Finitely generated nilpotent groups are finitely presented and residually finite

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Definition 1. Let G be a group. G is said to be *residually finite* if the intersection of all normal subgroups of G of finite index in G is trivial.

For a survey of results on residual finiteness and related properties, see Magnus, Karrass, and Solitar [6, Section 6.5].

We shall present a proof of the following well known theorem, which is important for Kharlampovich [4, 5]. See also O. V. Belegradek's review of [5] in Mathematical Reviews.

Theorem 2. *Let G be a finitely generated, nilpotent group. Then G is residually finite.*

Actually, we shall prove a more general result.

Definition 3. A group G is said to be *supersolvable* if $G = H_0 \supset H_1 \supset \cdots \supset H_n = 1$, where each H_i is a normal subgroup of G, and each of the quotient groups H_{i-1}/H_i is cyclic.

Lemma 4. *Let G be a finitely generated, nilpotent group. Then G is supersolvable.*

Proof. In any group G, the commutators of weight k on the generators of G generate G_k, the kth group in the lower central series of G. In particular, if G is finitely generated, then each G_k is finitely generated. Hence, each G_k/G_{k+1} is finitely generated Abelian, i.e., a direct product of finitely many cyclic groups. From this, the lemma follows easily. See also M. Hall [2, Theorem 10.2.4]. □

We shall prove:

Theorem 5. *Let G be a supersolvable group. Then G is residually finite.*

Theorem 5 is due to Hirsch [3, Theorem 3.25]. We follow the proof due to P. Hall as outlined by Gruenberg [1, Section 2.1]. Note also that our arguments and results below would remain valid if, in Definition 3, we were to replace "cyclic" by "finite or cyclic".

Lemma 6. *Every subgroup of a supersolvable group is supersolvable. Every quotient group of a supersolvable group is supersolvable.*

Proof. The proof is straightforward. See M. Hall [2, Theorem 10.5.1]. □

Lemma 7. *Let G be a supersolvable group with all elements of finite order. Then G is finite.*

Proof. If all of the elements of G are of finite order, then each of the quotient groups H_{i-1}/H_i in Definition 3 is finite. Hence G is finite. □

To prove Theorem 5, we proceed by induction on the length of the tower in Definition 3. The inductive step is given by the following lemma.

Lemma 8. *Let G be a group with a normal subgroup H such that G/H is cyclic, H is supersolvable, and H is residually finite. Then G is residually finite.*

Proof. Since G/H is cyclic, let $a \in G$ map to a generator of G/H.

Given $g \in G$, we must find a normal subgroup N of G such that G/N is finite and $g \notin N$. There are two cases.

Case 1: $g \notin H$. If G/H is finite, we are done by setting $N = H$. Otherwise, G/H is infinite cyclic, so let $n \neq 0$ be such that a^n and g have the same image in G/H. Let K be the subgroup of G/H generated by the image of a^{2n}. Clearly the image of a^n does not belong to K. Let N be the preimage of K in G. Then $g \notin N$, and $|G/N| = |(G/H)/K| = 2n$.

Case 2: $g \in H$. Since H is residually finite, let M be a normal subgroup of H such that $g \notin M$ and H/M is finite. Set $m = |H/M|$, and let H^m be the subgroup of H generated by h^m, $h \in H$. For all $h \in H$ we have $h^m \in M$, hence $H^m \subseteq M$.

A typical element of H^m is $h_1^m \cdots h_k^m$ where $h_i \in H$. For all $b \in G$ we have

$$b^{-1}(h_1^m \cdots h_k^m)b = (b^{-1}h_1b)^m \cdots (b^{-1}h_kb)^m \in H^m.$$

Thus H^m is a normal subgroup of G. Taking the quotient group by H^m, we may safely assume that $H^m = 1$. We still have $g \notin M$, hence $g \neq 1$, and $g \in H$.

Since all elements of H are of order dividing m, and since H is supersolvable, we see by Lemma 7 that H is finite. If in addition G/H is finite, then G is finite, so we are done by setting $N = 1$. Otherwise, G/H is infinite cyclic. However, since H is finite, the automorphism group of H is finite, hence the automorphism of H given by $h \mapsto a^{-1}ha$ is of finite order. Thus, for some $n > 0$, we have $a^{-n}ha^n = h$ for all $h \in H$, i.e., a^n commutes with h for all $h \in H$. Let N be the subgroup of G generated by a^n. Then N is a normal subgroup of G, and $g \notin N$. Moreover, $|G/N| = n|H|$ is finite. Our lemma is now proved. □

This completes the proof of Theorem 5, and of Theorem 2.

As an application, we shall show that the word problem of a finitely generated nilpotent group is decidable. (We say "decidable" rather than "solvable," in order to avoid confusion with solvable groups.)

