Math 597B – Homework #6

April 20, 2007

A set $U \subseteq 2^{\mathbb{N}}$ is said to be *dense* if $U \cap N_\sigma \neq \emptyset$ for all $\sigma \in 2^{<\mathbb{N}}$. A point $X \in 2^{\mathbb{N}}$ is said to be *weakly generic* if $X \in U$ for all dense Σ^0_1 sets $U \subseteq 2^{\mathbb{N}}$. A point $X \in 2^{\mathbb{N}}$ is said to be *weakly random* if $X \notin P$ for all Π^0_1 sets $P \subseteq 2^{\mathbb{N}}$ of measure 0.

1. (10 points) The concept of weak genericity was used in class to solve some of the problems which were in the Homework #4 problem set. Specifically, we sketched proofs of the following facts:

   (a) The set $\{X \in 2^{\mathbb{N}} \mid X \text{ is weakly generic}\}$ is dense.

   (b) If X is weakly generic, then X is weakly random.

   (c) If X is weakly generic, then X does not obey the Strong Law of Large Numbers, i.e.,

   $$\lim_{n\to\infty} \frac{|\{m < n \mid X(m) = 1\}|}{n} \neq \frac{1}{2}.$$

   In fact, the limit in question does not exist, because the lim sup is 1 and the lim inf is 0.

   (d) We can find $X, Y \in 2^{\mathbb{N}}$ such that $X \oplus Y$ is weakly generic yet $X \equiv_T Y$.

   Please work out the detailed proofs of these facts.

2. (10 points) Given a point $X \in 2^{\mathbb{N}}$, we write $\dim(X)$ = the *effective Hausdorff dimension* of X, defined by

   $$\dim(X) \;=\; \liminf_{n\to\infty} \frac{K(X\restriction n)}{n} \;=\; \liminf_{n\to\infty} \frac{C(X\restriction n)}{n}.$$

   Prove the following:

   (a) For all X we have $0 \leq \dim(X) \leq 1$.

   (b) For all r in the interval $0 \leq r \leq 1$, there exists X such that $\dim(X) = r$.

   (c) If X is random, then $\dim(X) = 1$.

   (d) If X is weakly generic, then $\dim(X) = 0$.

3. (5 points) Define X to be *2-random* if $X \notin \bigcap_n U_n$ whenever $U_n \subseteq 2^{\mathbb{N}}$ is uniformly Σ^0_2 with $\mu(U_n) \leq 1/2^n$ for all n.

   Prove that X is 2-random if and only if X is random relative to $0'$.

4. (5 points) Prove that if X is 2-random then $X' \equiv_T X \oplus 0'$.

5. (5 points) In Problem 4, what if we assume only that X is strongly random?