Thm}_{\mathsf{Q}} = \{\#(B) \mid \mathsf{Q} \vdash B\}$ via f. In other words, $n \in H \Leftrightarrow f(n) \in \mathrm{Thm}_{\mathsf{Q}}$. Since H is nonrecursive, it follows that $\mathrm{Thm}_{\mathsf{Q}}$ is nonrecursive, i.e., Q is undecidable. □

Remark 1.19. It follows from Theorems 1.18 and 1.11 that Q is incomplete, but this was obvious anyway. We are now going to show that any consistent theory which contains Q is incomplete.

Definition 1.20. Two sets $A, B \subseteq \mathbb{N}$ are said to be *recursively inseparable* if there is no recursive set X such that $A \subseteq X$ and $X \cap B = \emptyset$.

Remark 1.21. Let $A, B \subseteq \mathbb{N}$ be recursively enumerable (i.e., Σ^0_1), and disjoint and recursively inseparable. It is well known that such a pair of sets exists. Let A, B be defined by Σ_1 formulas $\exists y\,\Phi$ and $\exists z\,\Psi$ be Σ_1 respectively. Thus $A = \{m \mid \exists y\,\Phi[x/\underline{m}]\}$ and $B = \{m \mid \exists z\,\Psi[x/\underline{m}]\}$ where Φ and Ψ are Δ_0 formulas. Let $\Phi^* \equiv \Phi \wedge \neg\,(\exists z \leq y)\,\Psi$. This is again a Δ_0 formula. Note that $A = \{m \mid \exists y\,\Phi^*[x/\underline{m}]\}$. The passage from Φ to Φ^* is known as Rosser's Trick.

Theorem 1.22. Let T be a consistent theory which contains Q. Then T is undecidable.

Proof. Let A, B, Φ, Ψ, Φ^* be as in Remark 1.21. Let

$$A^* = \{m \in \mathbb{N} \mid T \vdash \exists y\,\Phi^*[x/\underline{m}]\}.$$

As before, A^* is reducible to Thm_T. Thus, it will suffice to show that A^* is not recursive.

Obviously $A^* \supseteq A$, because for all $m \in A$ we have $\mathsf{Q} \vdash \exists y\,\Phi^*[x/\underline{m}]$, hence $T \vdash \exists y\,\Phi^*[x/\underline{m}]$, hence $m \in A^*$.

We claim that $A^* \cap B = \emptyset$. To see this, suppose $m \in A^* \cap B$. Because $m \in A^*$ we have $T \vdash \exists y\,\Phi^*[x/\underline{m}]$, i.e.,

$$T \vdash \exists y\,(\Phi[x/\underline{m}] \wedge \neg\,(\exists z \leq y)\,\Psi[x/\underline{m}]). \tag{1}$$

At the same time, because $m \in B$ we have $\exists z\,\Psi[x/\underline{m}]$ so let $n \in \mathbb{N}$ be such that $\Psi[x/\underline{m}, z/\underline{n}]$ holds. Then $\mathsf{Q} \vdash \Psi[x/\underline{m}, z/\underline{n}]$, hence

$$T \vdash \Psi[x/\underline{m}, z/\underline{n}]. \tag{2}$$

Combining (1) and (2) we obtain

$$T \vdash \exists y\,(\Phi[x/\underline{m}] \land \underline{n} \nleq y) \tag{3}$$

and from (3) and Lemma 1.15 it follows that

$$T \vdash \exists y\,(\Phi[x/\underline{m}] \land (y = \underline{0} \lor \cdots \lor y = \underline{n})),$$

i.e.,

$$T \vdash \Phi[x/\underline{m}, y/\underline{0}] \lor \cdots \lor \Phi[x/\underline{m}, y/\underline{n}]. \tag{4}$$

On the other hand, because $m \in B$ and $A \cap B = \emptyset$ we have $m \notin A$, hence $\neg \exists y\, \Phi[x/\underline{m}]$ holds, hence for all $k \in \mathbb{N}$ we have $\neg \Phi[x/\underline{m}, y/\underline{k}]$ hence $\mathsf{Q} \vdash \neg \Phi[x/\underline{m}, y/\underline{k}]$ hence $T \vdash \neg \Phi[x/\underline{m}, y/\underline{k}]$ so in particular

$$T \vdash \neg\,(\Phi[x/\underline{m}, y/\underline{0}] \lor \cdots \lor \Phi[x/\underline{m}, y/\underline{n}]). \tag{5}$$

Now (4) and (5) contradict our assumption that T is consistent. This proves our claim.

We have seen that $A^* \supseteq A$ and $A \cap B = \emptyset$. Since A and B are recursively inseparable, it follows that A^* is nonrecursive. Hence Thm_T is nonrecursive, i.e., T is undecidable. □

Theorem 1.23. Let T be a recursively axiomatizable, consistent theory which contains Q. Then T is undecidable and incomplete.

Proof. Immediate from Theorems 1.11 and 1.22. □

Corollary 1.24. Each of the theories Z_1, Z_2, ZFC, ... is undecidable and incomplete.

