

# Math 597B – Homework #5

April 13, 2007

1. A set  $A \subseteq \mathbb{N}$  is said to be *simple* if

- (a)  $A$  is recursively enumerable,
- (b)  $\mathbb{N} \setminus A$  is infinite, and
- (c)  $\mathbb{N} \setminus A$  includes no infinite recursively enumerable set.

Let  $A$  be a simple, recursively enumerable set. Prove that  $A$  is neither recursive nor many-one complete.

2. For  $m, n \in \mathbb{N}$  let us write  $m \ll n$  to mean that  $f(m) < n$  where  $f$  is some fixed fast-growing recursive function. For instance, we could take  $f(m) =$  a stack of 2's of height  $m$ .

For  $n \in \mathbb{N}$  let us write  $C(n) = C(\underbrace{\langle 1, \dots, 1 \rangle}_n)$ .

Prove that the set  $\{n \in \mathbb{N} \mid C(n) \ll n\}$  is simple.

3. Prove that  $C(\tau) \leq |\tau| + O(1)$ .

This means:  $(\exists c) (\forall \tau \in 2^{<\mathbb{N}}) (C(\tau) \leq |\tau| + c)$ .

4. Prove that  $C(\tau) \leq K(\tau) + O(1)$ .

5. Prove the following series of inequalities:

- (a)  $K(\tau) \leq 2C(\tau) + O(1)$ .
- (b)  $K(\tau) \leq C(\tau) + 2 \log C(\tau) + O(1)$ .
- (c)  $K(\tau) \leq C(\tau) + \log C(\tau) + 2 \log \log C(\tau) + O(1)$ .
- (d) Et cetera.

Here  $\log n$  is defined to be the base 2 logarithm of  $n$ .

6. (a) Prove that  $K(|\tau|) \leq K(\tau) + O(1)$ .

(b) Prove Zambella's Theorem: For each  $c$ , the cardinality of

$$\{\tau \mid |\tau| = n, K(\tau) \leq K(|\tau|) + c\}$$

is bounded by a constant which is independent of  $n$ .

(c) In fact, this constant is  $O(2^c)$ .