Math 597B – Homework #3

February 23, 2007

Recall that $\vee$ and $\wedge$ denote, respectively, *join* (i.e., the least upper bound operator) and *meet* (i.e., the greatest lower bound operator) in the partial ordering of Turing degrees. A Turing degree $\mathbf{c}$ is said to be *join-reducible* (respectively *meet-reducible*) if there exist Turing degrees $\mathbf{a} \neq \mathbf{c}$ and $\mathbf{b} \neq \mathbf{c}$ such that $\mathbf{a} \vee \mathbf{b} = \mathbf{c}$ (respectively $\mathbf{a} \wedge \mathbf{b} = \mathbf{c}$).

Note: For an arbitrary pair of Turing degrees $\mathbf{a} = \deg_T(f)$ and $\mathbf{b} = \deg_T(g)$, we always have $\mathbf{a} \vee \mathbf{b} = \deg_T(f \oplus g)$. On the other hand, $\mathbf{a} \wedge \mathbf{b}$ may or may not exist. See Problem 7 below.

1. Prove that $\mathbf{0}$ is meet-reducible.

2. By relativizing Problem 1, prove that every Turing degree is meet-reducible.

3. Prove that $\mathbf{0}'$ is join-reducible.

4. By relativizing Problem 3 and applying the Friedberg Jump Theorem, prove that every Turing degree $\mathbf{c} \geq \mathbf{0}'$ is join-reducible.

   Note: It can be shown that not every Turing degree $\mathbf{c} > \mathbf{0}$ is join-reducible.

5. Prove the following theorem.

   Given an ascending sequence of Turing degrees

   $$\mathbf{c}_1 < \mathbf{c}_2 < \cdots < \mathbf{c}_n < \cdots$$

   we can find a pair of Turing degrees $\mathbf{a}, \mathbf{b}$ such that for all $\mathbf{d}$, $\exists n\,(\mathbf{d} \leq \mathbf{c}_n)$ if and only if $\mathbf{d} \leq \mathbf{a}$ and $\mathbf{d} \leq \mathbf{b}$.

6. Use the result of Problem 5 to prove that no ascending sequence of Turing degrees has a least upper bound.

7. For any pair of Turing degrees $\mathbf{a}, \mathbf{b}$ as in Problem 5, prove that $\mathbf{a} \wedge \mathbf{b}$ does not exist.