Proof. Let T be any of these recursively axiomatizable theories. Clearly Q is *interpretable* in T, i.e., we can find a definitional extension T' of T which contains Q. Then, by Theorems 1.22 and 1.23, T' is undecidable and incomplete. Using known results on definitional extensions, it follows that T is undecidable and incomplete. □

2 The Incompleteness Theorems of Gödel

In this section we sketch a proof of Gödel's First and Second Incompleteness Theorems. I presented this proof in December 2009 in a course at Penn State. In this presentation, the role of Robinson's theory Q is perhaps a bit unusual.

Lemma 2.1. Let $f : \mathbb{N} \to \mathbb{N}$ be a recursive function. Then, we can find a Σ_1 formula F in the language of Q which *represents* f in the sense that

$$\mathsf{Q} \vdash \forall x\,(x = \underline{f(m)} \Leftrightarrow F[w/\underline{m}])$$

for each $m \in \mathbb{N}$. The free variables of F are w and x.

Proof. Since f is recursive, the graph of f is Σ_1-definable over the natural numbers. Let $\exists y_1 \cdots \exists y_k \, \Phi$ be a Σ_1 formula with free variables w and x which defines f over $\mathbb{N}$. Thus for all $m, n \in \mathbb{N}$ we have that $f(m) = n$ if and only if $\exists y_1 \cdots \exists y_k \, \Phi[w/\underline{m}, x/\underline{n}]$ holds in $\mathbb{N}$. Let F be the following Σ_1 formula:

There exists z such that

1. $z =$ the least z such that $(\exists x, y_1, \ldots, y_k \leq z) \, \Phi$,
2. $x =$ the least $x \leq z$ such that $(\exists y_1, \ldots, y_k \leq z) \, \Phi$.

Note that the free variables of F are w and x. We can use our lemmas concerning Q to show that F has the desired property. □

Lemma 2.2 (Self-Reference Lemma)**.** Let L be a recursive language which includes the language of Q. Let A be an L-formula. Let T be an L-theory which includes Q. Let x be a number variable. Then, we can find an L-formula B such that

$$T \vdash B \Leftrightarrow A[x/\underline{\#(B)}].$$

Remark 2.3. The free variables of B are those of A except for x. In particular, if x is the only free variable of A, then B is an L-sentence.

Proof of Lemma 2.2. Let w be a number variable different from x and which does not occur in A. For all $m \in \mathbb{N}$ let $d(m) = \mathrm{sub}(m, \#(w), \mathrm{num}(m))$. Note that $d : \mathbb{N} \to \mathbb{N}$ is a primitive recursive function and for all L-formulas C we have $d(\#(C)) = \#(C[w/\underline{\#(C)}])$. Let D be a Σ_1 formula which represents d as in Lemma 2.1. The free variables of D are w and x and for each $m \in \mathbb{N}$ we have

$$\mathsf{Q} \vdash \forall x \, (x = \underline{d(m)} \Leftrightarrow D[w/\underline{m}]).$$

Let A^* be the formula $\forall x \, (D \Rightarrow A)$, and let B be the formula $A^*[w/\underline{\#(A^*)}]$. Since $d(\#(A^*)) = \#(B)$, we have

$$\mathsf{Q} \vdash \forall x \, (x = \underline{\#(B)} \Leftrightarrow D[w/\underline{\#(A^*)}]).$$

It follows that

$$T \vdash A[x/\underline{\#(B)}] \Leftrightarrow \forall x \, (D[w/\underline{\#(A^*)}] \Rightarrow A),$$

i.e.,

$$T \vdash A[x/\underline{\#(B)}] \Leftrightarrow A^*[w/\underline{\#(A^*)}],$$

i.e.,

$$T \vdash A[x/\underline{\#(B)}] \Leftrightarrow B.$$

This completes the proof. □

Definition 2.4 (the provability predicate)**.** Let L be a recursive language, and let T be a recursively axiomatizable L-theory. By Theorem 1.10 the set $\mathrm{Thm}_T = \{\#(B) \mid T \vdash B\}$ is recursively enumerable, i.e., Σ^0_1. By Corollary 1.7 let Pvbl_T be a Σ_1 formula which defines Thm_T over $\mathbb{N}$. Thus $\mathrm{Thm}_T = \{m \in \mathbb{N} \mid \mathrm{Pvbl}_T[x/\underline{m}]\}$. Here x is number variable, the only free variable of Pvbl_T. The Σ_1 formula Pvbl_T is called *the provability predicate* for T.

Lemma 2.5. Let L be a recursive language which includes the language of Q. Let T be a recursively axiomatizable L-theory. Then, we can find a sentence $G = G_T$ such that

$$\mathsf{Q} \vdash G \Leftrightarrow \neg\, \mathrm{Pvbl}_T[x/\underline{\#(G)}].$$

Moreover G is of the form $\neg\, S$ where S is a generalized Σ_1 sentence.

