Math 597B – Homework #2

February 19, 2007

Endow $\mathbb{N}$ with the discrete topology.

Endow $\mathbb{N}^{\mathbb{N}}$ with the product topology.

1. Prove that $U \subseteq \mathbb{N}^{\mathbb{N}}$ is open if and only if U is $\Sigma_1^{0,g}$ for some $g \in \mathbb{N}^{\mathbb{N}}$.

2. Prove that $P \subseteq \mathbb{N}^{\mathbb{N}}$ is G_δ (i.e., $P = \bigcap_{n=1}^{\infty} U_n$ for some sequence of open sets $U_1, U_2, \ldots$) if and only if P is $\Pi_2^{0,g}$ for some $g \in \mathbb{N}^{\mathbb{N}}$.

3. Clearly the total functional $\Psi : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ given by

$$\Psi(f)(x) = f(x) + f(x+1)$$

is computable. Write an oracle program which computes Ψ.

4. Consider a total functional $\Phi : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$. Prove that Φ is continuous if and only if Φ is computable relative to some fixed oracle $g \in \mathbb{N}^{\mathbb{N}}$. This means that there exists $e \in \mathbb{N}$ such that, for all $f \in \mathbb{N}^{\mathbb{N}}$ and $x \in \mathbb{N}$,

$$\Phi(f)(x) = \varphi_e^{(1), f \oplus g}(x).$$

5. What is the appropriate generalization of the previous result to partial functionals $\Phi :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$? State and prove this generalization.