Math 597B – Homework #4

March 21, 2007

1. Assume that $A \in 2^{\mathbb{N}}$ is random (in the sense of Martin-Löf). Let $f(n)$ be a total recursive function such that

$$\limsup_{n \to \infty} f(n) = \infty .$$

   Let U_n, $n = 1, 2, \ldots$, be a uniformly Σ^0_1 sequence of subsets of $2^{\mathbb{N}}$ such that $\mu(U_n) \le 1/f(n)$ for all n. Prove that $A \notin \bigcap_{n=1}^{\infty} U_n$.

2. Assume that $A \in 2^{\mathbb{N}}$ is random. Prove that

$$\lim_{n \to \infty} \frac{|\{m < n \mid A(m) = 1\}|}{n} = \frac{1}{2} .$$

3. In Problem 2, does your proof yield an estimate on the rate of convergence to 1/2?

4. In Problem 2, what if we assume only that A is weakly random?

5. Assume that $A \oplus B \in 2^{\mathbb{N}}$ is random. Prove that the Turing degrees $\mathbf{a} = \deg_T(A)$ and $\mathbf{b} = \deg_T(B)$ are incomparable.

6. In Problem 5, what if we assume only that $A \oplus B$ is weakly random?

7. Let $\mathbf{a} = \deg_T(A)$ where A is strongly random. Prove that $\mathbf{a} \wedge \mathbf{0}' = \mathbf{0}$.

   Note: It can be shown that this does not hold if we assume only that A is random. In fact, we can find a random A such that $A \le_T 0'$.