Proof. This is essentially the special case of Lemma 2.2 with $T = \mathsf{Q}$ and $A = \neg\, \mathrm{Pvbl}_T$. However, we have to slightly modify the proof of Lemma 2.2. Instead of A^* use the logically equivalent formula $\neg\, S^*$ where S^* is generalized Σ_1. Explicitly, S^* is $\exists x\, \exists y\, \exists z\, (\Phi \wedge \Psi)$ where D is $\exists y\, \Phi$ and Pvbl_T is $\exists z\, \Psi$ and Φ and Ψ are Δ_0 formulas. Then G is $(\neg\, S^*)[w/\underline{\#(\neg\, S^*)}]$ and the proof goes through as before. Note that G is identical to $\neg\, S$ where S is the generalized Σ_1 sentence $S^*[w/\underline{\#(\neg\, S^*)}]$. □

Theorem 2.6 (The First Incompleteness Theorem)**.** Let L be a recursive language which includes the language of Q. Let T be a recursively axiomatizable L-theory which includes Q. If T is consistent, then $T \nvdash G_T$.

Proof. Let $G = G_T$ and suppose $T \vdash G$. Then $\mathrm{Pvbl}_T[x/\underline{\#(G)}]$ is true. Since the axioms of Q are true, it follows by Lemma 2.5 that $\neg\, G$ is true. Moreover $\neg\, G$ is logically equivalent to a generalized Σ_1 sentence, so by Theorem 1.17 we have $\mathsf{Q} \vdash \neg\, G$. Since $T \supseteq \mathsf{Q}$ it follows that $T \vdash \neg\, G$. Thus T is inconsistent. □

Remark 2.7. Theorem 2.6 is a refinement of Gödel's First Incompleteness Theorem. The sentence G_T is known as *the Gödel sentence* for T. Note that G_T, although not provable in T, is true. This is because, by Lemma 2.5, G_T is equivalent to $T \nvdash G_T$, and the latter statement is true in view of Theorem 2.6.

Remark 2.8. Next we turn to Gödel's Second Incompleteness Theorem. In order to prove the Second Incompleteness Theorem, we need an additonal condition on T. The condition that we need is *adequacy* as formulated in the following definition. It can be shown that each of the theories Z_1, Z_2, ZFC, etc., is adequate. It is not clear whether Q is adequate. It is conceivable that the adequacy of Q may depend on the choice of the provability predicate $\mathrm{Pvbl}_{\mathsf{Q}}$.

Definition 2.9 (adequacy)**.** Let L be a recursive language which includes the language of Q. Let T be a recursively axiomatizable L-theory which includes Q. We say that T is *adequate* if

$$T \vdash S \Rightarrow \mathrm{Pvbl}_T[x/\underline{\#(S)}]$$

for all generalized Σ_1 sentences S. Note that, in view of Theorem 1.17, all sentences of the form $S \Rightarrow \mathrm{Pvbl}_T[x/\underline{\#(S)}]$ where S is a generalized Σ_1 sentence are true. Hence, it is reasonable to expect these sentences to be provable in T.

Definition 2.10 (the consistency sentence)**.** Let L be a recursive language which includes the language of Q. Let T be a recursively axiomatizable L-theory which includes Q. Let $G = G_T$ be the Gödel sentence for T. Recall that

G is of the form $\neg S$ where S is a generalized Σ_1 sentence. Let Con_T be the sentence $\neg(\mathrm{Pvbl}_T[x/\underline{\#(S)}] \wedge \mathrm{Pvbl}_T[x/\underline{\#(\neg S)}])$. Note that T is consistent if and only if Con_T is true. The sentence Con_T is known as *the consistency sentence* for T.

Theorem 2.11 (the Second Incompleteness Theorem)**.** Let L be a recursive language which includes the language of Q. Let T be a recursively axiomatizable L-theory which includes Q and is adequate. If T is consistent, then $T \nvdash \mathrm{Con}_T$.

Proof. Write $G = G_T$. By Theorem 2.6 we have $T \nvdash G$. Therefore, it suffices to show that $T \vdash \mathrm{Con}_T \Rightarrow G$, i.e., $T \vdash S \Rightarrow \neg\,\mathrm{Con}_T$. Here $G = \neg S$ where S is a generalized Σ_1 sentence. By adequacy of T we have

$$T \vdash S \Rightarrow \mathrm{Pvbl}_T[x/[\underline{\#(S)}]. \tag{6}$$

On the other hand, by Lemma 2.5 we have

$$T \vdash G \Leftrightarrow \neg\,\mathrm{Pvbl}_T[x/\underline{\#(G)}],$$

i.e.,

$$T \vdash S \Leftrightarrow \mathrm{Pvbl}_T[x/\underline{\#(\neg S)}]. \tag{7}$$

Combining (6) and (7) we have

$$T \vdash S \Rightarrow (\mathrm{Pvbl}_T[x/[\underline{\#(S)}] \wedge \mathrm{Pvbl}_T[x/\underline{\#(\neg S)}]),$$

i.e., $T \vdash S \Rightarrow \neg\,\mathrm{Con}_T$, Q.E.D. □

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