Theorem 9. *Let G be a finitely presented group. If G is residually finite, then the word problem of G is decidable.*

Proof. Let $\langle a_1, \dots, a_n \mid R_1 = \cdots = R_k = 1\rangle$ be a finite presentation of G. Then $G = F/N$, where F is the free group on $a_1, \dots, a_n$ and N is the smallest normal subgroup of F containing $R_1, \dots, R_k$. Clearly N is recursively enumerable. Given a word $W \in F$ we can effectively search for a finite group H and a finite mapping $\phi : \{a_1, \dots, a_n\} \to H$ such that $\phi(R_1) = \cdots = \phi(R_k) = 1$ and $\phi(W) \neq 1$. If $W \notin N$, then we will find such a pair H, ϕ, since G is residually finite. Conversely, if such a pair H, ϕ exists for a given $W \in F$, then we have a homomorphism $\phi : F \to H$ whose kernel includes N and not W, hence $W \notin N$. Thus $F \setminus N$ is recursively enumerable. It follows that N is recursive, i.e., the word problem of G is decidable. $\square$

Theorem 10. *Let G be a supersolvable group. Then G is finitely presented with decidable word problem.*

Proof. By Theorem 5, G is residually finite. By the general Theorem 13 below, G is finitely presented. Hence, by Theorem 9, the word problem of G is decidable. $\square$

Corollary 11. *Let G be a finitely generated, nilpotent group. Then G is finitely presented with decidable word problem.*

Proof. This is immediate from Lemma 4 and Theorem 10. $\square$

Remark 12. On the other hand, Kharlampovich [4] has constructed a solvable, finitely presented group with undecidable word problem. It follows by Theorems 9 and 10 and Corollary 11 that such a group cannot be residually finite, or supersolvable, or nilpotent.

Theorem 13. *Let G be a group. Assume that $G = H_0 \supset H_1 \supset \cdots \supset H_n = 1$ where each H_{i+1} is a normal subgroup of H_i. If each of the quotient groups H_i/H_{i+1} is finitely presented, then G is finitely presented.*

Proof. The proof is by induction on n, the induction step being given by the following lemma. $\square$

Lemma 14. *Let G be a group, and let N be a normal subgroup of G. If N and G/N are finitely presented, then G is finitely presented.*

Proof. Let $\langle a_1, \dots, a_m \mid R_1 = \cdots = R_s = 1\rangle$ be a presentation of N. Let $\langle x_1, \dots, x_n \mid W_1 = \cdots = W_t = 1\rangle$ be a presentation of $H = G/N$. Choose $\overline{x}_1, \dots, \overline{x}_n \in G$ which map to $x_1, \dots, x_n \in H$ respectively. For each a_i and $\overline{x}_j$ we have $a_i \in N$, hence $\overline{x}_j^{-1} a_i \overline{x}_j \in N$, say $\overline{x}_j^{-1} a_i \overline{x}_j = A_{ij}$ where A_{ij} is a word on $a_1, \dots, a_m$. Given a word $W = x_{j_1}^{e_1} \cdots x_{j_l}^{e_l}$ on $x_1, \dots, x_n$, let us write $\overline{W} = \overline{x}_{j_1}^{e_1} \cdots \overline{x}_{j_l}^{e_l}$. For each word W_k in our presentation of H, we have $W_k = 1$ in H, hence $\overline{W}_k \in N$, say $\overline{W}_k = B_k$ where B_k is a word on $a_1, \dots, a_m$. Then $\langle a_1, \dots, a_m, \overline{x}_1, \dots, \overline{x}_n \mid R_1 = \cdots = R_s = 1, \overline{x}_j^{-1} a_i \overline{x}_j = A_{ij}, i = 1, \dots, m, j =$

$1, \ldots, n, \overline{W}_k = B_k, k = 1, \ldots, t\rangle$ is a presentation of G. See also M. Hall [2, Sections 15.1 and 15.4]. $\square$

The proofs of Theorems 10 and 13 and Corollary 11 are now complete.

Remark 15. Note that all of our proofs above go through in RCA_0. Thus, for any recursively enumerable class C of finitely presented groups, if we can prove in RCA_0 that each group in the class C is residually finite, then the word problems of all of the groups in C are uniformly primitively recursively decidable, and this decidability result is again provable in RCA_0. All of this is a consequence of the known metamathematics of RCA_0 (see Simpson [7]). In particular, this applies to the class of supersolvable groups, and to the class of finitely generated, nilpotent groups.

Remark 16. (This remark was added April 8, 2005.) Modifying the proof of Lemma 14, we can prove a variant of Theorem 13 with "finitely presented" replaced by "finitely generated with decidable word problem." Thus, Theorem 10 and Corollary 11 could have been proved without reference to residual finiteness.

References

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