Math 597B – Homework #5

April 13, 2007

1. A set $A \subseteq \mathbb{N}$ is said to be *simple* if

   (a) A is recursively enumerable,

   (b) $\mathbb{N} \setminus A$ is infinite, and

   (c) $\mathbb{N} \setminus A$ includes no infinite recursively enumerable set.

   Let A be a simple, recursively enumerable set. Prove that A is neither recursive nor many-one complete.

2. For $m, n \in \mathbb{N}$ let us write $m \ll n$ to mean that $f(m) < n$ where f is some fixed fast-growing recursive function. For instance, we could take $f(m) =$ a stack of 2's of height m.

   For $n \in \mathbb{N}$ let us write $C(n) = C(\langle \underbrace{1, \ldots, 1}_{n} \rangle)$.

   Prove that the set $\{n \in \mathbb{N} \mid C(n) \ll n\}$ is simple.

3. Prove that $C(\tau) \leq |\tau| + O(1)$.

   This means: $(\exists c)\,(\forall \tau \in 2^{<\mathbb{N}})\,(C(\tau) \leq |\tau| + c)$.

4. Prove that $C(\tau) \leq K(\tau) + O(1)$.

5. Prove the following series of inequalities:

   (a) $K(\tau) \leq 2C(\tau) + O(1)$.

   (b) $K(\tau) \leq C(\tau) + 2 \log C(\tau) + O(1)$.

   (c) $K(\tau) \leq C(\tau) + \log C(\tau) + 2 \log \log C(\tau) + O(1)$.

   (d) Et cetera.

   Here $\log n$ is defined to be the base 2 logarithm of n.

6. (a) Prove that $K(|\tau|) \leq K(\tau) + O(1)$.

   (b) Prove Zambella's Theorem: For each c, the cardinality of

   $$\{\tau \mid |\tau| = n, K(\tau) \leq K(|\tau|) + c\}$$

   is bounded by a constant which is independent of n.

   (c) In fact, this constant is $O(2^c)$.